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General inequalities, consequences and applications

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Abstract

We present a variety of sharp inequalities of integral, polynomial, coefficient, binomial, exponential, and other types. In particular, we prove inequalities for convolutions with weights generated by the gamma and beta functions. Applications include hypergeometric series, fractional integrals, bi-hermitian forms, and univalent functions. A family of positive definite kernels and related transformations naturally arise in the study. The formulation in terms of positive definite matrices is discussed as well. The research is associated with author's recent result on general inequalities with binomial weights.

Keywords: General inequalities; Binomial coefficients; Gamma function; Beta function; Convolutions; Combinatorial inequalities; Bi-hermitian forms; Fractional integrals; Integral transformations; Hypergeometric series; Confluent hypergeometric functions; Positive definite matrices and kernels; General polynomials; Bernstein polynomials; Exponentiation; Univalent functions

1. Introduction

The paper centers around Theorem A, our recent result on general inequalities for complex vectors and positive real parameters. This result is established in [8] in terms of the binomial coefficients $d_n(\alpha)$, which arise from the expansion

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} d_n(\alpha) z^n. \tag{1}$$

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Theorem A implies a variety of sharp inequalities of coefficient, combinatorial, exponential, integral, and polynomial types. Actually, the effectiveness of exponential inequalities in the theory of univalent functions (cf. [4,7,9,16]) motivated the development of inequality (3) of Theorem A as the subtler and more general inequality applicable to a wider spectrum of problems. Note that for each $n = 1, 2, \ldots,$ the classical Cauchy (or Cauchy–Schwarz) inequality

$$\left|\sum_{k=0}^{n} a_k b_k\right|^2 \leqslant \sum_{k=0}^{n} |a_k|^2 \cdot \sum_{k=0}^{n} |b_k|^2, \tag{2}$$

where a_k, b_k are arbitrary complex numbers, is a trivial case of (3). The equality in (2) holds if and only if vectors $\mathbf{a} = (a_0, \ldots, a_n)$ and $\bar{\mathbf{b}} = (\bar{b}_0, \ldots, \bar{b}_n)$ are proportional. Of particular interest are those applications of Theorem A and its consequences one cannot cope with through (2) and other known inequalities (see, e.g., [2,3,12]).

Theorem A [8]. *Let* $\mathbf{a} = (a_0, \ldots, a_n)$ *and* $\mathbf{b} = (b_0, \ldots, b_n)$ *be non-zero complex vectors* $(n = 1, 2, \ldots)$. *Then for any numbers* $\alpha, \beta > 0$ *and* $\lambda \geqslant 0$, *the following inequality holds*:

$$\begin{aligned} &d_n(\lambda + \alpha + \beta) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha + \beta)} \left|\sum_{l=0}^{k} a_l b_{k-l}\right|^2 \\ &\quad \leqslant \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \beta)}{d_k(\alpha)} |a_k|^2 \cdot \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \alpha)}{d_k(\beta)} |b_k|^2. \end{aligned} \tag{3}$$

For $\lambda > 0$, *the equality in* (3) *holds if and only if* $a_k = \eta^k d_k(\alpha) a_0$ *and* $b_k = \eta^k d_k(\beta) b_0$ $(|\eta| = 1;\ k = 1, \ldots, n)$. *The case* $\lambda = 0$ *in* (3) *corresponds to the Cauchy–Schwarz inequality and the equality holds if and only if* $d_{n-k}(\beta) a_k = c d_k(\alpha) \bar{b}_{n-k}$ *for all* $k \leqslant n$ *and a constant* c.

The purpose of this paper is to present some new tools associated with Theorem A which are available for various applications. In addition to the theory of functions and inequalities, the results may be of interest in such fields as approximation theory, mathematical physics, matrix theory, discrete mathematics, and probability/statistics. A generalization of Theorem A, some of its consequences and applications are given in [8,9]. The basic development is discussed here. First of all, we focus on a weighted convolution inequality, which is the limiting case of (3) as $n \to \infty$ (Theorem B), then we give its bi-hermitian equivalent (Theorem C) and consider hypergeometric, fractional integral and other applications of this result (Corollaries 1–3). Our proof involves Eulerian integrals of the first and second kinds (i.e. the beta and gamma functions) and Bernstein polynomials. Recall that the Bernstein polynomial of order n of a function $h(x)$, $x \in [0,1]$, is defined by the formula

$$B_n(h;t) = \sum_{k=0}^{n} h(k/n)\binom{n}{k} t^k (1-t)^{n-k}. \tag{4}$$

We use the famous Bernstein theorem on polynomial sequences which has a probabilistic interpretation (see, e.g., [14]).

Bernstein's Theorem (1912–1913). *For a function h, which is continuous on $[0,1]$, the relation $\lim_{n\to\infty} B_n(h;t) = h(t)$ holds uniformly on $[0,1]$.*

Remark 1. In fact, for $g(x) = |h(x) - h(t)|$, $B_n(g;t) \to 0$ uniformly with respect to $t \in [0,1]$ as $n \to \infty$ (details can be found in [14] or [18]).

Also we present inequality (3) in an equivalent way which helps us to view it differently. We discuss the convolution, kernel and coefficient forms of this inequality as well as its matrix, polynomial, exponential and operator-theoretic aspects. A convolution integral equivalent of inequality (3) is given by Theorem D. Its proof involves the integral identity

$$\int_E \left|\phi(t)\right|^2 d\mu(t) \int_E \left|\psi(t)\right|^2 d\mu(t) - \left|\int_E \phi(t)\psi(t)\,d\mu(t)\right|^2 = \frac{1}{2}\int_E\int_E \left|\phi(t)\overline{\psi(\tau)} - \phi(\tau)\overline{\psi(t)}\right|^2 d\mu(t)\,d\mu(\tau), \tag{5}$$

where μ is a positive measure supported on a set E and $\phi, \psi \in L_2(E,\mu)$. It is well known [12] that such identity implies the integral versions of inequality (2). In particular, it gives the Bunyakovskii (or Bunyakovskii–Schwarz) inequality

$$\left|\int_0^1 g(t)h(t)\,dt\right|^2 \leqslant \int_0^1 \left|g(t)\right|^2 dt \int_0^1 \left|h(t)\right|^2 dt, \tag{6}$$

which turns out to be a limiting case of inequality (9) of Theorem B.

The connection of binomial coefficients (1) with Eulerian integrals is well known:

$$d_n(\alpha) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} \quad (n = 0, 1, \ldots), \tag{7}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt \quad \text{and} \quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \tag{8}$$

(we restrict ourselves to the case when $\alpha, \beta > 0$). This connection is used in the proof of Theorem B and also it leads to a general integral form of inequality (3) with $\lambda > 0$ (Theorem E). Its polynomial case is given in Corollary 4. A Taylor coefficient restatement of Theorem A and its consequences (Theorem F, Corollary 5) are followed by the multipolynomial generalization of Theorem A and the related results (Theorem 1, Corollaries 6 and 7). Theorems 2–5 deal with the matrices, kernels and transformations generated by Theorems A and E. Some pure binomial inequalities are given in Theorem 6. A curious symmetric case of (3) and its consequence are considered in Corollaries 8 and 9. Theorem 7 deals with a "differentiation" of (3) in the Cauchy–Schwarz case. General exponential inequalities implied by Theorem A (Theorem 8) in combination with Theorem F and Corollary 5 produce quasiexponential inequalities for formal power series and also they give exponential inequalities for formal derivatives (Corollaries 10 and 11). Theorem 9 gives a parametrized coefficient inequality for univalent functions.

The class of all polynomials $p(z)$ of degree at most n ($n \geqslant 1$) is denoted by $\mathcal{P}_n$. The coefficient of z^n in the Taylor series expansion about $z = 0$ of a function (or formal power series) $f(z)$ is denoted by $\{f\}_n$.

2. Inequalities for convolutions, fractional integrals, bi-hermitian forms and special functions

Theorem B gives a convolution inequality with weights generated by the gamma and beta functions. Also Theorem B can be viewed as a theorem on fractional integrals. We establish this result as the limiting case of inequality (3) when $n \to \infty$.

Theorem B. *Let $\phi(t)$ and $\psi(t)$ be complex-valued continuous functions on* $[0, 1]$. *Then for any numbers $\alpha, \beta, \lambda > 0$, the following inequality holds*:

$$\int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1}\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\phi(\tau t)\psi\bigl(\tau(1-t)\bigr)\,dt\right|^2 d\tau$$
$$\leqslant \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\lambda+\alpha+\beta)}{\Gamma(\lambda+\alpha)\Gamma(\lambda+\beta)\Gamma(\alpha+\beta)}$$
$$\times \int_0^1 (1-\tau)^{\lambda+\beta-1}\tau^{\alpha-1}\left|\phi(\tau)\right|^2 d\tau \int_0^1 (1-t)^{\lambda+\alpha-1}t^{\beta-1}\left|\psi(t)\right|^2 dt. \tag{9}$$

The equality in (9) *holds if $\phi(t) = \phi(0)e^{i\gamma t}$ and $\psi(t) = \psi(0)e^{i\gamma t}$ for $t \in$* $[0, 1]$ *and any real γ.*

Proof. For $n = 1, 2, \ldots,$ we use inequality (3) with $a_k = d_k(\alpha)\phi(k/n)$ and $b_k = d_k(\beta)\psi(k/n)$, $k \leqslant n$, and divide its both sides by $[d_n(\lambda+\alpha+\beta)]^2$. Then we use formulas (7) and (8). We have

$$\int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1}\sum_{k=0}^{n}\binom{n}{k}\tau^k(1-\tau)^{n-k}$$
$$\times\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{l=0}^{k}\binom{k}{l}t^l(1-t)^{k-l}\phi(l/n)\psi\big((k-l)/n\big)\,dt\right|^2 d\tau$$
$$\leqslant \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\lambda+\alpha+\beta)}{\Gamma(\lambda+\alpha)\Gamma(\lambda+\beta)\Gamma(\alpha+\beta)}$$
$$\times\int_0^1 (1-\tau)^{\lambda+\beta-1}\tau^{\alpha-1}\sum_{k=0}^{n}\binom{n}{k}\tau^k(1-\tau)^{n-k}\big|\phi(k/n)\big|^2\,d\tau$$
$$\times\int_0^1 (1-t)^{\lambda+\alpha-1}t^{\beta-1}\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\big|\psi(k/n)\big|^2\,dt. \tag{10}$$

By definition (4), the sums under the integral signs on the right-hand side of (10) are equal to $B_n(|\phi|^2;\tau)$ and $B_n(|\psi|^2;t)$ for each τ and t $(0\leqslant\tau,t\leqslant 1)$, correspondingly. Bernstein's theorem allows us to show that the limit of the right-hand side of (10) for $n\to\infty$ is equal to the right-hand side of (9). It takes a little more effort to find the limit of the left-hand side of (10) as $n\to\infty$. Note that for each $k\leqslant n$ and $\tau\in[0,1]$,

$$\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{l=0}^{k}\binom{k}{l}t^l(1-t)^{k-l}\phi(l/n)\psi\big((k-l)/n\big)\,dt\right|^2$$
$$=\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\phi(\tau t)\psi\big(\tau(1-t)\big)\,dt\right|^2 + A_k(\tau), \quad \text{where} \tag{11}$$

$$A_k(\tau) = 2\Re\left\{C_k(\tau)\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\overline{\phi(\tau t)\psi\big(\tau(1-t)\big)}\,dt\right\} + \big|C_k(\tau)\big|^2 \quad \text{and} \tag{12}$$

$$C_k(\tau) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{l=0}^{k}\binom{k}{l}t^l(1-t)^{k-l}$$
$$\times\big[\phi(l/n)\psi\big((k-l)/n\big)-\phi(\tau t)\psi\big(\tau(1-t)\big)\big]\,dt. \tag{13}$$

Let $|\phi|, |\psi|\leqslant M$ on $[0,1]$ and $L = 2M^2\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$. Then (13) and (12) give

$$\big|C_k(\tau)\big|^2\leqslant L\big|C_k(\tau)\big|, \qquad \big|A_k(\tau)\big|\leqslant 2L\big|C_k(\tau)\big|, \quad \text{and} \tag{14}$$

$$\begin{aligned}\bigl|C_k(\tau)\bigr| \leqslant \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{l=0}^{k}\binom{k}{l}t^l(1-t)^{k-l}\\ \times\Bigl[\bigl|\bigl(\phi(l/n)-\phi(\tau t)\bigr)\psi\bigl((k-l)/n\bigr)\bigr|\\ +\bigl|\bigl(\psi\bigl((k-l)/n\bigr)-\psi\bigl(\tau(1-t)\bigr)\bigr)\phi(\tau t)\bigr|\Bigr]\,dt.\end{aligned}$$

Hence

$$\bigl|C_k(\tau)\bigr| \leqslant M\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{l=0}^{k}\binom{k}{l}\bigl[g(l/n)t^l(1-t)^{k-l}+h(l/n)t^{k-l}(1-t)^l\bigr]\,dt, \tag{15}$$

where

$$g(x)=\bigl|\phi(x)-\phi(\tau t)\bigr|,\qquad h(x)=\bigl|\psi(x)-\psi\bigl(\tau(1-t)\bigr)\bigr| \tag{16}$$

for $x\in[0,1]$.

