THE ROTATION THEOREM FOR STARLIKE UNIVALENT FUNCTIONS

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1. **Introduction.** Let S denote the class of functions

$$F(z) = z + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular and univalent in $|z| < 1$, and map each circle $|z| \leqq r < 1$ onto a region starlike with respect to the origin. Let Σ denote the class of functions

$$\Phi(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n} \tag{1.2}$$

which are regular, except for the simple pole, and univalent in $|\zeta| > 1$ and map each circle $|\zeta| \geqq \rho > 1$ onto a region whose complement is starlike with respect to the origin.

In this paper we shall find upper bounds for

$$R_1 = R_1(z) = \arg F'(z), \qquad R_2 = R_2(\zeta) = \arg \Phi'(\zeta), \tag{1.3}$$

and

$$T_1 = T_1(z) = \arg F(z) - \arg z, \quad T_2 = T_2(\zeta) = \arg \Phi(\zeta) - \arg \zeta \tag{1.4}$$

for $F(z) \in S$ and $\Phi(\zeta) \in \Sigma$. For R_1, T_1, and T_2 the bounds obtained will be sharp.

The problem of finding the sharp upper bound for R_1 was first attacked by Bieberbach [2][1] who obtained the estimate

$$|R_1(z)| \leqq 2 \arc \sin r + \arc \sin \frac{2r}{1 + r^2}, \qquad |z| = r, \tag{1.5}$$

which is sharp only in the limit as $r \to 1$. Stroganoff [8] proved that the sharp bound for R_1 is attained for a function of the form $z/(1-z)^2$ for an appropriate value of z, but his proof is rather long and complicated. In §2 we shall present a proof of this result, which is both shorter and simpler than the one given by Stroganoff.

Birnbaum [3] obtained upper bounds for R_2 and T_2. In §3 we obtain the sharp bound for T_2, and an improved bound for R_2, and our method brings to light the difficulty of finding the sharp bound

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

for R_2. The sharp bound for T_1 is easy to obtain, and is added for completeness.

Finally, we note that Urazbaev [9] has extended Stroganoff's proof to the subclass of S, of functions which are k-wise symmetric with respect to the origin. In §4 we note that our method is equally valid for this subclass, and we state the result without details.

Bounds for R_j and T_j are of some geometric interest, since R_j is the rotation of a lineal element at z under the transformation, and T_j is the angular rotation, under the transformation, of the point z as viewed from the origin.

2. **The bound for R_1.** We begin with two elementary lemmas.

LEMMA 1. *Let u_1, u_2, v_1, v_2, α_1, α_2 be positive and such that $u_1<u_2$. Then the inequalities*

$$\frac{v_1}{u_1} \leqq \frac{v_2}{u_2} \tag{2.1}$$

and

$$\frac{\alpha_1 v_1/u_1 + \alpha_2 v_2/u_2}{\alpha_1+\alpha_2} \leqq \frac{\alpha_1 v_1 + \alpha_2 v_2}{\alpha_1 u_1 + \alpha_2 u_2} \tag{2.2}$$

each imply the other, and the equality sign occurs in (2.1) *if and only if it occurs in* (2.2).

LEMMA 2. *Suppose that for the positive quantities α_j, u_j, v_j, $j=1, 2, \cdots, n$, the following conditions are satisfied*:

$$\frac{v_1}{u_1} \geqq \frac{v_j}{u_j}, \quad v_1 \leqq v_j, \qquad j = 2, 3, \cdots, n, \tag{2.3}$$

$$u_j > 1/2, \qquad i = 1, 2, 3, \cdots, n, \tag{2.4}$$

and

$$\sum_{j=1}^{n} \alpha_j = 2. \tag{2.5}$$

Then

$$\frac{\sum_{j=1}^{n} \alpha_j v_j}{\left(\sum_{j=1}^{n} \alpha_j u_j\right) - 1} \leqq \frac{2v_1}{2u_1 - 1}, \tag{2.6}$$

and the equality sign occurs in (2.6) *only if it occurs in both parts of* (2.3) *for all indices j.*

The proofs of these two lemmas are direct and simple and we omit them.

Robertson [7, p. 376] has proved that if $f(z) \in C$, that is, if $f(z) = z + \cdots$ is regular and univalent in $|z| < 1$, and maps $|z| < 1$ onto a convex region, then there exists a sequence of functions $f_n(z)$, of the form[2]

$$f_n(z) = \int_0^z \prod_{j=1}^{n} (1 - te^{i\theta_j})^{-\alpha_j} dt \tag{2.7}$$

where θ_j is real, $\alpha_j > 0$, $j = 1, 2, \cdots, n$, and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2$, such that $f_n(z) \to f(z)$, the convergence being uniform in any closed circle $|z| \leqq r < 1$.

