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MATHEMATICS

Averaging Operator on Countable Directed Graphs and Its Inversion Formula

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1. GENERAL PROBLEM OF RECOVERING FUNCTIONS FROM KNOWN AVERAGES

Let X be a countable set with a fixed set M of finite or countable subsets covering X. The averaging of a function $f(x)$ on X is an operator I that maps $f(x)$ on X to linear combinations of its values on elements of each subset $m \in M$:

$$f(x) \to (If)(m) = \sum_{y \in m} K(m, y)f(y), \quad m \in M.$$

Here, $K(m, y)$ is a fixed weight function on the set of pairs $y \in X$ and $m \in M$, where $y \in m$. Specifically, for $K(m, y) \equiv 1$, the averaging operator has the form $(If)(m) = \sum_{y \in m} f(y)$.

The recovery problem is to find conditions under which the original function $f(x)$ on the "set of points" X can be recovered from its average values $(If)(m)$ and derive an explicit inversion formula expressing f in terms of If. This problem is a discrete analogue of ones in classical integral geometry (see, e.g., [1]) concerning the recovery of functions on manifolds from their integrals over some family of submanifolds, for example, the Radon problem of recovering functions in a plane from their integrals on straight lines.

2. STATEMENT OF THE PROBLEM

We consider a special case of the general recovery problem, namely, when the bijection $x \mapsto m_x$ between points $x \in X$ and elements $m_x \in M$ is defined. It is assumed that $x \in m_x$ and $x \notin m_y$ for any $y \in m_x$, $y \neq x$.

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The bijection $x \mapsto m_x$ defines on X the structure of a directed graph whose vertices are the points of X and whose edges are directed from each point x to all the points of m_x other than x. Thus, all the definitions and results in this paper can be formulated in terms of directed graphs.

Since X is identified with M, we can interpret I as an operator from a suitable space L of complex-valued functions on X to another space of functions on X, i.e., as an operator of the form

$$(If)(x) = K(x, x)f(x) + \sum_{x \prec y} K(x, y)f(y), \tag{1}$$

where the symbol $x \prec y$ means that there exists a directed edge from x to y. In other words, $K(x, y)$, $x \neq y$, is a fixed function on the set of directed edges of the graph.

In what follows, we assume that $K(x, x) \neq 0$ for all $x \in X$. Then it is natural to set $K(x, x) \equiv 1$. The averaging operator then has the form $I = E + A$, where E is the identity operator and

$$(Af)(x) = \sum_{x \prec y} K(x, y)f(y), \tag{2}$$

with a fixed function $K(x, y)$ on the set of directed edges of the graph.

Assume that the space L of averaged functions on X is invariant under the operator A and, hence, under the algebra $U_0(A)$ of all polynomials $P(A) = \sum_{k=0}^{n} c_k A^k$ in A, where $A^0 = E$ is the identity operator. Then simultaneously with the inversion of I, it is natural to solve the following problem: describe all the invertible operators of $U_0(A)$ and find explicit inversion formulas for them.

Below, this problem is solved under additional conditions imposed on the directed graph X.

3. ADDITIONAL CONDITIONS ON THE PROPERTIES OF THE OPERATOR A

The path on X from a point x to a point y is defined as an arbitrary sequence of points

$$x = x_1 \prec x_2 \prec \ldots \prec x_n = y, \quad n = 1, 2, \ldots,$$

and the number $n - 1$ is called the length of this path.

The following three conditions are imposed on the graphs under consideration.

Condition 1. Each vertex of the graph issues at least one edge.

Condition 2. There are only a finite number of entering and exiting directed edges at each vertex x.

Condition 3. For any vertices $x, y \in X$, the lengths of the directed paths from x to y are bounded.

These conditions imply the following properties of the operator A defined by (2) in the space of functions on X.

(i) Sum (2) contains a finite nonzero number of terms; therefore, the operator A is well defined on the set of all functions.

(ii) The operator A maps compactly supported functions on X (i.e., those vanishing outside a finite set of points) again to compactly supported functions.

Condition 3 implies that X does not contain any cycles, i.e., paths with a common beginning and end and, hence, there are no paths with repeated points. Conditions 2 and 3 imply that, for any points $x, y \in X$, the set of points belonging to at least one path from x to y is at most finite.

Denote by $\mathcal{F}$ the space of all compactly supported complex-valued functions on X. Since the operator A preserves $\mathcal{F}$, all the polynomials $P(A) = \sum_{k=0}^{n} c_k A^k$ in the operator A with complex-valued coefficients $c_k \in \mathbb{C}$ also possess this property. Let $U_0(A)$ denote the commutative algebra of all such polynomials. The algebra $U_0(A)$ contains the averaging operator $I = E + A$.