It follows from (11) that the left-hand side of (10) is equal to

$$\int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1}\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\phi(\tau t)\psi\bigl(\tau(1-t)\bigr)\,dt\right|^2 d\tau + D_n, \tag{17}$$

where

$$|D_n| \leqslant \int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1}\sum_{k=0}^{n}\binom{n}{k}\tau^k(1-\tau)^{n-k}\bigl|A_k(\tau)\bigr|\,d\tau.$$

By (14) and (15), we obtain

$$\begin{aligned}|D_n| &\leqslant 2L\int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1}\sum_{k=0}^{n}\binom{n}{k}\tau^k(1-\tau)^{n-k}\bigl|C_k(\tau)\bigr|\,d\tau\\ &\leqslant 2LM\int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{k=0}^{n}\binom{n}{k}\tau^k(1-\tau)^{n-k}\\ &\quad\times\sum_{l=0}^{k}\binom{k}{l}\bigl[g(l/n)t^l(1-t)^{k-l}+h(l/n)t^{k-l}(1-t)^l\bigr]\,dt\,d\tau.\end{aligned}$$

Hence

$$|D_n| \leqslant 2LM \int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \times \sum_{l=0}^{n} \binom{n}{l} \big[g(l/n)(\tau t)^l (1-\tau t)^{n-l} + h(l/n)\big(\tau(1-t)\big)^l \big(1-\tau(1-t)\big)^{n-l}\big]\,dt\,d\tau. \tag{18}$$

By definition (4) and (18), we have

$$|D_n| \leqslant 2LM \int_0^1 (1-\tau)^{\lambda-1}\tau^{\alpha+\beta-1} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\big[B_n(g;\tau t) + B_n\big(h;\tau(1-t)\big)\big]\,dt\,d\tau.$$

According to Remark 1 after Bernstein's theorem and by (16),

$$\lim_{n\to\infty} D_n = 0.$$

Thus by (17), the limit of the left-hand side of (10) as $n \to \infty$ equals the left-hand side of (9). □

Remark 2. Inequalities (3) and (10) show that as $\lambda \to 0$ the limit of inequality (9) divided by $\Gamma(\lambda)$ corresponds to the Bunyakovskii–Schwarz inequality:

$$\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\phi(t)\psi(1-t)\,dt\right|^2 \leqslant \int_0^1 (1-\tau)^{\beta-1}\tau^{\alpha-1}\big|\phi(\tau)\big|^2\,d\tau \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}\big|\psi(t)\big|^2\,dt.$$

The equality here holds for continuous functions ϕ and ψ if and only if $\phi(t)$ and $\overline{\psi}(1-t)$ are proportional on $[0,1]$.

In addition, the statement of Theorem B is correct for measurable functions $\phi(t)$ and $\psi(t)$ since measurable functions can be approximated by continuous functions.

We give an equivalent (bi-hermitian) form of Theorem B in terms of shifted factorials. This result has an interesting connection with some completely monotonic functions involving the gamma function [10].

Theorem C. *For any complex numbers* a_n, b_n $(n = 0, 1, \ldots, N;\ N = 0, 1, \ldots)$ *and non-negative numbers* α, β, λ $(\alpha + \beta > 0)$, *the following inequality holds*:

$$\sum_{n,l,k,m=0}^{N} a_n \bar{a}_l b_k \bar{b}_m \frac{(\alpha)_n (\alpha)_l (\beta)_k (\beta)_m}{(\alpha+\beta)_{n+k} (\alpha+\beta)_{l+m}} \frac{(\alpha+\beta)_{n+l+k+m}}{(\alpha+\beta+\lambda)_{n+l+k+m}}$$
$$\leqslant \sum_{n,l=0}^{N} a_n \bar{a}_l \frac{(\alpha)_{n+l}}{(\alpha+\beta+\lambda)_{n+l}} \sum_{k,m=0}^{N} b_k \bar{b}_m \frac{(\beta)_{k+m}}{(\alpha+\beta+\lambda)_{k+m}}. \tag{19}$$

For non-zero vectors $\mathbf{a} = (a_0, \ldots, a_N)$ *and* $\mathbf{b} = (b_0, \ldots, b_N)$, *the equality in* (19) *holds if* $\alpha\beta = 0$ *or* $a_n = b_n = 0$ $(n = 1, \ldots, N)$. *The case* $\lambda = 0$ *corresponds to the Bunyakovskii–Schwarz inequality.*

Proof. We take polynomial functions ϕ and ψ in Theorem B:

$$\phi(t) = \sum_{n=0}^{N} a_n t^n \quad \text{and} \quad \psi(t) = \sum_{k=0}^{N} b_k t^k.$$

Then we use (9) and (8). □

Even the case $N = 1$ in Theorem C is not obvious: the difference of the right and left parts of (19) divided by $\alpha\beta/(\alpha + \beta + \lambda)_2$ is equal to

$$\frac{1}{\alpha+\beta+\lambda} \left| a_0 \bar{b}_1 + a_1 \bar{b}_0 + \frac{\alpha+\beta+2}{\alpha+\beta+\lambda+2} a_1 \bar{b}_1 \right|^2$$
$$+ \frac{\lambda}{(\alpha+\beta)(\alpha+\beta+\lambda)} \left| a_0 b_1 - a_1 b_0 + \frac{\alpha-\beta}{\alpha+\beta+\lambda+2} a_1 b_1 \right|^2$$
$$+ \lambda \frac{(\lambda+1)((\alpha)_2 + (\beta)_2) + (\alpha+\beta)_3}{(\alpha+\beta)_2 (\alpha+\beta+\lambda+2)(\alpha+\beta+\lambda+1)_3} |a_1 b_1|^2.$$

For $\lambda > 0$ and $(|a_0| + |a_1|)(|b_0| + |b_1|) \neq 0$, this expression equals 0 if and only if $a_1 = b_1 = 0$. For $\lambda = 0$, it equals 0 if and only if $a_0 \bar{b}_1 + a_1 \overline{(b_0 + b_1)} = 0$.

To apply Theorem B to some special functions we use two types of asymmetric integral averages defined by the normalized measures

$$t^{\alpha-1}(1-t)^{\beta-1} B^{-1}(\alpha, \beta)\, dt.$$

Some cases of such averages are well known. See, e.g., [1,6,19] for the classical results of Kummer, Euler, and Bateman, and for details and references on fractional integration. Also see [5] for symmetric averages and related results.

Let F be a complex function of n variables $z_k \in D_k$ $(k = 1, \ldots, n)$. For each k, D_k is a plane set which is star-like with respect to $z_k = 0$ (if $z \in D_k$ then $tz \in D_k$, $0 \leqslant t < 1$).

For any $\alpha, \beta > 0$, integral transformations $\mathbf{A}[F; \alpha, \beta]$ and $\mathbf{G}[F; \alpha, \beta]$ are defined by the formulas:

$$\mathbf{A}\big[F(z_1, \ldots, z_n); \alpha, \beta\big] = B^{-1}(\alpha, \beta) \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} F(tz_1, \ldots, tz_n)\, dt \tag{20}$$

and for $n = 2$

$$\mathbf{G}\big[F(z_1, z_2); \alpha, \beta\big] = B^{-1}(\alpha, \beta) \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} F\big(tz_1, (1-t)z_2\big)\, dt, \tag{21}$$

provided that the integrals exist. Transformation $\mathbf{A}$ for $n = 1$ is expressed in terms of the Erdélyi–Kober fractional integral operator which is defined by the formula (see, e.g., [19]):

$$I^{\gamma,\delta}[F](z) = \frac{1}{z^{\gamma+\delta}\Gamma(\gamma)} \int_0^z (z-t)^{\gamma-1} t^{\delta} F(t)\, dt. \tag{22}$$

We have from (20):

$$\mathbf{A}\big[F(z); \alpha, \beta\big] = B^{-1}(\alpha, \beta) z^{-\alpha-\beta+1} \int_0^z t^{\alpha-1}(z-t)^{\beta-1} F(t)\, dt.$$

Hence, by (22),

$$\mathbf{A}\big[F(z); \alpha, \beta\big] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \big(I^{\beta,\alpha-1}F\big)(z). \tag{23}$$

Here are some known examples of transformations $\mathbf{A}$ and $\mathbf{G}$ resulting in the confluent hypergeometric function ${}_1F_1$, classical Gauss hypergeometric function ${}_2F_1$, Appell hypergeometric function F_3, and Horn function Ξ_1 [6]:

$$\mathbf{A}[e^z; \alpha, \beta] = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{(\alpha+\beta)_n n!} = {}_1F_1(\alpha, \alpha+\beta; z);$$

$$\mathbf{A}\big[(1-z)^{-\gamma}; \alpha, \beta\big] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\gamma)_n z^n}{(\alpha+\beta)_n n!} = {}_2F_1(\alpha, \gamma; \alpha+\beta; z),$$

where $|z| < 1$;

$$\mathbf{G}\big[e^{z+\zeta};\alpha,\beta\big]=e^{\zeta}\sum_{n=0}^{\infty}\frac{(\alpha)_n(z-\zeta)^n}{(\alpha+\beta)_n n!}=e^{\zeta}_1F_1(\alpha,\alpha+\beta;z-\zeta);$$

$$\mathbf{G}\big[(1-z)^{-\gamma}(1-\zeta)^{-\delta};\alpha,\beta\big]=\sum_{n,m=0}^{\infty}\frac{(\alpha)_n(\beta)_m(\gamma)_n(\delta)_m z^n\zeta^m}{(\alpha+\beta)_{n+m}n!m!}$$
$$=F_3(\alpha,\beta,\gamma,\delta,\alpha+\beta;z,\zeta),$$

where $|z|, |\zeta|<1$;

$$\mathbf{G}\big[(1-z)^{-\gamma}e^{\zeta};\alpha,\beta\big]=\sum_{n,m=0}^{\infty}\frac{(\alpha)_n(\beta)_m(\gamma)_n z^n\zeta^m}{(\alpha+\beta)_{n+m}n!m!}=\Xi_1(\alpha,\beta,\gamma,\alpha+\beta;z,\zeta),$$

where $|z|<1$.

