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NOTE

Estimating the Argument of Some Analytic Functions

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We consider a class of analytic functions that are closely related to approximate conformal mappings of simply connected domains onto the unit disk. Using a result of Warschawski, we improve upon our estimates of the argument in the considered class for the important case when the mapping is nearly circular. This new estimate is asymptotically sharp.

1. INTRODUCTION

Let GS [5, 4] be the class of functions f, $f(0)=f'(0)-1=0$, $f(z)/z\neq 0$, analytic in E: $|z|<1$ and such that the boundary of $f(E)$ belongs to a nondegenerate annulus $0<m_f\leqslant |w|\leqslant M_f<\infty$. For $f\in GS$ let $l(z)=\operatorname{Re}(zf'(z)/f(z))$ and $\varDelta_f=\min\{1-\inf_{z\in E} l(z), \sup_{z\in E} l(z)-1\}$. The subclass $GS(\varDelta)$ of GS consists of functions f such that $\varDelta_f\leqslant\varDelta$. It was proved in [5] that if $f\in GS(\varDelta)$, $0<\varDelta<\infty$, and

$$\rho=\rho_f=\frac{M_f}{m_f}, \tag{1}$$

then for $z\in E$

$$\left|\arg\frac{f(z)}{z}\right|\leqslant 2\,\varDelta\cos^{-1}(\rho^{-1/(2\varDelta)}). \tag{2}$$

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If $\rho^{1/\Delta}$ is close enough to 1, then inequality (2) can be considerably improved due to a theorem by Warschawski [7], which is in turn a generalization of a theorem by Marchenko [6]. The corresponding result, which is sharp with respect to $\log \rho$, is presented in Theorem 1. This theorem confirms three equivalent conjectures from [5].

THEOREM 1. *Let $f \in GS(\Delta)$, $\Delta \in (0, \infty)$. Then for $z \in E$ and $\delta \in (1, e)$,*

$$\left| \arg \frac{f(z)}{z} \right| \leqslant C \log \rho \log \frac{\Delta}{\log \rho}, \tag{3}$$

provided $\rho \in (1, \delta^{\Delta})$, where ρ is defined by (1) and $C = C(\delta) < \infty$ depends only on δ.

In particular, $C(1.001) < 9$ and $\tilde{C} = \limsup_{\delta \to 1+} C(\delta) \leqslant \pi$.

The function $f(z) = z/(1 - rz)^{i\tau}$ (where $r \in (0, 1)$, $\tau = \Delta(1 - r^2)/r$ and $\Delta \in (0, \infty)$) belongs to the class $GS(\Delta)$. For this function $\log \rho = 2\tau \sin^{-1} r$ and $\sup_{z \in E} |\arg(f(z)/z)| = \tau \log(1/(1 - r))$ (cf. [5]). It shows that inequality (3) is sharp with respect to $\log \rho$ ($\rho \to 1+$), $C(1.001) > 2/5$ and $\tilde{C} \geqslant 1/\pi$. The last inequality may also be derived from Ferrand's example [1] and from the example given by Gaier [2] who used Marchenko's theorem mentioned above. In her paper [1], Ferrand announced without proof a version of Marchenko's theorem which may allow one to show that $\tilde{C} = 1/\pi$. Furthermore, we note that the extremal growth in (3) is realized in particular by convex functions (see Gaier's convex function in [2] and the convex function generated by the example above in [5]).

2. PROOF OF THEOREM 1

The proof of Theorem 1 is based on the following theorem from [7].

THEOREM 2. *Let G be a simply connected region that contains the origin and whose boundary is contained in the ring*

$$1 \leqslant |w| \leqslant 1 + \varepsilon$$

for some ε, $0 < \varepsilon < \log(8/\pi)$. Also, let λ be a number such that any two points in G, whose distance is less than ε, may be connected by an arc in G whose diameter does not exceed λ. If $w = \varphi(z)$ maps E conformally onto G such that $\varphi(0) = 0$, $\varphi'(0) > 0$, then

$$|\varphi(z) - z| \leqslant [\pi \log(1/\varepsilon) + a(\varepsilon)]\, \varepsilon + 2\lambda[1 + (\varepsilon\pi e^{\varepsilon}/\lambda)^{1/3}], \tag{4}$$

where $a(\varepsilon)=1+\varepsilon^3+e^{\varepsilon}+\pi e^{\varepsilon}[1+\log 4+4(1+\varepsilon)^2-\varepsilon\log\varepsilon]$.

