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Author(s): William B. Gordon
Reviewed work(s):
Source: *Mathematics Magazine*, Vol. 69, No. 2 (Apr., 1996), pp. 118-120
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2690666
Accessed: 15/11/2011 11:14

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outlined here should simplify greatly what had been a difficult homework problem for a course in nonlinear dynamics.

Acknowledgements. This work was supported by an NSERC Research Grant. The author is an Alfred P. Sloan Foundation Fellow. I thank Steven Strogatz for providing me with a pre-publication copy of his paper with Partha Saha and for helpful correspondence.

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Period Three Trajectories of the Logistic Map

WILLIAM B. GORDON
Naval Research Laboratory
Code 5311
Washington, DC 20375

A recent Note in this MAGAZINE [3] was concerned with locating the "tangent bifurcation" to the logistic map

$$f(x) = rx(1-x). \tag{1}$$

From graphical considerations, this problem amounts to finding the smallest value of $r \in (0,4)$ for which the map f has a non-trivial 3-periodic orbit. In [3] this value is shown to be $r = r_1$, where

$$r_1 = 1 + 2\sqrt{2} \approx 3.8284\,2712\,4746. \tag{2}$$

The purpose of this note is to show how the result (2) can be easily obtained by exploiting the fact that every 3-periodic sequence $\{x(n)\}$ can be written in the form

$$x(n) = \mu + \beta\omega^n + \bar{\beta}\bar{\omega}^n, \tag{3}$$

where μ and β are constants, ω is a complex cube root of unity, and the overbars indicate complex conjugation. We shall also give an upper bound for the r-values that support stable 3-periodic orbits, *viz.*, $r = r_2$, where

$$r_2 = 1 + \sqrt{\left[\frac{11}{3} + \left(\frac{1915}{54} + \frac{5\sqrt{201}}{2}\right)^{1/3} + \left(\frac{1915}{54} - \frac{5\sqrt{201}}{2}\right)^{1/3}\right]}$$
$$\approx 3.8414\,9900\,7543. \tag{4}$$

A different proof of (2) is given in this current issue of the MAGAZINE by Bechhoefer [1]. We also note that (3) can be viewed as a discrete Fourier transform representation of the 3-periodic orbit $x(n)$; and that discrete Fourier transform techniques have been used in the study of periodic orbits of the Hénon map [2].

In our proof of (2) we substitute (3) into the relation

$$x(n+1) - f(x(n)) = 0, \tag{5}$$

and use the identities $\omega^2 = \overline{\omega}$ and $\overline{\omega}^2 = \omega$, to express the left-hand side of (5) as a linear combination of the three functions $\{1, \omega^n, \overline{\omega}^n\}$. Setting the coefficients of these three functions equal to zero produces the system

$$\begin{aligned} 2\beta\overline{\beta} &= (1 - 1/r)\mu - \mu^2 \\ \overline{\beta}^2 &= (1 - 2\mu - \omega/r)\beta \\ \beta^2 &= (1 - 2\mu - \overline{\omega}/r)\overline{\beta}. \end{aligned} \tag{6}$$

Multiplying the last two equations together and substituting the result into the first equation gives a quadratic equation in μ, *viz.*,

$$(1 - 1/r)\mu - \mu^2 = 2|1 - 2\mu - \omega/r|^2.$$

Solutions to this quadratic are

$$\mu = \frac{3r + 1 \pm \sqrt{r^2 - 2r - 7}}{6r}. \tag{7}$$

The smallest possible value of $r \in (0, 4)$ for which 3-periodic orbits are possible is therefore the positive root of $r^2 - 2r - 7 = 0$; i.e., $r = 1 + 2\sqrt{2}$. This completes our proof of (2).

Let $D = D(r)$ denote the derivative of the third iterate $f^3(x)$ evaluated at a 3-periodic orbit $\{x(n)\}$. In [1] and [3] the value of r_1 is calculated by solving the equation $D(r) = +1$. We shall now calculate the value of r_2 by solving $D(r) = -1$. First, we express $D(r)$ as an explicit function of r. To this end we have

$$D(r) = r^3(1 - 2x(0))(1 - 2x(1))(1 - 2x(2)),$$

or

$$D(r) = r^3(1 - 2A + 4B - 8C), \tag{8}$$

where for notational ease we set

$$\begin{aligned} A &= x(0) + x(1) + x(2), \\ B &= x(0)x(1) + x(1)x(2) + x(2)x(0), \\ C &= x(0)x(1)x(2). \end{aligned} \tag{9}$$

Using (3) to express A and B in terms of μ and β gives

$$A = 3\mu \quad \text{and} \quad B = 3(\mu^2 - |\beta|^2). \tag{10}$$

Now we use a trick. Instead of using (3) to calculate C directly, we use (1) and the 3-periodicity of $\{x(n)\}$ to express C as a function of A and B. From (1),

$$x(n+1)/x(n) = r(1 - x(n)).$$

Multiplying the three equations obtained by setting $n = 0$, 1, and 2, and using the fact that $x(3) = x(0)$, we get

$$1 = r^3(1 - A + B - C).$$

Hence,

$$C = 1 - A + B - 1/r^3. \tag{11}$$

Substituting (11) into (8) we get an expression for $D(r)$ in terms of A and B; using (10) we get an expression for $D(r)$ in terms of μ and $|\beta|^2$. We now use (6) and (7) to express $D(r)$ explicitly as a function of r. The values of r that lead to *stable* orbits can be shown to correspond to the choice of the minus sign in (7), and with this choice we get

$$D(r) = r(2-r)\sqrt{(r^2-2r-7)} - (r^2-2r-8). \tag{12}$$

Now r_2 is a root of the equation $D(r) = -1$. Clearing out the square root in this equation gives the sixth-degree polynomial equation

$$0 = H(r) \equiv r^6 - 6r^5 + 4r^4 + 24r^3 - 14r^2 - 36r - 81. \tag{13}$$

It turns out that $H(r)$ is symmetric about $r = 1$, with

$$H(1+t) = H(1-t) = t^6 - 11t^4 + 37t^2 - 108.$$

Setting $s = t^2$ in the equation $H(1+t) = 0$ gives a cubic equation in s with only one real note, *viz.*,

$$s = \frac{11}{3} + \left(\frac{1915}{54} + \frac{5\sqrt{201}}{2}\right)^{1/3} + \left(\frac{1915}{54} - \frac{5\sqrt{201}}{2}\right)^{1/3}.$$

Equation (4) is then obtained by setting $r_2 = 1 + t = 1 + \sqrt{s}$.

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