Corollary 1. *Let $f(z)$ and $g(\zeta)$ be continuous complex functions on plane sets D_z and D_ζ correspondingly which are star-like with respect to the origin. Then for any $\alpha,\beta,\lambda>0$, $z\in D_z$, and $\zeta\in D_\zeta$, the following inequality holds*:

$$\mathbf{A}\big[\big|\mathbf{G}\big[f(z)g(\zeta);\alpha,\beta\big]\big|^2;\alpha+\beta,\lambda\big]\leqslant\mathbf{A}\big[\big|f(z)\big|^2;\alpha,\beta+\lambda\big]\mathbf{A}\big[\big|g(\zeta)\big|^2;\beta,\alpha+\lambda\big].\quad(24)$$

The equality in (24) *holds if $f(z)=f(0)e^{i\gamma z}$, $g(\zeta)=g(0)e^{i\gamma\zeta}$, and γz and $\gamma\zeta$ are real and equal to each other.*

Proof. We take $\phi(t)=f(tz)$ and $\psi(t)=g(t\zeta)$ in Theorem B. Then (9), (8), (20), and (21) imply (24). □

We consider three special cases of Corollary 1.

Case 1. Let $f(z)=e^z$ and $g(\zeta)=e^\zeta$ in (24). Then we have the following inequality for confluent hypergeometric functions:

$$\mathbf{A}\big[\big|e^{\zeta}_1F_1(\alpha,\alpha+\beta;z-\zeta)\big|^2;\alpha+\beta,\lambda\big]$$
$$\leqslant {}_1F_1(\alpha,\alpha+\beta+\lambda;2\Re z)\,{}_1F_1(\beta,\alpha+\beta+\lambda;2\Re\zeta).\quad(25)$$

The equality in (25) holds for any z and ζ if $\alpha\beta=0$.

In particular, for $z=\zeta=x/2$, we obtain a triangle type inequality for $\log {}_1F_1$:

$$ {}_1F_1(\alpha+\beta,\alpha+\beta+\lambda;x)\leqslant {}_1F_1(\alpha,\alpha+\beta+\lambda;x)\,{}_1F_1(\beta,\alpha+\beta+\lambda;x).\quad(26)$$

Here $\alpha,\beta,\lambda\geqslant 0$ ($\alpha+\beta+\lambda>0$), and x is any real number. Inequality (26) turns into the equality for all x if $\alpha\beta=0$. Also the equality in (26) holds if $x=0$. Let $V(x)$ be defined by the formula

$$V(x)=\frac{{}_1F_1(\alpha+\beta,\alpha+\beta+\lambda;x)}{{}_1F_1(\alpha,\alpha+\beta+\lambda;x)\,{}_1F_1(\beta,\alpha+\beta+\lambda;x)}.$$

One can use the asymptotic expansions for confluent hypergeometric functions [6] to show that

$$V(x) = \frac{\Gamma(\alpha+\lambda)\Gamma(\beta+\lambda)}{\Gamma(\lambda)\Gamma(\alpha+\beta+\lambda)}\bigl[1+O\bigl(|x|^{-1}\bigr)\bigr], \quad \text{as } x \to -\infty\ (\lambda > 0) \quad \text{and}$$

$$V(x) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta+\lambda)} x^{\alpha+\beta+\lambda} e^{-x}\bigl[1+O\bigl(x^{-1}\bigr)\bigr], \quad \text{as } x \to \infty\ (\alpha, \beta > 0).$$

Case 2. For $f(x) = (1-x)^{-\gamma}$ and $g(u) = (1-u)^{-\delta}$ in (24), where γ, δ are real and $x, u \in (-1, 1)$, we have the following inequality in terms of the classical and Appell hypergeometric functions:

$$\mathbf{A}\bigl[F_3^2(\alpha, \beta, \gamma, \delta, \alpha+\beta; x, u); \alpha+\beta, \lambda\bigr] \leqslant {}_2F_1(\alpha, 2\gamma; \alpha+\beta+\lambda; x)\,{}_2F_1(\beta, 2\delta; \alpha+\beta+\lambda; u).$$

Case 3. If $f(x) = (1-x)^{-\gamma}$ and $g(\zeta) = e^{\zeta}$ (γ is real, $x \in (-1, 1)$) then inequality (24) involves the confluent, Gauss and Horn hypergeometric functions:

$$\mathbf{A}\bigl[\bigl|\Xi_1(\alpha, \beta, \gamma, \alpha+\beta; x, \zeta)\bigr|^2; \alpha+\beta, \lambda\bigr] \leqslant {}_2F_1(\alpha, 2\gamma; \alpha+\beta+\lambda; x)\,{}_1F_1(\beta, \alpha+\beta+\lambda; 2\Re\zeta).$$

We state the next corollary in terms of the Erdélyi–Kober fractional operator.

Corollary 2. *Let $f(z)$ be continuous function on a plane domain D which is star-like with respect to the origin. Then for any $\alpha, \beta, \lambda > 0, \delta \geqslant 0$, and $z \in D$, the following inequality holds*:

$$I^{\lambda,\alpha+\beta+2\delta-1}\bigl(\bigl|I^{\beta+\delta,\alpha-1} f\bigr|^2\bigr)(z) \leqslant \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\lambda)\Gamma(\beta+2\delta)}{\Gamma(\alpha+\beta)\Gamma^2(\beta+\delta)\Gamma(\alpha+\beta+\lambda+2\delta)}\bigl(I^{\beta+\lambda,\alpha-1}|f|^2\bigr)(z). \tag{27}$$

The equality in (27) *holds if f is a constant function and $\delta = 0$.*

Proof. Take $\phi(t) = f(tz)$ and $\psi(t) = t^{\delta}$ in Theorem B and use (22). □

The Theorem B or Corollary 2 approach can be applied to the generalized hypergeometric function defined by

$${}_pF_q\bigl[(a)_p; (b)_q; z\bigr] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p}(a_k)_n}{\prod_{k=1}^{q}(b_k)_n} \frac{z^n}{n!} \tag{28}$$

for $p \leqslant q+1$. We use a known property [1] of functions (28) in the **A**-form (see (20)):

$$\mathbf{A}\big[{}_pF_q\big[(a)_p;(b)_q;z\big];\alpha,\beta\big] = {}_{p+1}F_{q+1}\big[(a)_{p+1};(b)_{q+1};z\big],$$

where $a_{p+1}=\alpha$ and $b_{q+1}=\alpha+\beta$. Then, for example, we have the following inequality:

$$\begin{aligned}&\mathbf{A}\big[\big|{}_{p+1}F_{q+1}\big[(a)_{p+1};(b)_{q+1};z\big]\big|^2;\alpha+\beta+2\delta,\lambda\big]\\&\quad\leqslant \mathbf{A}\big[\big|{}_pF_q\big[(a)_p;(b)_q;z\big]\big|^2;\alpha,\beta+\lambda\big]\frac{B(\alpha,\beta)B(\alpha,\beta+2\delta)}{B^2(\alpha,\beta+\delta)},\end{aligned} \tag{29}$$

where $a_{p+1}=\alpha$, $b_{q+1}=\alpha+\beta+\delta$, $\alpha,\beta,\lambda>0$, and $\delta \geqslant 0$. One can use (23) to restate inequality (29) in the Erdélyi–Kober terms.

A combination of (9) and (6) gives weakened though simpler versions.

Corollary 3. *Under the conditions of Theorem B, the following inequality holds*:

$$\begin{aligned}&\left|\int_0^1 (1-t)^{\alpha-1}\phi(t)\int_0^t (t-\tau)^{\lambda-1}\tau^{\beta-1}\psi(\tau)\,d\tau\,dt\right|^2\\&\quad\leqslant \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma^2(\lambda)}{\Gamma(\lambda+\alpha)\Gamma(\lambda+\beta)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\lambda+\beta-1}\big|\phi(\tau)\big|^2\,d\tau\int_0^1 (1-t)^{\lambda+\alpha-1}t^{\beta-1}\big|\psi(t)\big|^2\,dt.\end{aligned} \tag{30}$$

The equality in (30) *holds if* $\phi(t)$ *and* $\psi(t)$ *are constant functions on* $[0,1]$.

Proof. Replace $\phi(t)$ by $\phi(1-t)$ in (9) and apply inequality (6) to its left-hand side. Then use the substitution $t=(1-x)/(1+y-x)$ and $\tau = 1+y-x$, where $0\leqslant y\leqslant x\leqslant 1$. □

In particular, let $\beta=\lambda=1$ and $\int_0^t \psi(\tau)\,d\tau = h(t)$ in (30). Then we have

$$\left|\int_0^1 (1-t)^{\alpha-1}\phi(t)h(t)\,dt\right|^2 \leqslant \frac{1}{\alpha}\int_0^1 t(1-t)^{\alpha-1}\big|\phi(t)\big|^2\,dt\int_0^1 (1-\tau)^{\alpha}\big|h'(\tau)\big|^2\,d\tau. \tag{31}$$

Inequality (31) holds for any $\alpha>0$ and any complex-valued functions ϕ and h, $h(0)=0$, which are continuous and continuously differentiable correspondingly on $[0,1]$. The equality in (31) holds if $\phi(t)$ and $h(t)/t$ are constant functions on $[0,1]$.

Theorem D is both a consequence and generalization of Theorem A. Theorem A corresponds to the case of Theorem D when the parameter measure μ is just one point mass on each subinterval $[s_k,s_{k+1})$ $(k=0,1,\dots,n)$ of the basic interval $[0,s)$.

Theorem D. *Let μ be a positive measure supported on an interval $[0, s)$, and let $\phi(t)$ and $\psi(t)$ be complex-valued functions in $L_2([0, s), \mu)$. Then for any numbers $\alpha, \beta > 0$, $\lambda \geqslant 0$, and $n = 0, 1, 2, \ldots,$ the following inequality holds*:

$$d_n(\lambda+\alpha+\beta)\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\alpha+\beta)}\left|\int\limits_{[0,s_{k+1})}\phi(t)\psi(s_{k+1}-t)\,d\mu(t)\right|^2 \leqslant \sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\beta)}{d_k(\alpha)}u_k\cdot\sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\alpha)}{d_k(\beta)}v_k, \tag{32}$$

where $s_k = sk/(n+1)$ $(k \leqslant n+1)$ and

$$u_k = \int\limits_{[s_k,s_{k+1})}\left|\phi(t)\right|^2 d\mu, \qquad v_k = \int\limits_{[s_k,s_{k+1})}\left|\psi(t)\right|^2 d\mu \qquad (k \leqslant n).$$

If at least one product $u_k v_l$ $(k, l = 0, 1, \ldots, n)$ is not equal to zero, then the equality in (32) *holds if and only if the following conditions satisfy for each $k \leqslant n$*:

(1) *for every $l \leqslant k$, functions $\phi(t)$ and $\overline{\psi}(s_{k+1}-t)$ are proportional for $t \in [s_l, s_{l+1})$ with a possible exception for a set $H \subset [s_l, s_{l+1})$ with $\mu H = 0$;*
(2) *all numbers $z_l = \int_{[s_l,s_{l+1})}\phi(t)\psi(s_{k+1}-t)\,d\mu(t)$ $(l = 0, \ldots, k)$ have equal arguments*;
(3) *$u_k = u_0 d_k^2(\alpha)$, $v_k = v_0 d_k^2(\beta)$ if $\lambda \neq 0$, otherwise $d_{n-k}^2(\beta)u_k = c d_k^2(\alpha)v_{n-k}$, where c is a positive constant.*

