Modern Euclidean Geometry (250261) – Philadelphia University – Dr. Amin Witno

AXIOMS OF EUCLIDEAN GEOMETRY

Based on the book *Euclidean and Non-Euclidean Geometries* by Marvin J. Greenberg, 1994

The Original Euclid's Postulates (5)

1. For every point A and for every point B not equal to A there exists a unique line that passes through A and B.
2. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and such that segment CD is congruent to segment BE.
3. For every point O and every point A not equal to O, there exists a circle with center O and radius OA.
4. All right angles are congruent to each other.
5. (Euclid's Parallel Postulate) For every line l and for every point P that does not lie on l, there exists a unique line m passing through P that is parallel to l.

Incidence Axioms (3)

1. Given 2 distinct points there is a unique line incident with them.
2. Given a line there exist at least 2 distinct points incidence with it.
3. There exist 3 distinct points not incident with the same line.

Incidence Propositions

1. If 2 distinct lines are not parallel then they have a unique common point.
2. There exist 3 distinct lines which are not concurrent.
3. For every line there is at least one point not incidence with it.
4. For every point there is at least one line not incidence with it.
5. For every point there exist at least 2 lines incidence with it.

Betweenness Axioms (4)

1. If $A*B*C$ then also $C*B*A$ and A, B, C are distinct collinear points.
2. Given 2 points P and Q there exist 3 points A, B, C such that $P*A*Q$ and $P*Q*B$ and $C*P*Q$.
3. Given 3 collinear points, only one of them can be between the other two.
4. (Plane Separation) For every line l and for every 3 points A, B, C not on l,
   (a) If A, B are on the same side of l and B, C are on the same side of l, then A, C are on the same side of l.
   (b) If A, B are on the opposite sides of l and B, C are on the opposite sides of l, then A, C are on the same side of l. Corollary:
   (c) If A, B are on the opposite sides of l and B, C are on the same side of l, then A, C are on the opposite sides of l.

Betweenness Propositions

1. $AB> \cap BA> = AB$ and $AB> \cup BA> = <AB>$
2. Every line gives exactly two mutually exclusive half-planes.
3. (a) Given $A*B*C$ and $A*C*D$ then $B*C*D$ and $A*B*D$
   (b) Given $A*B*C$ and $B*C*D$ then $A*B*D$ and $A*C*D$

4. Line Separation Property
5. Given $A*B*C$ then
   (a) $AB \cup BC = AC$
   (b) $AB \cap BC = \{ B \}$
   (c) $BA> \cap BC> = \{ B \}$
   (d) $AB> = AC>$
6. Pasch's Theorem
7. Given $\angle CAB$ and a point D on the line BC, then D belongs to the interior of $\angle CAB$ if and only if $B*D*C$.
8. If D is in the interior of $\angle CAB$ then
   (a) so is all of ray AD except A itself
   (b) the opposite of ray AD is completely in the exterior
   (c) if $C*A*E$ then B is in the interior of $\angle DAE$
9. Crossbar Theorem

Congruence Axioms (6)

1. Given segment AB and any ray with vertex C, there is a unique point D on this ray such that $AB \approx CD$.
2. If $AB \approx CD$ and $AB \approx DF$ then $CD \approx DF$.
3. Given $A*B*C$ and $D*E*F$, if $AB \approx DE$ and $BC \approx EF$ then $AC \approx DF$.
4. Given $\angle D$ and any ray AB there is a unique ray AC on each half-plane of the line AB such that $\angle BAC \approx \angle D$.
5. If $\angle A \approx \angle B$ and $\angle A \approx \angle C$ then $\angle B \approx \angle C$.
6. (SAS Criterion) If 2 sides and the included angle of a triangle are congruent to those of another triangle, respectively, then the two triangles are congruent.

Congruence Propositions

1. Segment Subtraction
2. Segment Ordering
3. Supplements of congruent angles are congruent.
4. All vertical angles are congruent to each other.
5. An angle congruent to a right angle is a right angle.
6. Given a line l and a point P there exists a line through P perpendicular to l.
7. ASA Criterion
8. Isosceles Triangle Theorem
9. Angle Addition
10. Angle Subtraction
11. Angle Ordering
12. SSS Criterion
13. All right angles are congruent to each other.

Continuity Axioms (2)

1. (Circular Continuity Principle) If a circle has one point inside and one point outside another circle, then the two circles intersect in two points.
2. (Archimedes' Axiom) Given segment CD and any ray AB there is a number n and a point E on this ray such that $n \times CD \approx AE \geq AB$.

Parallelism Axiom (1)

- (Hilbert's Parallel Axiom) Given a line l and a point P not on l, there is at most one line through P which is parallel to l.

Theorems in Neutral Geometry:

1. Alternate Interior Angle Theorem and its corollaries:
   (a) Two lines perpendicular to another line are parallel.
   (b) Given a line l and a point P not on l, there is a unique line through P which is perpendicular to l.
   (c) Given a line l and a point P not on l, there exists a line through P which is parallel to l.
2. SAA Criterion
3. Every segment has a unique midpoint.
4. Every segment has a unique perpendicular bisector.
5. Every angle has a unique bisector.
6. Given ΔABC, $AB > BC$ if and only if $\angle C > \angle A$.
7. Given ΔABC and ΔDEF with $AB \approx DE$ and $BC \approx EF$, then $AC < DF$ if and only if $\angle B < \angle E$.
8. Triangle Inequality Theorem
9. Saccheri-Legendre Theorem
10. If there is one triangle with angle sum = 180º then a rectangle exists.
11. If a rectangle exists then every triangle has angle sum = 180º.
12. If there is one triangle with angle sum < 180º then every triangle has angle sum < 180º.

Note: Using Euclid's Parallel Postulate it can be proved that in Euclidean Geometry the angle sum of any triangle = 180º. In Hyperbolic Geometry angle sum of any triangle always < 180º whereas in Elliptic Geometry > 180º.

13. Euclid's Parallel Postulate is equivalent to each of the following statements:
    (a) If two lines are cut by a transversal such that two interior angles of the same side have degree sum < 180º then the two lines intersect on this same side.
    (b) Hilbert's Parallel Axiom
    (c) If a line intersects l then it intersects any line which is parallel to l.
    (d) The converse of the Alternate Interior Angle Theorem
    (e) If $l_1 // l_2$ and $m \perp l_1$ then $m \perp l_2$.
    (f) If $l_1 // l_2$ and $m_1 \perp l_1$ and $m_2 \perp l_2$ then either $m_1 = m_2$ or $m_1 // m_2$.

Hyperbolic Axiom (1)

- There exists a line l and a point P not on l such that there are at least two lines through P which are parallel to l.

Theorems in Hyperbolic Geometry:

1. There are no rectangles.
2. Universal Hyperbolic Theorem
3. For every line l and a point P not on l, there are infinitely many lines through P which are parallel to l.
4. The angle sum of any triangle < 180º.
5. AAA Criterion

Note: It can be proved that, if Euclidean Geometry is consistent then
(a) so is Hyperbolic Geometry
(b) the Parallel Axiom is independent from the other axioms.