By Alexander's theorem [1] if $f(z) \in C$, then $F(z) = zf'(z) \in S$, and conversely by integrating such an expression, if $F(z) \in S$, then $f(z) \in C$. To obtain bounds for R_1 it is therefore sufficient to examine functions of the form

$$F(z) = \frac{z}{\prod_{j=1}^{n} (1 - e^{i\theta_j} z)^{\alpha_j}} \tag{2.8}$$

subject to the same conditions as (2.7). An easy computation, using (2.8), (2.5), and the fact that arg $F'(z) = \Im \log F'(z)$, yields

$$R_1(z) \doteq \sum_{j=1}^{n} \alpha_j \Im \log \frac{1}{1 - ze^{i\theta_j}} + \Im \log \left(\sum_{j=1}^{n} \frac{\alpha_j}{1 - ze^{i\theta_j}} - 1 \right). \tag{2.9}$$

We next observe that it is sufficient to consider (2.9) only for $0 < z < 1$. For if $z_0 = re^{i\theta_0}$ is a point at which arg $F'(z)$ is a maximum, then $F_1(z) = e^{-i\theta_0} F(ze^{i\theta_0}) \in S$, and $F_1'(r) = F'(re^{i\theta_0})$. The problem, then, is to determine the variables n, θ_j, α_j, subject to the restrictions already made, so that for fixed r

$$R_1(r) = \sum_{j=1}^{n} \alpha_j \Im \log \frac{1}{1 - re^{i\theta_j}} + \Im \log \left(\sum_{j=1}^{n} \frac{\alpha_j}{1 - re^{i\theta_j}} - 1 \right) \tag{2.10}$$

is a maximum.

Now the function $w = (1 - rz)^{-1} = u + iv$ maps $|z| \leqq 1$ onto a circle K in the w-plane with diameter end points $1/(1-r)$ and $1/(1+r)$

[2] This is of course the Schwarz-Christoffel transformation, and it has frequently been used as the basis of investigations. For further references see [5].

$>1/2$. Setting $w(e^{i\theta_j})=u_j+iv_j$, equation (2.10) becomes

$$(2.11)\quad R_1(r) = \sum_{j=1}^{n} \alpha_j \text{ arc tan } \frac{v_j}{u_j} + \text{arc tan } \frac{\sum_{j=1}^{n} \alpha_j v_j}{\left(\sum_{j=1}^{n} \alpha_j u_j\right) - 1} = A + B.$$

Since the circle K lies entirely in the half-plane $\Re(w)>1/2$, and since the log functions in (16) are determined by the requirement $\log 1=0$, it follows that in (2.11) the arc tan function lies between $-\pi/2$ and $\pi/2$.

The points $w_j=u_j+iv_j$ lie on the boundary of K, and it is clear that to maximize $R_1(r)$ given by (2.11) one should first take $v_j>0$ for $j=1, 2, \cdots, n$. Secondly, if for some j, $u_j>1/(1-r^2)$, the center of K, then w_j can be replaced by a point for which v_j is the same but $u_j<1/(1-r^2)$, and both A and B will be increased by this change. We can now number the distinct points w_j so that $1/2<u_1<u_2< \cdots <u_n$ and, since the points are on the boundary of K, $0<v_1<v_2< \cdots <v_n$. We form the ratios of v_j/u_j, $j=1, 2, \cdots, n$, and note that either the inequality (2.1) of Lemma 1 holds, or the inequalities (2.3) of Lemma 2.

If the inequality (2.1) occurs, we leave unaltered the points w_j, $j=3, 4, \cdots, n$, and their associated θ_j, α_j, and we replace w_1 and w_2 by the point $w_{12}=\alpha_1u_1+\alpha_2u_2+i(\alpha_1v_1+\alpha_2v_2)$ which lies *inside* K, and associate with this point $\alpha_{12}=\alpha_1+\alpha_2$. Since arc tan q is convex downward for $q>0$,

$$(2.12)\quad \begin{aligned} &\alpha_1 \text{ arc tan } \frac{v_1}{u_1} + \alpha_2 \text{ arc tan } \frac{v_2}{u_2} \\ &\qquad < (\alpha_1 + \alpha_2) \text{ arc tan } \frac{\alpha_1 v_1/u_1 + \alpha_2 v_2/u_2}{\alpha_1 + \alpha_2}. \end{aligned}$$

Combining this with the inequality (2.2), it follows that, in (2.11), A is increased by the replacement of w_1 and w_2 by w_{12}, and it is obvious that B is unchanged. We may now replace w_{12} by a point on the boundary of K leaving u_{12} fixed but increasing v_{12}, and this replacement increases both A and B. This point determines a θ_{12} which with α_{12} determines a new function $F(z)$ of the form (2.8) with only $n-1$ terms in the product, and for which $R_1(r)$ is larger than for the original $F(z)$. This process may be repeated, and after a finite number of such steps, we have either an $F(z)$ of the form (2.8) with only one term in the denominator, or we arrive at an $F(z)$ for which all the conditions of Lemma 2 are satisfied, with inequality in (2.3).