Proof. We prove (32) for a given n. Since

$$\left|\int\limits_{[0,s_{k+1})}\phi(t)\psi(s_{k+1}-t)\,d\mu(t)\right| \leqslant \sum_{l=0}^{k}\left|\int\limits_{[s_l,s_{l+1})}\phi(t)\psi(s_{k+1}-t)\,d\mu(t)\right|$$

for any $k \leqslant n$, we have by (5):

$$\left|\int\limits_{[0,s_{k+1})}\phi(t)\psi(s_{k+1}-t)\,d\mu(t)\right| \leqslant \sum_{l=0}^{k}\sqrt{u_l v_{k-l}}.$$

Inequality (32) is implied by these inequalities for $k = 0, 1, \ldots, n$ and inequality (3) with $a_k = \sqrt{u_k}$ and $b_k = \sqrt{v_k}$ $(k \leqslant n)$. The equality statement in Theorem D follows from that of Theorem A and from identity (5). □

3. Integral inequalities with a kernel

Now we use the polynomial kernel $\mathcal{K}^n_{\lambda,\alpha,\beta}$ introduced in [8]:

$$\mathcal{K}^n_{\lambda,\alpha,\beta}(z,\zeta,u,v) = \int_0^1\int_0^1\int_0^1\int_0^1 \big[(1-\tau_1)(1-\tau_2)\big]^{\lambda-1}(\tau_1\tau_2)^{\alpha+\beta-1}\big[t_1(1-t_2)\big]^{\alpha-1}\big[(1-t_1)t_2\big]^{\beta-1} \times \Big[(1-\tau_1 t_1+\tau_1 t_1 z\bar\zeta)^n(1-\tau_2 t_2+\tau_2 t_2 u\bar v)^n - \big(1-\tau_1+\tau_1\big(t_1 z+(1-t_1)u\big)\big((1-t_2)\bar\zeta+t_2\bar v\big)\big)^n\Big]\,d\tau_1\,d\tau_2\,dt_1\,dt_2,$$

where z,ζ,u,v are complex variables; α,β,λ are positive parameters; and n is a natural number. Note that this kernel is a polynomial of degree at most n with respect to each of four variables $z,\bar\zeta,u,\bar v$. The following symmetry properties are obvious:

$$\overline{\mathcal{K}^n_{\lambda,\alpha,\beta}(z,\zeta,u,v)} = \mathcal{K}^n_{\lambda,\alpha,\beta}(\zeta,z,v,u), \qquad \mathcal{K}^n_{\lambda,\alpha,\beta}(z,\zeta,u,v) = \mathcal{K}^n_{\lambda,\beta,\alpha}(u,v,z,\zeta).$$

Theorem E. *Let $\boldsymbol\mu$ and $\boldsymbol\nu$ be finite complex measures supported on some sets E and H, correspondingly, and let $\phi(x)$, $x\in E$, and $\psi(y)$, $y\in H$, be complex-valued functions. If for a natural n, finite integrals $\int_E \phi^k(x)\,d\boldsymbol\mu(x)$ and $\int_H \psi^k(y)\,d\boldsymbol\nu(y)$ exist for each $k = 1,\dots,n$, then the inequality*

$$\int_E\int_E\int_H\int_H \mathcal{K}^n_{\lambda,\alpha,\beta}\big(\phi(x_1),\phi(x_2),\psi(y_1),\psi(y_2)\big)\,d\boldsymbol\mu(x_1)\,d\overline{\boldsymbol\mu(x_2)}\,d\boldsymbol\nu(y_1)\,d\overline{\boldsymbol\nu(y_2)} \geqslant 0 \tag{33}$$

holds for any numbers $\lambda,\alpha,\beta>0$.

If $\boldsymbol\mu(E)\boldsymbol\nu(H)\neq 0$ then the equality in (33) *holds if and only if*

$$\frac{1}{\boldsymbol\mu(E)}\int_E \phi^k(x)\,d\boldsymbol\mu(x) = \frac{1}{\boldsymbol\nu(H)}\int_H \psi^k(y)\,d\boldsymbol\nu(y) = \eta^k \quad (|\eta|=1;\ k=1,\dots,n); \tag{34}$$

otherwise the equality holds if and only if at least all integrals $\int_E \phi^k(x)\,d\boldsymbol\mu(x)$ $(k\leqslant n)$ or all integrals $\int_H \psi^k(y)\,d\boldsymbol\nu(y)$ $(k\leqslant n)$ are equal to zero.

Proof. A restatement of Theorem A is used. We replace a_k and b_k for $k=1,\dots,n$ in inequality (3) by $d_k(\alpha)a_k$ and $d_k(\beta)b_k$ correspondingly and divide both sides of this inequality by $[d_n(\lambda+\alpha+\beta)]^2$. The equality for $\mathbf{a},\mathbf{b}\neq\mathbf{0}$ and $\lambda>0$ will take place if and only if $a_k=\eta^k a_0$ and $b_k=\eta^k b_0$ $(|\eta|=1;\ k=1,\dots,n)$. Then we set $a_k=\int_E\phi^k(x)\,d\boldsymbol\mu(x)$ and $b_k=\int_H\psi^k(y)\,d\boldsymbol\nu(y)$ $(k\leqslant n)$, and use formulas (7) and (8). □

It is easy to see that the statement of Theorem E and that of Theorem A with $\lambda>0$ are equivalent. Two cases of inequality (33) are considered in [8]. To prove one of them we have replaced the kth components of vectors **a** and **b** in (3) by $d_k(\alpha)\sum_{j=1}^{N_a} z_j a_j^k$ and $d_k(\beta)\sum_{m=1}^{N_b} \zeta_m b_m^k$ for each $k = 0, 1, \ldots, n$ and some complex numbers a_j, z_j, b_m, ζ_m ($j = 1, \ldots, N_a$; $m = 1, \ldots, N_b$). This procedure corresponds to the discrete measure choice for $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in (33). Another case considered in [8] is a restatement of (3) with $\lambda \neq 0$ in terms of two functions f and g which are analytic in the closed unit disk. This result is implied by (33) with $x = y = z$, $d\boldsymbol{\mu}(x) = f(z)\,dz/z$, $d\boldsymbol{\nu}(y) = g(z)\,dz/z$, $E = H = \{z\colon |z| = 1\}$, $\phi(x) = \psi(y) = z^{-1}$. In particular, we have the following corollary for polynomials.

Corollary 4. *For* $n = 1, 2, \ldots,$ *any polynomials* $p(z), q(z) \in \mathcal{P}_n$, *and any numbers* $\alpha, \beta, \lambda > 0$, *the following inequality holds*:

$$\int\limits_{|z|=1}\int\limits_{|\zeta|=1}\int\limits_{|u|=1}\int\limits_{|v|=1} p(z)\overline{p(\zeta)}q(u)\overline{q(v)}\mathcal{K}^n_{\lambda,\alpha,\beta}(\zeta, z, v, u)\frac{dz}{z}\frac{d\zeta}{\zeta}\frac{du}{u}\frac{dv}{v} \geqslant 0.$$

If $p(0)q(0) \neq 0$ *then the equality here holds if and only if*

$$\frac{p(z)}{p(0)} = \frac{q(z)}{q(0)} = 1 + \eta z + \cdots + (\eta z)^n$$

for some η, $|\eta| = 1$; *otherwise the equality holds if and only if at least one of polynomials* p *or* q *is identically zero.*

4. Coefficient and multipolynomial inequalities

Inequality (3) may be considered as an inequality for the Taylor coefficients of two functions (or formal power series) $f(z) = a_0 + a_1 z + \cdots$ and $g(z) = b_0 + b_1 z + \cdots$ and the corresponding Taylor coefficients of their product $f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)z + \cdots$. Furthermore one may consider $m > 2$ functions (e.g., polynomials) and use induction on m to obtain a more general result. First we give the suitable restatement of Theorem A and its consequences.

Theorem F. *Let* $f(z)$ *and* $g(z)$ *be arbitrary formal Taylor series expansions about* $z = 0$ *and let* n *be a natural number* ($n \geqslant 1$) *such that at least one product* $\{f\}_k\{g\}_l$ *is not equal to zero for* $0 \leqslant k, l \leqslant n$. *Then for any numbers* $\alpha, \beta > 0$, *the following inequalities hold*:

$$d_n(\lambda + \alpha + \beta)\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\alpha + \beta)}\big|\{fg\}_k\big|^2 \leqslant \sum_{k=0}^{n}\frac{d_{n-k}(\lambda + \beta)}{d_k(\alpha)}\big|\{f\}_k\big|^2 \cdot \sum_{k=0}^{n}\frac{d_{n-k}(\lambda + \alpha)}{d_k(\beta)}\big|\{g\}_k\big|^2, \tag{35}$$

where λ *is any nonnegative number, and*

$$\left|\{fg\}_n\right|^2 \leqslant \sum_{k=0}^{n} \frac{d_{n-k}(\lambda_1+\lambda_2+\beta)}{d_k(\alpha)} \left|\left\{f(z)(1-\epsilon z)^{\lambda_1}\right\}_k\right|^2 \times \sum_{k=0}^{n} \frac{d_{n-k}(\lambda_1+\lambda_2+\alpha)}{d_k(\beta)} \left|\left\{g(z)(1-\epsilon z)^{\lambda_2}\right\}_k\right|^2, \tag{36}$$

where λ_1 and λ_2 are any real numbers with $\lambda_1+\lambda_2 \geqslant 0$ and ϵ is any complex number with $|\epsilon| = 1$.

For $\lambda > 0$, the equality in (35) *holds if and only if $\{f\}_k = \{f\}_0\{(1-\eta z)^{-\alpha}\}_k$ and $\{g\}_k = \{g\}_0\{(1-\eta z)^{-\beta}\}_k$ for some η, $|\eta| = 1$ $(k = 1, \dots, n)$. For $\lambda_1+\lambda_2 > 0$, the equality in* (36) *holds if and only if $\{f\}_k = \{f\}_0\{(1-\epsilon z)^{-\alpha-\lambda_1}\}_k$ and $\{g\}_k = \{g\}_0\{(1-\epsilon z)^{-\beta-\lambda_2}\}_k$ $(k = 1, \dots, n)$.*

Proof. Inequality (35) and its equality conditions are implied by Theorem A. One uses the Cauchy–Schwarz inequality and replaces λ by $\lambda_1 + \lambda_2$, and $f(z)$ and $g(z)$ by $f(z)(1-\epsilon z)^{\lambda_1}$ and $g(z)(1-\epsilon z)^{\lambda_2}$, correspondingly, to obtain inequality (36) from (35). The equality statement in (36) follows as well. □

The next corollary is an immediate consequence of Theorem F. Here we deal with the Taylor coefficients of a formal power series f and related formal series f', $\log f$, etc.

