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Remarks on the Grunsky norm and *p*th root transformation

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

We show that the norm of the Grunsky operator generated by a univalent function does not decrease with a pth root transformation, $p \geqslant 2$. The result is sharp for each p. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let S be the class of functions $f(z)$ that are analytic and univalent in the unit disk $E=\{z\colon |z|<1\}$, and normalized by the conditions $f(0)=0$, $f'(0)=1$. Each function $f\in S$ generates its Grunsky operator

$$G_f = \{\sqrt{nm}\alpha_{n,m}\}_{n,m=1}^{\infty} : \ell^2 \to \ell^2,$$

where $\alpha_{n,m}$ are determined by the expansion

$$\log\frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{n,m=0}^{\infty} \alpha_{n,m} z^n \zeta^m \quad (z,\zeta\in E).$$

Here ℓ^2 is the Hilbert space of all square-summable complex sequences $X=(x_1,x_2,\ldots)$, with the norm

$$\|X\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}.$$

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The Grunsky operator is an important tool in the theory of conformal mappings. The norm of this linear operator (the Grunsky norm) $\| G_f \|, f \in S$, is at most 1. It is used when estimating the Taylor coefficients and other traditional functionals on various classes of univalent functions, e.g. those with a quasiconformal extension (see e.g. [7, Chs. 3 and 9] and [6,3,4]). The following representation of the Grunsky norm was established in [2] (see also [3]).

Theorem 1. *Let* $w = f(z) \in S$ *and*

$$N(f) = \{q: q(w) \not\equiv \text{const is analytic in } \bar{\mathbb{C}} \setminus f(E)\}.$$

For $q \in N(f)$ *let*

$$q \circ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad 0 < \rho_q < |z| < 1,$$

and

$$\| f \|_q = \left[\sum_{n=1}^{\infty} n|a_n|^2 \Big/ \sum_{n=1}^{\infty} n|a_{-n}|^2 \right]^{1/2}.$$

Then the Grunsky norm $\| G_f \|$ *can be found by the formula*

$$\| G_f \| = \sup_{q \in N(f)} \| f \|_q. \tag{1}$$

Some properties of the Grunsky norm are immediate consequences of Theorem 1. For instance, we have the following inequality for an r-contraction of $f \in S$ which generalizes Pommerenke's observation [7, p. 287, Example 9.2].

Corollary. *Let* $f \in S$ *and* $f_r(z) = r^{-1} f(rz)$, $r \in (0,1)$. *Then*

$$\| G_{f_r} \| \leqslant \| G_f \| r^2. \tag{2}$$

The equality in (2) *holds for all* r *for the function*

$$f(z) = z/(1 - tz)^2 \quad (z \in E, \text{parameter } t \in \bar{E}). \tag{3}$$

Note that for function (3), $\| G_f \| = |t|^2$ and $f_r(z) = z/(1 - rtz)^2$. Hence $\| G_{f_r} \| = |t|^2 r^2 = \| G_f \| r^2$.

Remark. Inequality (2) can be derived from the Grunsky norm definition directly.

In this paper, we apply Theorem 1 to prove a sharp Grunsky norm inequality arising from the pth root transformations.

2. *P*-symmetric functions

A function $g(z)$ is said to be p-symmetric ($p = 2, 3, \ldots$) in E if for every z in E

$$g(e^{2\pi i/p} z) = e^{2\pi i/p} g(z).$$

According to Gronwall's theorem (see e.g. [1, pp. 18–19]), $g(z)$ is regular and p-symmetric in E if and only if it has a power series expansion of the form

$$g(z)=\sum_{n=0}^{\infty} c_{np+1}z^{np+1}, \quad z\in E.$$

It follows that a function $g\in S$ is p-symmetric if and only if g is the pth root transformation of a function $f\in S$, i.e.

$$g(z)=\sqrt[p]{f(z^p)}. \tag{4}$$

It is natural to ask how the properties of functions f and g, connected by (4) for some integer $p\geqslant 2$, are related. We shall consider three simple examples.

Clearly, g is bounded if and only if f is bounded. Also it is easy to see that for each $z\in E$

$$z^p f'(z^p)/f(z^p)=zg'(z)/g(z). \tag{5}$$

Taking into account Nevanlinna's condition for starlikeness and (5) we conclude that g is starlike if and only if f is starlike [1, p. 111 and Problem 30, p. 131].

Now let g be a convex p-symmetric function in S, $p\geqslant 2$. Then for f defined by (4) we have

$$p[1+f''(z^p)z^p/f'(z^p)]=1+zg''(z)/g'(z)+(p-1)zg'(z)/g(z), \quad z\in E. \tag{6}$$

Study's condition for convexity [1, p. 111] together with Nevanlinna's condition imply that f is convex as well. However, the latter property of the p-symmetric functions cannot be reversed. Namely, let f be a convex function in S such that

$$\mathrm{Re}\{p[1+f''(z)z/f'(z)]-(p-1)zf'(z)/f(z)\}<0$$

for some integer $p\geqslant 2$ and some $z\in E$, for example, $f(z)=z/(1-z)$ (see Problems 15 and 31 in [1, pp. 129 and 132]). According to Study's condition, (5) and (6), the corresponding function g is not convex.