Proof of Theorem 1. Inequality (2) gives $|\arg(f(z)/z)|\leqslant\pi\Delta$. Hence, it is enough to prove (3) for a fixed value of $\delta\in(1,e)$ (for example, $\delta=1.001$).

Let $g(z)=z(f(z)/z)^{1/\Delta}$ and $h(z)=z(f(z)/z)^{-1/\Delta}$. Since

$$\Delta\frac{zg'(z)}{g(z)}=\frac{zf'(z)}{f(z)}+\Delta-1 \qquad \text{and} \qquad \Delta\frac{zh'(z)}{h(z)}=-\frac{zf'(z)}{f(z)}+\Delta+1,$$

the R. Nevanlinna condition (see e.g. [3, Ch. 8]) implies that g or h is a starlike function. We denote this starlike function by v. Clearly, $v\in GS(1)$. We have $|w|\geqslant m_v$ (for any $w\in\partial v(E)$ and for either $m_v=m_f^{1/\Delta}$ or $M_f^{-1/\Delta}$),

$$\rho_v=\rho^{1/\Delta} \qquad \text{and} \qquad \left|\arg\frac{f(z)}{z}\right|=\Delta\left|\arg\frac{v(z)}{z}\right|. \tag{5}$$

Using Theorem 2 with $\varphi=v/m_v$, $\varepsilon=\rho_v-1$, and $\lambda=3\varepsilon$, we obtain for $z\in E$

$$|v(z)/m_v-z|\leqslant B(\rho_v-1), \tag{6}$$

where $B(\varepsilon)=\varepsilon[\pi\log(1/\varepsilon)+a(\varepsilon)+6(1+(e^{\varepsilon}\pi/3)^{1/3})]$ and $a(\varepsilon)$ is defined by (4).

Taking into account the condition $\rho_v-1<0.001$ and the inequality (2) with $\Delta=1$ we get

$$\left|\log\left[\frac{v(z)}{m_v z}\right]\right|\leqslant\sqrt{(\log\rho_v)^2+4[\cos^{-1}(\rho_v^{-1/2})]^2}=\alpha(\rho_v)<1.$$

Next, setting $\zeta=\log\left[\dfrac{v(z)}{m_v z}\right]$ and using the inequalities

$$|\operatorname{Im}\zeta|\leqslant\frac{2-|\zeta|}{2(1-|\zeta|)}\,|e^{\zeta}-1|, \quad \zeta\in E,$$

and (6) we obtain

$$\left|\arg\frac{v(z)}{z}\right|\leqslant\frac{2-\alpha(\rho_v)}{2(1-\alpha(\rho_v))}\,B(\rho_v-1).$$

The required result (3) follows from equations (5) and the last inequality after some calculations. ∎

Recently Gaier (private communication) has found some bounds of the form $L(\beta)(\log\rho)^{\beta}$ ($\beta\in[\frac{2}{3},1)$; $L(\frac{2}{3})=(\frac{3}{2}\pi)^{1/3}$; $L(\beta)\to\infty$ as $\beta\to 1-$) for $|\arg(f(z)/z)|$, where f is a starlike function in the class GS. His method differs from ours and leads to an improvement in our estimates for some values of ρ.

REFERENCES

1. J. Ferrand, Sur la déformation analytique d'un domaine, *C.R. Acad. Sci. Paris* **221** (1945), 132–134.
2. D. Gaier, On conformal mapping of nearly circular regions, *Pacific J. Math.* **12** (1962), 149–162.
3. A. W. Goodman, "Univalent Functions," Vol. 1, Polygonal Publ. House, Washington, NJ, 1983.
4. A. Z. Grinshpan, Hausdorff's moment sequences and exponential convolutions, *Methods Appl. Anal.* **3**, No. 1 (1996), 31–45.
5. A. Z. Grinshpan and E. B. Saff, Estimating the argument of approximate conformal mappings, *Complex Variables* **26** (1994), 191–202.
6. A. R. Marchenko, Sur la représentation conforme, *C.R. Dokl. Acad. Sci. USSR* **1** (1935), 289–290.
7. S. E. Warschawski, On the degree of variation in conformal mapping of variable regions, *Trans. Amer. Math. Soc.* **69** (1950), 335–356.

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