Corollary 5. *Let $f(z) = 1 + \{f\}_1 z + \cdots$ be a formal Taylor series. Then for any $\alpha > 0$ and $n = 1, 2, \dots,$ the following inequalities hold*:

$$d_n(\lambda+\alpha+1) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha+1)} \left|\{f'\}_k\right|^2 \leqslant \sum_{k=0}^{n} \frac{d_{n-k}(\lambda+1)}{d_k(\alpha)} \left|\{f\}_k\right|^2 \cdot \sum_{k=0}^{n} d_{n-k}(\lambda+\alpha) \left|\left\{(\log f)'\right\}_k\right|^2, \tag{37}$$

where λ is any nonnegative number, and

$$\left|\{f'\}_n\right|^2 \leqslant \sum_{k=0}^{n} \frac{d_{n-k}(\lambda_1+1)}{d_k(\alpha)} \left|\left\{f(z)(1-\epsilon z)^{\lambda_1}\right\}_k\right|^2 \times \sum_{k=0}^{n} \frac{d_{n-k}(\lambda_1+\lambda_2+\alpha)}{d_k(1-\lambda_2)} \left|\left\{\left(\log f(z)\right)'(1-\epsilon z)^{\lambda_2}\right\}_k\right|^2, \tag{38}$$

where λ_1 and λ_2 are any numbers such that $\lambda_1+\lambda_2 \geqslant 0$, $\lambda_2 < 1$, and ϵ is any number with $|\epsilon| = 1$.

For $\lambda > 0$, the equality in (37) *holds if and only if $\{f\}_k = \{(1-\eta z)^{-\alpha}\}_k$ for some η, $|\eta| = 1$ $(k = 1, \dots, n)$. For $\lambda_1+\lambda_2 > 0$, the equality in* (38) *holds if and only if $\{f\}_k = \{(1-\epsilon z)^{-\alpha-\lambda_1}\}_k$ $(k = 1, \dots, n)$.*

Remark 3. The cases $\lambda = 0$ in (35) and (37), and $\lambda_1 + \lambda_2 = 0$ in (36) and (38) correspond to the Cauchy–Schwarz inequality. Inequalities (35) and (37) for $\lambda = 0$ are identical with the corresponding inequalities (36) and (38) for $\lambda_1 = \lambda_2 = 0$.

Theorem 1. *Let $p_j(z) \in \mathcal{P}_n$ $(n \geqslant 1)$ and let α_j be arbitrary positive numbers* $(j = 1, \dots, m;\ m > 2)$. *Then for any $\lambda \geqslant 0$, the following inequality holds*:

$$d_n^{(m-1)}\left(\lambda + \sum_{j=1}^{m} \alpha_j\right) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda)}{d_k(\sum_{j=1}^{m} \alpha_j)} \left| \left\{ \prod_{j=1}^{m} p_j \right\}_k \right|^2 \leqslant \prod_{j=1}^{m} \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \sum_{l \neq j} \alpha_l)}{d_k(\alpha_j)} \left| \{p_j\}_k \right|^2. \tag{39}$$

If $\prod_{j=1}^{m} p_j(0) \neq 0$, then the equality in (39) *holds if and only if*

$$p_j(z) = p_j(0) \sum_{k=0}^{n} d_k(\alpha_j)(\eta z)^k \quad (|\eta| = 1;\ j = 1, \dots, m);$$

otherwise the equality holds if and only if at least one of polynomials $p_j(z)$ is identically zero.

Proof. We use induction on m. Inequality (39) for $m = 2$ and, in addition, the case of equality in it for $\lambda > 0$ are implied by Theorem F. We show that the statement of Theorem 1 holds for $m > 2$ if it is valid for $(m - 1)$. One considers the non-trivial case when no polynomial $p_j(z)$ $(j \leqslant m)$ is identically zero. We have the following inequality from Theorem F with $f(z) = \prod_{j=1}^{m-1} p_j(z)$, $g(z) = p_m(z)$, $\alpha = \sum_{j=1}^{m-1} \alpha_j$, and $\beta = \alpha_m$:

$$\begin{aligned} &d_n^{(m-1)}\left(\lambda + \sum_{j=1}^{m} \alpha_j\right) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda)}{d_k(\sum_{j=1}^{m} \alpha_j)} \left| \left\{ \prod_{j=1}^{m} p_j \right\}_k \right|^2 \\ &\quad \leqslant d_n^{(m-2)}\left(\lambda + \sum_{j=1}^{m} \alpha_j\right) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \alpha_m)}{d_k(\sum_{j=1}^{m-1} \alpha_j)} \left| \left\{ \prod_{j=1}^{m-1} p_j \right\}_k \right|^2 \\ &\quad \times \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \sum_{j=1}^{m-1} \alpha_j)}{d_k(\alpha_m)} \left| \{p_m\}_k \right|^2. \end{aligned} \tag{40}$$

It follows from (40) that inequality (39) holds for any $m \geqslant 3$ if it is valid for $(m - 1)$. Now we prove the equality statement in Theorem 1. Note that our induction step uses

$(\lambda + \alpha_m) > 0$ instead of λ. By the induction hypothesis, the equality in (39) is possible only if

$$p_j(z) = p_j(0) \sum_{k=0}^{n} d_k(\alpha_j)(\eta z)^k \quad (|\eta| = 1;\ j = 1, \ldots, m-1).$$

If $\lambda = 0$, the equality statement in Theorem F implies that

$$d_k\left(\sum_{j=1}^{m-1} \alpha_j\right) \overline{\{p_m\}}_{n-k} = c d_{n-k}(\alpha_m) \left\{\prod_{j=1}^{m-1} p_j\right\}_k$$

for all $k \leqslant n$ and a constant c. Hence

$$\left\{\prod_{j=1}^{m-1} p_j\right\}_k = \prod_{j=1}^{m-1} p_j(0) d_k\left(\sum_{j=1}^{m-1} \alpha_j\right) \eta^k$$

for all k and $\overline{\{p_m\}}_k = c d_k(\alpha_m) \eta^{n-k} \prod_{j=1}^{m-1} p_j(0)$ with

$$c = \overline{p_m(0)\eta^n} \bigg/ \prod_{j=1}^{m-1} p_j(0).$$

It follows that $p_m(z) = p_m(0) \sum_{k=0}^{n} d_k(\alpha_m)(\eta z)^k$.

The equality condition in (39) follows immediately if $\lambda > 0$. □

Corollary 6 implied by Theorem 1 gives a multipolynomial version of inequality (36).

Corollary 6. *Let* $p_j(z) \in \mathcal{P}_n$ $(n \geqslant 1)$ *and let* α_j *be arbitrary positive numbers* $(j = 1, \ldots, m;\ m > 2)$. *Then for any real numbers* λ_j, $\sum_{j=1}^{m} \lambda_j \geqslant 0$, *the following inequality holds*:

$$\left|\left\{\prod_{j=1}^{m} p_j\right\}_n\right|^2 \leqslant d_n^{(2-m)}\left(\sum_{j=1}^{m}(\alpha_j + \lambda_j)\right) \times \prod_{j=1}^{m} \sum_{k=0}^{n} \frac{d_{n-k}(\sum_{j=1}^{m} \lambda_j + \sum_{l \neq j} \alpha_l)}{d_k(\alpha_j)} \left|\left\{p_j(z)(1-z)^{\lambda_j}\right\}_k\right|^2.$$

If $\prod_{j=1}^{m} p_j(0) \neq 0$, *then the equality here holds if and only if*

$$p_j(z) = p_j(0) \sum_{k=0}^{n} d_k(\alpha_j + \lambda_j) z^k \quad (j = 1, \ldots, m);$$

otherwise the equality holds if and only if at least one of polynomials $p_j(z)$ is identically zero.

Note that an integral analog of inequality (39) for any n and m can be obtained in the similar way as in the case of inequality (3). The only difference is that the Dirichlet formula

$$\int\limits_{\substack{t_1,\ldots,t_m>0\\ t_1+\cdots+t_m\leqslant 1}}\cdots\int t_1^{\alpha_1-1}\ldots t_m^{\alpha_m-1}\,dt_1\ldots dt_m=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots+\alpha_m+1)}$$

$(\alpha_1,\ldots,\alpha_m>0)$ should be used instead of Eq. (8). In particular, in Theorem 1 we choose $\lambda=0$ and polynomials $p_j(z)$ which are equal to the Hadamard products

$$q_j(z)*(1-z)^{-\alpha_j}=\sum_{k=0}^{n}\{q_j\}_k d_k(\alpha_j)z^k$$

with some polynomials $q_j\in\mathcal{P}_n$ $(j=1,\ldots,m)$, and hence we have Corollary 7.

Corollary 7. *Let $q_j(z)\in\mathcal{P}_n$ $(n\geqslant 1)$ and let α_j be arbitrary positive numbers $(j=1,\ldots,m;\ m>2)$. Then the following inequality holds*:

$$\left|\int\limits_{|z_1|=1}\cdots\int\limits_{|z_m|=1}\int\limits_{\substack{t_1,\ldots,t_m>0\\ t_1+\cdots+t_m\leqslant 1}}\cdots\int\left(\sum_{j=1}^{m}t_j\overline{z_j}\right)^n\prod_{j=1}^{m}\left(t_j^{\alpha_j-1}q_j(z_j)\,dt_j\,\frac{dz_j}{z_j}\right)\right|^2$$
$$\leqslant\frac{\Gamma^{m-2}(\sum_{j=1}^m\alpha_j)}{(n+\sum_{j=1}^m\alpha_j)^2}\cdot\prod_{j=1}^{m}\left[\frac{\Gamma(\alpha_j)}{\Gamma(\sum_{l\neq j}\alpha_l)}Q_j\right],$$

where

$$Q_j=\int\limits_{|z|=1}\int\limits_{|\zeta|=1}q_j(z)\overline{q_j(\zeta)}\int\limits_0^1(1-t)^{\sum_{l\neq j}\alpha_l-1}t^{\alpha_j-1}(1-t+t\bar{z}\zeta)^n\,dt\,\frac{dz}{z}\,\frac{\overline{d\zeta}}{\bar{\zeta}}.$$

If $\prod_{j=1}^m q_j(0)\neq 0$, then the equality here holds if and only if

$$\frac{q_j(z)}{q_j(0)}=1+\eta z+\cdots+(\eta z)^n\quad(|\eta|=1;\ j=1,\ldots,m);$$

otherwise the equality holds if and only if at least one of polynomials $q_j(z)$ is identically zero.

Remark 4. One can use the Dirichlet average notation [5] to describe the inequality in Corollary 7. Also note that there the case $m = 2$ corresponds to the Cauchy–Schwarz inequality.

5. Generated matrices, kernels and transformations

A family of positive definite matrices with "binomial" entries results from Theorem A. Given a non-zero complex vector $\mathbf{b} = (b_0, \ldots, b_n)$ and positive numbers λ, α, β, we define a matrix $\mathcal{B} = \mathcal{B}(\mathbf{b}, \lambda, \alpha, \beta) = [B_{l,m}]$ $(l, m = 0, 1, \ldots, n)$ by the formulas:

$$B_{l,m} = -d_n(\lambda + \alpha + \beta) \sum_{k=\max(l,m)}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha + \beta)} \overline{b_{k-l}} b_{k-m} \quad (l \neq m),$$

$$B_{l,l} = \frac{d_{n-l}(\lambda + \beta)}{d_l(\alpha)} \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \alpha)}{d_k(\beta)} |b_k|^2 - d_n(\lambda + \alpha + \beta) \sum_{k=l}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha + \beta)} |b_{k-l}|^2.$$

We call a complex vector $\mathbf{b} = (b_0, \ldots, b_n)$ and a positive number β *admissible* if they satisfy the following condition: there exists no η, $|\eta| = 1$, such that $b_k = \eta^k d_k(\beta) b_0$ for $k = 1, \ldots, n$. This condition is necessary for matrix $\mathcal{B}$ to be positive definite. Note that $\mathcal{B}$ is positive semi-definite if $b_k = \eta^k d_k(\beta) b_0$ for some η, $|\eta| = 1$, and all $k \leqslant n$. In this case $\mathcal{B}\mathbf{a} = \mathbf{0}$ for vector $\mathbf{a}$ with components $\eta^k d_k(\alpha)$ $(0 \leqslant k \leqslant n)$. Also it is easy to see that in the Cauchy–Schwarz case $(\lambda = 0)$, matrix $\mathcal{B}$ is positive semi-definite for any vector $\mathbf{b}$. In fact, $\mathcal{B}(\mathbf{b}, 0, \alpha, \beta)\mathbf{a} = \mathbf{0}$ if components of vector $\mathbf{a}$ are equal to $d_k(\alpha)\overline{b_{n-k}}/d_{n-k}(\beta)$ $(0 \leqslant k \leqslant n)$.