By definition, each polynomial $P(A)$ of degree n is an averaging operator of the form

$$(P(A)f)(x) = \sum K_P(x, y)f(y),$$

where the sum is taken over the set of points y on the paths of length $\le n$ starting from x. Specifically, if $(Af)(x) = \sum_{x \prec y} f(y)$, then, for any polynomial $P(A) = \sum_{k=0}^{n} c_k A^k$, we have

$$K_P(x, y) = \sum_{k=0}^{n} c_k W_k(x, y),$$

where $W_k(x, y)$ is the number of paths of length k from x to y.

The goal is to describe all invertible operators of $U_0(A)$ on $\mathcal{F}$ and their inverses. These inverse operators, except for the identity operator E, no longer belong to $U_0(A)$. Therefore, the problem is reduced to constructing of an extension $U(A)$ of $U_0(A)$ that contains, together with invertible operators from $U_0(A)$, their inverses.

4. SUBSETS $X_a \subset X$ AND THE ALGEBRA $U(A_a)$ OF OPERATORS IN SPACE OF FUNCTIONS ON X_a

The problem can be reduced to a similar one on subgraphs of the original graph.

Definition 1. Let X_a, $a \in X$ denote the subset of points $x \in X$ that are the ends of the paths starting at a, $\mathcal{F}_a$ be the space of all compactly supported functions on X_a, and A_a denote the restriction of the operator A to the subspace of functions on X_a; i.e.,

$$(A_a f)(x) = \sum_{x \prec y} K(x, y)f(y), \quad x \in X_a, \quad f \in \mathcal{F}_a. \quad (3)$$

The sets X_a are equipped with the natural structure of a subgraph of the original graph satisfying Conditions 1–3.

By definition, for any function f on X, the function $A_a f$ depends only on the restriction f_a of f to X_a; i.e., $A_a f = A_a f_a$. It follows that

$$(Af)_a = A_a f_a \text{ for any point } x \in X. \quad (4)$$

As in the case of X, the action of the operator A_a generates the action, on $\mathcal{F}_a$, of the algebra $U_0(A_a)$ of all polynomials $P(A_a)$ in A_a.

Given an arbitrary point $a \in X$, let $l(x)$ denote the maximal length of the paths from a to the point $x \in X_a$, and let $X^k = \{x \in X_a | \, l(x) \le k\}$; specifically, $X^0 = \{a\}$.

The subsets X^k thus defined are finite and form an increasing sequence, while their union is the entire set X_a.

Proposition 1. *The operator* A_a *given by* (3) *maps each function* f *supported by* X^k $(k > 0)$ *to a function supported by* X^{k-1} *and maps each function supported by* $X^0 = \{a\}$ *to zero.*

Proof. The second assertion of the theorem is obvious. It is sufficient to prove the first assertion for f supported at a single point y_0, where $l(y_0) > 0$. The definition of A_a implies that $(A_a f)(x)$, where f is an arbitrary compactly supported function and x is any point, is a linear combination of the values of f on a subset of points y such that $l(y) > l(x)$. Therefore, if f is supported at y_0 and $l(y) = k$ for $k > 0$, then $(A_a f)(x) = 0$ for any point $x \in X_a$ such that $l(x) \ge k$. Therefore, this function is supported by the subset X^{k-1}.

Corollary 1. *If $f \in \mathcal{F}_a$ is supported by the subset X^k, then $A_a^{k+1} f \equiv 0$.*

Theorem 1. *For every formal (i.e., not necessarily converging) power series $P(z) = \sum_{k=0}^{\infty} c_k z^k$, $c_k \in \mathbb{C}$, there is an operator $P(A_a)$ on the space $\mathcal{F}_a$ of compactly supported functions on X_a that is defined as*

$$(P(A_a)f)(x) = \sum_{k=0}^{\infty} c_k (A_a^k f)(x) \tag{5}$$

and maps $\mathcal{F}_a$ into itself.

Proof. For any function $f \in \mathcal{F}_a$, there exists n such that its support is contained in the subset X^n. Then, according to Proposition 1 (Corollary 1), we have $A_a^{n+1} f \equiv 0$. Therefore,

$$\sum_{k=0}^{\infty} c_k (A_a^k f)(x) = \sum_{k=0}^{n} c_k (A_a^k f)(x),$$

which implies that the operator $P(A_a)$ is well defined on $\mathcal{F}_a$ and preserves this space.

Like the formal series $P(z)$, the operators $P(A_a)$ form an algebra, which is denoted by $U(A_a)$, and the mapping $P(z) \mapsto P(A_a)$ is a homomorphism of algebras $\mathbb{C}[[z]] \to U(A_a)$, where $\mathbb{C}[[z]]$ is the algebra of all formal series. Therefore, the invertibility of the operator $P(A_a) \in U(A_a)$ is equivalent to that of the corresponding series $P(z)$. Thus, the following result holds.