It is possible that other properties of univalent functions are also not "improving" under the pth root transformations. In Section 3 we verify this occurrence in the case of the Grunsky norm.

3. An inequality for the Grunsky norm

We show that for functions f and g satisfying (4) $\|G_g\|$ can be estimated from below via $\|G_f\|$ but not from above.

Theorem 2. *Given a function $f\in S$ and an integer $p\geqslant 2$ let g be defined by* (4). *Then*

$$\|G_f\|\leqslant\|G_g\|. \tag{7}$$

This inequality is sharp for each p and over each subclass $\{f\in S\colon \|G_f\|=k\}$, $0\leqslant k\leqslant 1$.

Proof. Let $N(f)$ and $\|f\|_q$, $q\in N(f)$, be defined as in Theorem 1. We denote

$$N_p=\{v=q(w^p)\colon q(w)\in N(f)\},$$

where each function v is defined on $\bar{\mathbb{C}}\backslash g(E)$. It follows that

$$N_p \subset N(g) \quad \text{and} \quad \text{for } q \in N(f), \quad \|f\|_q = \|g\|_{q\circ w^p}.$$

According to Theorem 1, we obtain

$$\|G_f\| = \sup_{q\in N(f)} \|f\|_q = \sup_{q\in Np} \|g\|_q \leqslant \|G_g\|.$$

To show the sharpness of inequality (7) we use Krushkal's example [5]. He evaluated (in an equivalent form) the Grunsky norm of the sequence of p-symmetric functions generated by (3). In particular, he proved that for each even p, any $t \in E$, and

$$h(z) = z/(1 - tz^p)^{2/p}, \quad z \in E, \tag{8}$$

the Grunsky norm $\|G_h\|$ equals $|t|$. Let $f(z)$ be defined by (8) with $p = 2$ and $t = k$, $0 \leqslant k < 1$, i.e. $f(z) = z/(1 - kz^2)$, $z \in E$. Then $\|G_f\| = k$ and the Grunsky norm of the function $\sqrt[p]{f(z^p)}$ is the same for each $p = 2, 3, \ldots$. The case $\|G_f\| = 1$ is trivial. The proof is complete. □.

Remark. Inequality (7) with $p = 2$ was implicitly used in [4] to prove an inclusion theorem for bounded univalent functions.

It is sufficient to consider an unbounded function $f \in S$ with $\|G_f\| = k$, $0 \leqslant k < 1$, to see that in the general case there is no upper bound less than 1 for $\|G_g\|$ in Theorem 2. For example, let $f(z) = z/(1 - z)^{1+k}$ [3]. Since the function g defined by (4) has at least two logarithmic poles on the unit circle, it cannot have a quasiconformal extension onto $\bar{\mathbb{C}}$. By Pommerenke's theorem on quasiconformal extensions [7, p. 292], $\|G_g\| = 1$.

Given $f \in S$ the sequence $b_p = \|G_g\|$ $(p = 2, 3, \ldots)$, where g is defined by (4), is not necessarily nondecreasing. For example [5], for each odd $p \geqslant 3$ and $t \in E\backslash\{0\}$, the Grunsky norm of function (8) is less than $|t|$ (recall that the Grunsky norm of this function equals $|t|$ if p is even). The case $p = 3$ was established by Kühnau in 1981 (see e.g. [6, p. 135]). Now let $a = \limsup b_p$ $(p \to \infty)$. Theorem 2 implies that $\|G_f\| \leqslant b_p \leqslant a$ $(p = 2, 3, \ldots)$. If f, with $\|G_f\| < 1$, is a bounded function which has a q-quasiconformal extension $\tilde{f}$ onto $\bar{\mathbb{C}}$ with the normalization $\tilde{f}(\infty) = \infty$, then it follows that $a \leqslant q < 1$. The smallest possible value of q is likely to be equal to a for any function f of this type. The function f defined by (8) with $p = 3$ was the first known example (given by Kühnau) of a function such that the smallest possible value of q, which is equal to $|t|$, is greater than $\|G_f\|$. For details concerning the inequality $\|G_f\| \leqslant q$ and this example we refer the reader to the monograph [6, Part 2, Ch. 2] and its references.

References

[1] A.W. Goodman, Univalent Functions, vol. 1, Polygonal Publ. House, Washington, NJ, 1983.

[2] A.Z. Grinshpan, Univalent functions and regularly measurable mappings, Sibirsk. Mat. Zh. 27 (6) (1986) 50–64 (Russian); English transl. Siberian Math. J. 27 (6) (1986) 825–837.

[3] A.Z. Grinshpan, Univalent functions with logarithmic restrictions, Ann. Polon. Math. 55 (1991) 117–138.

[4] A.Z. Grinshpan, Ch. Pommerenke, The Grunsky norm and some coefficient estimates for bounded functions, Bull. London Math. Soc. 29 (1997) 705–712.
[5] S.L. Krushkal, The Grunsky coefficient conditions, Sibirsk. Mat. Zh. 28 (1) (1987) 138–145 (Russian); English transl. Siberian Math. J. 28 (1) (1987) 104–110.
[6] S.L. Krushkal, R. Kühnau, Quasiconformal mappings – new methods and applications, Nauka Sibirsk. Otdel., Novosibirsk, 1984 (in Russian).
[7] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.