Theorem 2. *Let a complex vector* $\mathbf{b} = (b_0, \ldots, b_n)$ *and number* $\beta > 0$ *be admissible* $(n \geqslant 1)$. *Then matrix* $\mathcal{B}(\mathbf{b}, \lambda, \alpha, \beta)$ *defined as above is positive definite for any* $\lambda, \alpha > 0$.

Proof. Let $\mathbf{a} = (a_0, \ldots, a_n)$ be a non-zero complex vector. We have

$$\begin{aligned}
\langle \mathcal{B}\mathbf{a}, \mathbf{a} \rangle &= \sum_{l,m=0}^{n} B_{l,m} a_m \overline{a_l} \\
&= \sum_{l=0}^{n} \frac{d_{n-l}(\lambda + \beta)}{d_l(\alpha)} |a_l|^2 \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \alpha)}{d_k(\beta)} |b_k|^2 \\
&\quad - d_n(\lambda + \alpha + \beta) \sum_{l,m=0}^{n} \sum_{k=\max(l,m)}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha + \beta)} \overline{b_{k-l}} b_{k-m} \overline{a_l} a_m \\
&= \sum_{l=0}^{n} \frac{d_{n-l}(\lambda + \beta)}{d_l(\alpha)} |a_l|^2 \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \alpha)}{d_k(\beta)} |b_k|^2 \\
&\quad - d_n(\lambda + \alpha + \beta) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha + \beta)} \left| \sum_{l=0}^{k} a_l b_{k-l} \right|^2.
\end{aligned}$$

It follows from Theorem A that $\langle \mathcal{B}\mathbf{a}, \mathbf{a}\rangle$ is strictly positive. □

Thus for matrix $\mathcal{B}$ in Theorem 2, transformation $\mathcal{B}\mathbf{a}$ of $(n+1)$-dimensional complex vectors $\mathbf{a}$ is one-to-one. An integral version of this transformation is implied by Theorem E. It shows that kernel $\mathcal{K}^n_{\lambda,\alpha,\beta}$ is a source of many positive definite kernels if conditions (34) are inconsistent. Let $\boldsymbol{\nu}$ be a non-trivial finite complex measure supported on some set H such that for some complex-valued function $\psi(y)$, $y \in H$, finite integrals $\int_H \psi^k(y)\,d\boldsymbol{\nu}(y)$ exist for $k = 1, \ldots, n$. Also let no η such that $|\eta| = 1$ and

$$\int_H \psi^k(y)\,d\boldsymbol{\nu}(y) = \eta^k \boldsymbol{\nu}(H) \quad (k = 1, \ldots, n)$$

exist. We call a measure $\boldsymbol{\nu}$, which satisfies the above conditions, *admissible*.

Let a function $\mathcal{K}^{\boldsymbol{\nu},n}_{\lambda,\alpha,\beta}$ be defined by the formula

$$\mathcal{K}^{\boldsymbol{\nu},n}_{\lambda,\alpha,\beta}(z,\zeta) = \int_H\int_H \mathcal{K}^n_{\lambda,\alpha,\beta}\big(z,\zeta,\psi(y_1),\psi(y_2)\big)\,d\boldsymbol{\nu}(y_1)\,d\overline{\boldsymbol{\nu}(y_2)}. \tag{41}$$

According to Theorem E, formula (41) gives a family of positive definite kernels. Note that $\mathcal{K}^{\boldsymbol{\nu},n}_{\lambda,\alpha,\beta}(z,\zeta)$ is a polynomial in two variables z and $\bar{\zeta}$ of degree at most n with respect to each of them. If $H = \{z\colon |z| = 1\}$, $y = z$, $\psi(y) = z^{-1}$, and $d\boldsymbol{\nu}(y) = q(z)\,dz/z$, where $q(z) \in \mathcal{P}_n$ $(n \geqslant 1)$ satisfies the condition $q(z) \neq q(0)(1 + \eta z + \cdots + (\eta z)^n)$ for any η, $|\eta| = 1$, then

$$\mathcal{K}^{\boldsymbol{\nu},n}_{\lambda,\alpha,\beta}(z,\zeta) = \mathcal{K}^{q,n}_{\lambda,\alpha,\beta}(z,\zeta) = \int_{|u|=1}\int_{|v|=1} \mathcal{K}^n_{\lambda,\alpha,\beta}(z,\zeta,v,u)q(u)\,du/u\overline{q(v)dv/v}.$$

Theorem 3. *Let a linear transformation $\mathcal{I}(p)$ on $\mathcal{P}_n$ $(n = 1, 2, \ldots)$ be defined by the formula*

$$\mathcal{I}(p)(z) = \int_{|\zeta|=1} p(\zeta)\mathcal{K}^{\boldsymbol{\nu},n}_{\lambda,\alpha,\beta}(z,\zeta)\frac{d\zeta}{\zeta},$$

where $\alpha, \beta, \lambda > 0$, and $\boldsymbol{\nu}$ is an admissible measure.

Then $\mathcal{I}(p)$ is a one-to-one transformation that maps $\mathcal{P}_n$ onto itself.

Proof. Definition (41) implies that $\mathcal{I}(p)(z)$ is a polynomial of degree at most n for any $p \in \mathcal{P}_n$. It is sufficient to show that polynomial $\mathcal{I}(p)(z)$ is identically zero only for the identically zero polynomial p. Theorem E implies that

$$\int_{|z|=1} \mathcal{I}(p)(z)\overline{p(z)}\frac{\overline{dz}}{\bar{z}} = \int_{|z|=1}\int_{|\zeta|=1} \overline{p(z)}p(\zeta)\mathcal{K}^{\boldsymbol{\nu},n}_{\lambda,\alpha,\beta}(z,\zeta)\frac{\overline{dz}}{\bar{z}}\frac{d\zeta}{\zeta} \geqslant 0$$

and the equality holds if and only if $p(z)$ is identically zero. Hence the null-space of $\mathcal{I}(p)$ is trivial. $\square$

The interconnection of Theorems 2 and 3 and that of transformations $\mathcal{B}\mathbf{a}$ and $\mathcal{I}(p)$ are transparent.

We shall consider another parametrized algebraic transformation which is generated by Theorem A though the condition $\lambda \neq 0$ is not the issue. However we shall use this condition to introduce an integral transformation defined on the products of polynomials from the class $\mathcal{P}_n$. Let $\mathcal{J}(p,q)$ be defined by

$$\mathcal{J}(p,q)(z,u) = \int\limits_{|\zeta|=1} \int\limits_{|v|=1} p(\zeta)q(v)\mathcal{K}^n_{\lambda,\alpha,\beta}(z,\zeta,u,v)\frac{d\zeta}{\zeta}\frac{dv}{v},$$

where $p, q \in \mathcal{P}_n$, $\alpha, \beta, \lambda > 0$, and $n = 1, 2, \ldots.$ It follows that $\mathcal{J}(p,q)(z,u)$ is a polynomial in two variables z and u of degree at most n with respect to each of them.

Theorem 4. *If function $\mathcal{J}(p,q)(z,u)$ defined as above is identically zero for some polynomials p and $q \in \mathcal{P}_n$, then at least one of these polynomials is identically zero.*

Proof. We have

$$\int\limits_{|z|=1} \int\limits_{|u|=1} \mathcal{J}(p,q)(z,u)\overline{p(z)}\,\overline{q(u)}\frac{dz}{z}\frac{du}{u}$$
$$= \int\limits_{|z|=1} \int\limits_{|\zeta|=1} \int\limits_{|u|=1} \int\limits_{|v|=1} \overline{p(z)}p(\zeta)\overline{q(u)}q(v)\mathcal{K}^n_{\lambda,\alpha,\beta}(z,\zeta,u,v)\frac{dz}{z}\frac{d\zeta}{\zeta}\frac{du}{u}\frac{dv}{v} = 0$$

since $\mathcal{J}(p,q)(z,u)$ is identically zero. If neither p nor q is identically zero, then according to Corollary 4 $p(0)q(0) \neq 0$. We apply Corollary 4 again, this time we do it for constant polynomials $p(0)$ and $q(0)$ and conclude that $\mathcal{J}(p,q)(0,0) \neq 0$ which is a contradiction. $\square$

The following transformation of pairs of $(n+1)$-dimensional complex vectors into the set of polynomials of two variables (equivalently, square matrices) is an algebraic analog of $\mathcal{J}(p,q)$. Let $\mathcal{A}(\mathbf{a},\mathbf{b})$ be defined by

$$\mathcal{A}(\mathbf{a},\mathbf{b})(z,u) = \sum_{k=0}^{n} d_{n-k}(\lambda+\beta)a_k z^k \sum_{l=0}^{n} d_{n-l}(\lambda+\alpha)b_l u^l$$
$$- d_n(\lambda+\alpha+\beta)\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\alpha+\beta)}\sum_{l=0}^{k} a_l b_{k-l}\sum_{m=0}^{k} d_m(\alpha)d_{k-m}(\beta)z^m u^{k-m},$$

where $\mathbf{a} = (a_0, \dots, a_n)$ and $\mathbf{b} = (b_0, \dots, b_n)$ are complex vectors, z and u are complex variables, $\alpha, \beta > 0$, $\lambda \geqslant 0$, and $n = 1, 2, \dots$. Some properties of transformations $\mathcal{A}(\mathbf{a}, \mathbf{b})$ are simple consequences of Theorem A.

Theorem 5. *Any transformation $\mathcal{A}(\mathbf{a}, \mathbf{b})$ defined as above satisfies the inequality*

$$\int\limits_{|z|=1} \int\limits_{|u|=1} \mathcal{A}(\mathbf{a}, \mathbf{b})(z, u) \sum_{k=0}^{n} \frac{\overline{a_k}}{d_k(\alpha)} z^{-k-1} \sum_{l=0}^{n} \frac{\overline{b_l}}{d_l(\beta)} u^{-l-1}\, dz\, du \leqslant 0. \tag{42}$$

The cases of equality in (42) *for non-zero vectors* **a** *and* **b** *correspond to the ones in* (3).

Function $\mathcal{A}(\mathbf{a}, \mathbf{b})(z, u)$ identically equals zero for some vectors **a** *and* **b** *if and only if at least one of these vectors is the zero vector.*

Proof. Inequality (42) is equivalent to inequality (3). If $\mathcal{A}(\mathbf{a}, \mathbf{b})(z, u)$ is identically equal to zero for some non-zero vectors **a** and **b** then (42) turns to the equality. We consider two cases $\lambda \neq 0$ and $\lambda = 0$.

Let $\lambda \neq 0$. It follows that both components a_0 of **a** and b_0 of **b** are not equal to zero. Then

$$\mathcal{A}(\mathbf{a}, \mathbf{b})(0, 0) = a_0 b_0 \big[d_n(\lambda + \beta) d_n(\lambda + \alpha) - d_n(\lambda + \alpha + \beta) d_n(\lambda) \big] \neq 0,$$

which is a contradiction.