Theorem 2. *The operator $P(A_a) = \sum_{k=0}^{\infty} c_k A_a^k$ is invertible if and only if $c_0 \neq 0$.*

Specifically, the original averaging operator $I = E + A_a$ associated with the polynomial $1 + z$ is invertible. Its inverse is associated with the series $(1 + z)^{-1} = \sum_{k=0}^{\infty} (-1)^k z^k$ and, hence, has the form

$$P(A_a) = \sum_{k=0}^{\infty} (-1)^k A_a^k.$$

Theorem 3. *If $K(x, y) > 0$ for all $x \prec y$, then the mapping $P(z) \mapsto P(A_a)$, where $(A_a f)(x) = \sum_{x \prec y} K(x, y) f(y)$, is an isomorphism between the algebra of formal power series $\mathbb{C}[[z]]$ and the operator algebra $U(A_a)$.*

Proof. Assume the opposite: the kernel of the mapping $P(z) \to P(A_a)$ contains a nonzero element $P_0(z)$; i.e., $P_0(A_a) = 0$. This element can be represented as $P_0(z) = z^k P_1(z)$, where $P_1(0) \neq 0$. Then $A_a^k P_1(A_a) = 0$. Since the operator $P_1(A_a)$ is invertible, it follows that $A_a^k = 0$; i.e., $(A_a^k f)(x) \equiv 0$ for any compactly supported function on X_a. Let us show that this is not possible. Indeed, let $x \in X_a$ be any point such that there exists a path longer than k from a to x, and let f be a function on X_a supported at x such that $f(x) > 0$. Then Condition 1 and the definition of A_a imply that the function $A_a^k f$ is nonzero.

Corollary 2. *Under the condition of the theorem, the mapping $G \to G(A_a)$ of the group G of invertible formal power series to the group $G(A_a)$ of invertible operators of the algebra $U(A_a)$ is an isomorphism.*

5. OPERATOR ALGEBRA $U(A)$ IN SPACE OF FUNCTIONS ON X

Note that any function f on X is uniquely defined by its restrictions f_a to all possible subsets X_a, $a \in X$. Therefore, we can write $f = \{f_a | \ a \in X\}$. The result below follows from (4).

Proposition 2. *With the accepted notation, the operator A on X is defined by*

$$Af = \{A_a f_a | \ a \in X\}. \tag{6}$$

Corollary 3. *For any element $P(A)$ of the algebra $U_0(A)$, we have*

$$P(A)f = \{P(A_a)f_a | \ a \in X\}. \tag{7}$$

Definition 2. The subspace of all functions f on X such that $f_a \in \mathcal{F}_a$ for any $a \in X$ (i.e., the function f_a is compactly supported on X_a) is called the space of test functions on X and is denoted by $\tilde{\mathcal{F}}$.

Obviously, $\tilde{\mathcal{F}} \supset \mathcal{F}$, where $\mathcal{F}$ is the space of compactly supported functions on X, but, generally speaking, $\tilde{\mathcal{F}} \neq \mathcal{F}$.

Theorem 4. *The space $\tilde{\mathcal{F}}$ is invariant under the operator A acting on the space of all functions on X, and, for any formal series $P(z)$, the operator $P(A)$ on $\tilde{\mathcal{F}}$ given by (7) is well defined.*

Proof. The invariance of $\tilde{\mathcal{F}}$ follows from (6), since the fact that f_a is compactly supported implies the same property for $A_a f_a$. The fact that (7) is well defined for any formal series $P(A)$ also follows from (6), since for any $a \in X$, the operator $P(A_a)$ is well defined in the space $\mathcal{F}_a$ of compactly supported functions on X_a and preserves this space.

Corollary 4. *Any operator $P(A)$ of the form $P(A) = \sum_{k=0}^{\infty} c_k A^k$ on $\tilde{\mathcal{F}}$ is invertible for $c_0 \neq 0$, and its inverse is the operator on $\tilde{\mathcal{F}}$ associated with the formal series $[P(z)]^{-1}$.*

Remark. Each analytic function $\varphi(z)$ that is regular in the neighborhood of 0 and such that $\varphi(0) \neq 0$ is associated with an invertible operator $\varphi(A)$, and the its inverse is associated with the function $(\varphi(z))^{-1}$. For example, the invertible operators e^A and $\cos A$ are defined on $\tilde{\mathcal{F}}$, and their inverses are e^{-A} and $(\cos A)^{-1}$, respectively.

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