For $\lambda = 0$, we have $d_{n-k}(\beta) a_k = c d_k(\alpha) \overline{b_{n-k}}$ $(0 \leqslant k \leqslant n)$, where c is a non-zero constant. Hence

$$\begin{aligned} \mathcal{A}(\mathbf{a}, \mathbf{b})(z, u)/c = {} & \sum_{k=0}^{n} d_k(\alpha) \overline{b_{n-k}} z^k \sum_{l=0}^{n} d_{n-l}(\alpha) b_l u^l \\ & - \sum_{k=0}^{n} \frac{d_k(\alpha)}{d_{n-k}(\beta)} |b_{n-k}|^2 \sum_{l=0}^{n} d_l(\alpha) d_{n-l}(\beta) z^l u^{n-l} \end{aligned}$$

is identically equal to zero. It follows that there is just one non-zero component of **b** since $\overline{b_k} b_l = 0$ if $k \neq l$. The final contradiction is obvious. □

6. Pure binomial and limiting cases

Theorem A and Theorem 1 imply the following pure binomial inequalities.

Theorem 6. *Let $\alpha_j > 0$ and β_j $(j = 1, \dots, m;\ m \geqslant 2)$ be arbitrary real numbers such that $\sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} \beta_j$. Then for any $\lambda \geqslant 0$ and $n = 1, 2, \dots,$ the following inequality holds*:

$$d_n^m \left(\lambda + \sum_{j=1}^{m} \alpha_j \right) \leqslant \prod_{j=1}^{m} \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \sum_{l \neq j} \alpha_l)}{d_k(\alpha_j)} d_k^2(\beta_j). \tag{43}$$

The equality in (43) *holds if and only if* $\beta_j = \alpha_j$ $(j = 1, \ldots, m)$, *except one additional equality case for* $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$ *when* $\lambda = 0$, $m = 2$, $n = 1$.

Proof. For $m = 2$, inequality (43) is the case of inequality (3) with $\alpha = \alpha_1$, $\beta = \alpha_2$, and $a_k = d_k(\beta_1)$, $b_k = d_k(\beta_2)$ for all $k \leqslant n$. We make use of the equality statement in Theorem A. One has the equality in (43) for $\lambda \neq 0$ if and only if $d_k(\beta_1) = \eta^k d_k(\alpha_1)$, $d_k(\beta_2) = \eta^k d_k(\alpha_2)$ for some η with $|\eta| = 1$ and $k = 1, \ldots, n$. Hence $\eta = 1$ and $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$. If $\lambda = 0$ and $n > 1$, we have the following equality conditions in (43):

$$d_{n-k}(\alpha_2)d_k(\beta_1) = cd_k(\alpha_1)d_{n-k}(\beta_2) \quad (0 \leqslant k \leqslant n)$$

for a non-zero constant c. Take $k = 0, 1$, and 2 to show that $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$. The case when $\lambda = 0$ and $n = 1$ is trivial.

For $m > 2$, inequality (43) is the case of inequality (39) with $\{p_j\}_k = d_k(\beta_j)$ $(k = 1, \ldots, n;\ j = 1, \ldots, m)$. The equality statement in (43) is implied by Theorem 1. □

Corollary 8. *The inequality*

$$d_n^2(\lambda + \alpha + \beta) \leqslant \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \beta)}{d_k(\alpha)} d_k^2(\beta) \cdot \sum_{k=0}^{n} \frac{d_{n-k}(\lambda + \alpha)}{d_k(\beta)} d_k^2(\alpha) \tag{44}$$

holds for any numbers $\alpha, \beta > 0$, $\lambda \geqslant 0$, *and* $n = 1, 2, \ldots$.

The equality in (44) *holds if and only if* $\alpha = \beta$, *except the identity case when* $\lambda = 0$ *and* $n = 1$.

Proof. Inequality (44) is a consequence of Theorem 6. Alternatively, (44) is the case of inequality (3) with $a_k = d_k(\beta)$ and $b_k = d_k(\alpha)$ for all $k \leqslant n$. For $\lambda \neq 0$, the equality in (44) holds if and only if $d_k(\beta) = \eta^k d_k(\alpha) = \eta^{2k} d_k(\beta)$ for some η with $|\eta| = 1$ and $k = 1, \ldots, n$. Hence $\eta = 1$ and $\alpha = \beta$. If $\lambda = 0$ and $n > 1$, we have $d_{n-k}(\beta)d_k(\beta) = cd_k(\alpha)d_{n-k}(\alpha)$ $(0 \leqslant k \leqslant n)$ for a non-zero constant c. Take $k = 0$ and $k = 1$ to show that $\alpha = \beta$. □

Below we use coefficients $d_{n,m}(\alpha)$ defined by the expansion

$$(1 - z)^{-\alpha} \log^m \frac{1}{1 - z} = \sum_{n=m}^{\infty} d_{n,m}(\alpha) z^n \quad (m = 0, 1, \ldots).$$

The binomial coefficients $d_n(\alpha)$ correspond to the case $m = 0$: $d_{n,0}(\alpha) = d_n(\alpha)$. Note that

$$d_{n,1}(\alpha) = \sum_{k=0}^{n-1} \frac{1}{n - k} d_k(\alpha);$$

in particular, $d_{n,1}(0) = 1/n$. It is easy to see that the derivative of $d_{n,m}(\alpha)$ with respect to α is equal to $d_{n,m+1}(\alpha)$. For example, we have $d_n'(\alpha) = d_{n,1}(\alpha)$ and $d_n''(\alpha) = d_{n,2}(\alpha)$.

Corollary 9. *The inequality*

$$2d_{n,1}^2(\lambda+2\alpha) \leqslant d_n(\lambda+2\alpha)\left[d_{n,2}(\lambda+2\alpha)+\sum_{k=1}^{n}\frac{d_{n-k}(\lambda+\alpha)}{d_k(\alpha)}d_{k,1}^2(\alpha)\right] \tag{45}$$

holds for any numbers $\alpha > 0$, $\lambda \geqslant 0$, *and* $n = 1, 2, \ldots$.

Proof. According to Corollary 8, the function

$$F(\beta)=\sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\beta)}{d_k(\alpha)}d_k^2(\beta)\cdot\sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\alpha)}{d_k(\beta)}d_k^2(\alpha)-d_n^2(\lambda+\alpha+\beta)$$

is not negative for $\beta > 0$. Note that $F(\alpha) = F'(\alpha) = 0$. Inequality (45) is equivalent to the nonnegativity of $F''(\alpha)$. □

Theorem 7. *For any complex vector* $\mathbf{x} = (x_0, \ldots, x_n)$ $(n = 1, 2, \ldots)$ *and any numbers* $\alpha, \beta > 0$*, the following inequality holds*:

$$\frac{d_{n,1}(\alpha+\beta)}{d_n(\alpha+\beta)}\left[\sum_{k=0}^{n}d_k(\alpha)d_{n-k}(\beta)|x_k|^2\right]^2+d_n(\alpha+\beta)\sum_{k=0}^{n-1}\frac{|\sum_{l=0}^{k}d_l(\alpha)d_{k-l}(\beta)x_l\overline{x_{n-k+l}}|^2}{(n-k)d_k(\alpha+\beta)}$$
$$\leqslant\sum_{k=0}^{n}d_k(\alpha)d_{n-k}(\beta)|x_k|^2\left[\sum_{k=0}^{n-1}d_k(\alpha)d_{n-k,1}(\beta)|x_k|^2+\sum_{k=1}^{n}d_{k,1}(\alpha)d_{n-k}(\beta)|x_k|^2\right]. \tag{46}$$

The equality in (46) *holds if and only if* $x_k = \eta^k x_0$ $(|\eta| = 1;\ k = 1, \ldots, n)$.

Proof. It is easy to check the statement of Theorem 7 for $n = 1$, since (46) in this case is equivalent to the following inequality:

$$\frac{\alpha\beta}{\alpha+\beta}\big(|x_0|^2-|x_1|^2\big)^2\geqslant 0.$$

One can use inequality (3) to prove (46) for $n \geqslant 2$, but something stronger is needed to confirm the case of equality in (46). Namely, define the function F_n by the formula:

$$\begin{aligned}F_n &= F_n(\lambda,\alpha,\beta;\mathbf{u},\mathbf{v})\\ &= d_n^{-2}(\lambda+\alpha+\beta)\sum_{k=0}^{n}d_{n-k}(\lambda+\beta)d_k(\alpha)|u_k|^2\cdot\sum_{k=0}^{n}d_{n-k}(\beta)d_k(\lambda+\alpha)|v_k|^2\\ &\quad -d_n^{-1}(\lambda+\alpha+\beta)\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\alpha+\beta)}\left|\sum_{l=0}^{k}d_l(\alpha)d_{k-l}(\beta)u_lv_{k-l}\right|^2,\end{aligned}$$

where $\mathbf{u} = (u_0, \ldots, u_n)$ and $\mathbf{v} = (v_0, \ldots, v_n)$ are arbitrary $(n+1)$-dimensional complex vectors. It follows from (3) with $a_k = d_k(\alpha)u_k$ and $b_k = d_k(\beta)v_k$ $(k = 0, 1, \ldots, n)$ that F_n is not negative for $\lambda \geqslant 0$ $(\alpha, \beta > 0)$, and that function $F(\lambda) = F_n(\lambda, \alpha, \beta; \mathbf{u}, \mathbf{v})$, where vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy the conditions

$$u_k = x_k, \quad v_k = \overline{x_{n-k}} \quad (k \leqslant n), \tag{47}$$

equals 0 for $\lambda = 0$. Inequality (46) is equivalent to the fact that

$$\lim_{\lambda \to 0+} F(\lambda)/\lambda \geqslant 0.$$

Now we use the recursive inequality from [8] (see Eq. (16) there)

$$\begin{aligned}(\lambda + \alpha + \beta)^2 F_n(\lambda, \alpha, \beta; \mathbf{u}, \mathbf{v}) \geqslant{}& \lambda(\lambda + \alpha + \beta) F_{n-1}(\lambda + 1, \alpha, \beta; \mathbf{u}', \mathbf{v}') \\ &+ \alpha(\lambda + \alpha) F_{n-1}(\lambda, \alpha + 1, \beta; \mathbf{u}'', \mathbf{v}') \\ &+ \beta(\lambda + \beta) F_{n-1}(\lambda, \alpha, \beta + 1; \mathbf{u}', \mathbf{v}''),\end{aligned}$$

where four n-dimensional vectors are defined as follows:

$$\begin{aligned}\mathbf{u}' &= (u_0, \ldots, u_{n-1}), \qquad \mathbf{v}' = (v_0, \ldots, v_{n-1}), \\ \mathbf{u}'' &= (u_1, \ldots, u_n), \quad \text{and} \quad \mathbf{v}'' = (v_1, \ldots, v_n).\end{aligned}$$

Let vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy conditions (47), and let function G_n be defined by the formula:

$$G_n(\alpha, \beta; \mathbf{x}) = \lim_{\lambda \to 0+} \frac{F_n(\lambda, \alpha, \beta; \mathbf{u}, \mathbf{v})}{\lambda}.$$

We divide the recursive inequality above by λ and run λ to $0+$. Hence

$$\begin{aligned}(\alpha + \beta)^2 G_n(\alpha, \beta; \mathbf{x}) \geqslant{}& (\alpha + \beta) F_{n-1}(1, \alpha, \beta; \mathbf{x}', \mathbf{x}''') + \alpha^2 G_{n-1}(\alpha + 1, \beta; \mathbf{x}'') \\ &+ \beta^2 G_{n-1}(\alpha, \beta + 1; \mathbf{x}'),\end{aligned}$$

where $\mathbf{x}' = (x_0, \ldots, x_{n-1})$, $\mathbf{x}'' = (x_1, \ldots, x_n)$, and $\mathbf{x}''' = (\overline{x_n}, \ldots, \overline{x_1})$.

Induction on n (or the equality statement in Theorem A for $\lambda \neq 0$) allows us to complete the proof. □

7. Exponential and quasiexponential inequalities

Exponential inequalities presented in Theorem 8 are implied by Theorem A. This result generalizes basic exponential inequalities developed by I.M. Milin [16, Chapter 2]. For $\alpha = (\beta - 1)$, Theorem 8 gives the most general form of the famous Lebedev–Milin exponential inequalities [15,16].

Theorem 8 [8]. *Let $f = 1 + \{f\}_1 z + \cdots$ be a formal Taylor series. Then for any numbers α, β with $0 < \alpha \leqslant \beta$ and $n = 1, 2, \ldots,$ the following inequality holds*:

$$\sum_{k=0}^{n} \frac{d_{n-k}(\beta-\alpha)}{d_k(\alpha)} \left|\{f\}_k\right|^2 \leqslant d_n(\beta) \exp\left[\frac{1}{d_n(\beta)} \sum_{k=1}^{n} d_{n-k}(\beta)\left(\frac{k}{\alpha}\left|\{\log f\}_k\right|^2 - \frac{\alpha}{k}\right)\right]. \quad (48)$$

The equality in (48) *holds if and only if*

$$\{f\}_k = \left\{(1-\eta z)^{-\alpha}\right\}_k \quad (|\eta| = 1;\ k = 1, \ldots, n).$$

As an immediate consequence of Theorem 8 we have the sum

$$\sum_{k=0}^{n} d_{n-k}(\beta-\alpha)\left|\{f\}_k\right|^2 / d_k(\alpha)$$

which is not greater than $d_n(\beta)$ if the bracketed expression in (48) is not positive. The celebrated theorem of L. de Branges (the proof of Milin's conjecture [16]) allows us to use this observation if we deal with univalent functions.

De Branges' Theorem [4]. *Let $g(z) = z + \{g\}_2 z^2 + \cdots$ be analytic and univalent in the unit disk $\{z\colon |z| < 1\}$. Then the inequality*

$$\sum_{k=1}^{l} (l+1-k)\left(k\left|\left\{\log\left[g(z)/z\right]\right\}_k\right|^2 - 4/k\right) \leqslant 0 \quad (49)$$

holds for each $l = 1, 2, \ldots$.

Theorem 9 [9]. *For any function $g(z) = z + \{g\}_2 z^2 + \cdots$ which is analytic and univalent in the unit disk and $n = 1, 2, \ldots,$ the following inequality holds*:

$$\sum_{k=0}^{n} \frac{d_{n-k}(b)}{d_k(2a)} \left|\left\{\left[\frac{g(z)}{z}\right]^a\right\}_k\right|^2 \leqslant d_n(2a+b), \quad (50)$$

where a and b are any numbers such that $a > 0$, $b \geqslant 0$, $2a + b \geqslant 2$.

The equality in (50) *takes place if and only if g is a rotation of the Koebe function*: $g(z) = z/(1-\eta z)^2$, $|\eta| = 1$.

Proof. We take $f(z) = [g(z)/z]^a$, $\alpha = 2a$ and $\beta = 2a + b$ in Theorem 8. Then

$$\sum_{k=1}^{n} d_{n-k}(\beta)\left(\frac{k}{\alpha}\left|\{\log f\}_k\right|^2 - \frac{\alpha}{k}\right)$$

$$= a\sum_{k=1}^{n} d_{n-k}(\beta)\left(\frac{k}{2}\left|\left\{\log\frac{g(z)}{z}\right\}_k\right|^2 - \frac{2}{k}\right)$$

$$= \frac{a}{2}\sum_{l=1}^{n} d_{n-l}(\beta-2)\sum_{k=1}^{l}(l+1-k)\left(k\left|\left\{\log\frac{g(z)}{z}\right\}_k\right|^2 - \frac{4}{k}\right).$$

The double sum above is not positive since $\beta \geqslant 2$ and each inner sum satisfies inequality (49). Thus (48) implies (50). In addition, the equality condition in (48) implies that the equality in (50) holds only if $|\{g\}_2| = 2$. Then the Bieberbach theorem on the second coefficient estimate (see, e.g., [16] or [9]) gives the equality condition in (50). □

The statement of Theorem 9 was given in [9] without proof. This theorem is a generalization of the well-known coefficient estimates for univalent functions conjectured by L. Bieberbach ($a = 1, b = 0$), M.S. Robertson ($a = 1/2, b = 1$) and Milin ($a \to 0+, b = 2$) and proved by de Branges [4]. Also the case of (50) with $a > 1$ and $b = 0$ was proved earlier. Its proof first appeared in the joint papers [13,17] at one and the same time (see [9] for further details and references).

A combination of Theorem F and Theorem 8 leads to quasiexponential inequalities for pairs of formal power series (cf. [9,11]).

Corollary 10. *Let $f(z) = 1 + \{f\}_1 z + \cdots$ and $g(z)$ be arbitrary formal Taylor series expansions about $z = 0$. Then for any numbers $\alpha, \beta > 0$ and $n = 1, 2, \ldots,$ the following inequalities hold*:

$$\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\alpha+\beta)}\left|\{fg\}_k\right|^2$$
$$\leqslant \sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\alpha)}{d_k(\beta)}\left|\{g\}_k\right|^2$$
$$\times \exp\left[\frac{1}{d_n(\lambda+\alpha+\beta)}\sum_{k=1}^{n} d_{n-k}(\lambda+\alpha+\beta)\left(\frac{k}{\alpha}\left|\{\log f\}_k\right|^2 - \frac{\alpha}{k}\right)\right], \tag{51}$$

where λ is any nonnegative number, and

$$\left|\{fg\}_n\right|^2 \leqslant d_n(\alpha+\beta)\sum_{k=0}^{n}\frac{d_{n-k}(\alpha)}{d_k(\beta)}\left|\{g(z)(1-\epsilon z)^{\lambda}\}_k\right|^2$$
$$\times \exp\left[\frac{1}{d_n(\alpha+\beta)}\sum_{k=1}^{n} d_{n-k}(\alpha+\beta)\left(\frac{k}{\alpha}\left|\{\log f\}_k + \frac{\lambda\epsilon^k}{k}\right|^2 - \frac{\alpha}{k}\right)\right], \tag{52}$$

where λ is any real number and ϵ is any complex number.

If at least one coefficient $\{g\}_k$ ($k \leqslant n$) is not zero, we have the equality conditions in (51) *and* (52) *as follows*: *The equality in* (51) *holds if and only if $\{f\}_k = \{(1-\eta z)^{-\alpha}\}_k$ and*

$\{g\}_k = \{g\}_0\{(1-\eta z)^{-\beta}\}_k$ *for some* η, $|\eta| = 1$ $(k = 1, \dots, n)$. *The equality in* (52) *holds if and only if* $\{f\}_k = \{(1-\epsilon z)^{\lambda}(1-\eta z)^{-\alpha}\}_k$ *and* $\{g\}_k = \{g\}_0\{(1-\epsilon z)^{-\lambda}(1-\eta z)^{-\beta}\}_k$ $(|\eta| = 1;\ k = 1, \dots, n)$.

Note that inequalities (51) and (52) are identical if $\lambda = 0$.

Inequality (37) of Corollary 5 with $\lambda = \beta - \alpha$, Theorem 8, and the elementary inequality $x \leqslant e^{x-1}$ give exponential inequalities involving derivatives of formal Taylor series.

Corollary 11. *Let* $f(z) = 1 + \{f\}_1 z + \cdots$ *be a formal Taylor series. Then for any numbers* α, β *with* $0 < \alpha \leqslant \beta$ *and* $n = 1, 2, \dots,$ *the following inequalities hold*:

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{d_{n-k}(\beta-\alpha)}{d_k(\alpha+1)} \left|\{f'\}_k\right|^2 \\
&\quad = \alpha \sum_{k=1}^{n+1} \frac{d_{n+1-k}(\beta-\alpha)}{d_k(\alpha)} k \left|\{f\}_k\right|^2 \\
&\quad \leqslant \sum_{k=1}^{n+1} d_{n+1-k}(\beta) k^2 \left|\{\log f\}_k\right|^2 \exp\left[\frac{1}{d_n(\beta+1)} \sum_{k=1}^{n} d_{n-k}(\beta+1) \left(\frac{k}{\alpha}\left|\{\log f\}_k\right|^2 - \frac{\alpha}{k}\right)\right] \\
&\quad \leqslant \alpha^2 d_n(\beta+1) \exp\left[\frac{1}{d_{n+1}(\beta)} \sum_{k=1}^{n+1} d_{n+1-k}(\beta) \left(1 + \frac{\beta/\alpha - 1}{n+1} k\right) \left(\frac{k}{\alpha}\left|\{\log f\}_k\right|^2 - \frac{\alpha}{k}\right)\right].
\end{aligned}$$

The equality in both inequalities holds if and only if

$$\{f\}_k = \left\{(1-\eta z)^{-\alpha}\right\}_k \quad (|\eta| = 1;\ k = 1, \dots, n).$$

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia Math. Appl., vol. 71, Cambridge University Press, Cambridge, 1999.

[2] E.F. Beckenbach, R.E. Bellman, Inequalities, Springer, Berlin, 1961.

[3] P. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York, 1995.

[4] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. 154 (1985) 137–152.

[5] B.C. Carlson, Special Functions of Applied Mathematics, Academic Press, New York, 1977.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Triconi, Higher Transcendental Functions, vol. 1, Krieger, New York, 1981.

[7] A.Z. Grinshpan, The Bieberbach conjecture and Milin's functionals, Amer. Math. Monthly 106 (3) (1999) 203–214.

[8] A.Z. Grinshpan, Inequalities generated by the gamma function, Adv. in Appl. Math. 29 (1) (2002) 115–125.

[9] A.Z. Grinshpan, Logarithmic geometry, exponentiation, and coefficient bounds in the theory of univalent functions and nonoverlapping domains, in: R. Kühnau (Ed.), Handbook of Complex Analysis, vol. 1: Geometric Function Theory, North-Holland, Amsterdam, 2002, pp. 273–332.

[10] A.Z. Grinshpan, M.E.H. Ismail, Completely monotonic functions involving the gamma function and its q-analogue, preprint, 2004.

[11] A.Z. Grinshpan, Ch. Pommerenke, The Grunsky operator and coefficient differences, Complex Variables 33 (1997) 113–127.
[12] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, 2nd ed., University Press, Cambridge, 1952.
[13] W.K. Hayman, J.A. Hummel, Coefficients of powers of univalent functions, Complex Variables 7 (1986) 51–70.
[14] G.G. Lorentz, Bernstein Polynomials, University of Toronto Press, Toronto, 1953.
[15] I.M. Milin, On the coefficients of univalent functions, Dokl. Akad. Nauk SSSR 176 (1967) 1015–1018 (in Russian); Soviet Math. Dokl. 8 (1967) 1255–1258.
[16] I.M. Milin, Univalent Functions and Orthonormal Systems, Nauka, Moscow, 1971 (in Russian); published in English by Amer. Math. Soc., Providence, RI, 1977.
[17] I.M. Milin, A.Z. Grinshpan, Logarithmic coefficients means of univalent functions, Complex Variables 7 (1986) 139–147.
[18] I.P. Natanson, Theory of Functions of a Real Variable, vol. 1, Frederick Ungar, New York, 1961.
[19] A.I. Zayed, Handbook of Function and Generalized Function Transformations, CRC Press, Boca Raton, 1996.