If the conditions of Lemma 2 are satisfied, replace w_j by w_1 for $j=2, 3, \cdots, n$, and α_1 by $\alpha_1+\alpha_2+\cdots+\alpha_n$. It is clear from inequality (2.3) that A is increased, and from inequality (2.6) that B is increased. This completes the proof that $R_1(r)$ assumes its maximum for a function of form $F(z)=z/(1-ze^{i\theta})^2$. Direct computation, or a simplification of (2.11), yields, for this function,

$$R_1(r) = \arctan \frac{r \sin \theta}{1 + r \cos \theta} + 3 \arctan \frac{r \sin \theta}{1 - r \cos \theta} \cdot \tag{2.13}$$

The extreme values occur in (2.13) for the roots of

$$Q(\cos \theta) = 2r \cos^2 \theta + 2(1 - r^2) \cos \theta - r(1 + r^2). \tag{2.14}$$

Since $Q(-1)=(r-2)(1-r^2)<0$, $Q(0)<0$, and $Q(1)=(2+r)(1-r^2)>0$, the extreme values of $R_1(r)$ are obtained for $\theta=\theta_0$ where

$$\cos \theta_0 = \frac{r^2 - 1 + (1 + 3r^4)^{1/2}}{2r} \cdot \tag{2.15}$$

Either the symmetry of the problem, or the fact that $R_1(r)$ is an odd function of θ, shows that the maximum corresponds to the first quadrant angle and the minimum corresponds to the fourth quadrant angle given by (2.15).

THEOREM 1. *If* $F(z)\in S$, *and* $0<|z|=r<1$, *then*

$$|\arg F'(z)| \leqq \arctan \frac{r \sin \theta_0}{1 + r \cos \theta_0} + 3 \arctan \frac{r \sin \theta_0}{1 - r \cos \theta_0} \tag{2.16}$$

where $0<\theta_0<\pi/2$ *is given by* (2.15). *This bound is sharp for each* r, *equality occurring for* $F_M(z)=z/(1-z)^2$ *at* $z=re^{i\theta_0}$.

3. **The bounds for T_1, T_2, and R_2.** From equation (2.8) with $z=re^{i\theta}$, we have

$$T_1(z) = -\sum_{j=1}^{n} \alpha_j \arg (1 - re^{i(\theta+\theta_j)}). \tag{3.1}$$

For $|z|\leqq 1$, the points $w=(1-rz)$ fill a circle of radius r and center at $w=1$, so that $|\arg (1-rz)|\leqq \arcsin r$. This gives

THEOREM 2. *If* $F(z)\in S$, *and* $O<|z|=r<1$, *then*

$$|\arg F(z) - \arg z| \leqq 2 \arcsin r. \tag{3.2}$$

This bound is sharp for each r, *equality occurring for* $F_M(z)$ *at* $z=re^{i\theta_1}$,

$\theta_1 = \text{arc cos } r$.

It is well known, and easy to prove, that the elements of S and Σ are related by

$$\Phi(\zeta) = \frac{1}{F(z)}, \qquad z = \frac{1}{\zeta} \tag{3.3}$$

and so $\Phi(\zeta)/\zeta = z/F(z)$. This gives immediately

THEOREM 3. *If* $\Phi(\zeta) \in \Sigma$, *and* $|\zeta| = \rho > 1$, *then*

$$\left| \arg \Phi(\zeta) - \arg \zeta \right| \leqq 2 \text{ arc sin } \rho^{-1}. \tag{3.4}$$

This bound is sharp for every $\rho > 1$, *equality occurring for* $\Phi_M(\zeta) = \zeta - 2 + \zeta^{-1}$.

Birnbaum [**3**] obtained $2 \text{ arc sin } \rho^{-1} + \pi/2$ for the right side of (3.4).

From equations (3.3) and (2.8) it follows that for each $\Phi(\zeta) \in \Sigma$ there is a sequence of functions of the form

$$\Phi_n(\zeta) = \zeta \prod_{j=1}^{n} \left(1 - \frac{e^{i\theta_j}}{\zeta}\right)^{\alpha_j} \tag{3.5}$$

where θ_j and α_j satisfy the same conditions as in equation (2.7), such that $\Phi_n(\zeta) \to \Phi(\zeta)$, the convergence being uniform in any ring $1 < \rho_1 \leqq |\zeta| \leqq \rho_2 < \infty$.

Just as for $R_1(r)$, it is sufficient to consider $\zeta = \rho > 1$. Setting $\rho = 1/r$, a computation from (3.5) yields

$$\begin{aligned} R_2(\rho) &= \sum_{j=1}^{n} \alpha_j \Im \log (1 - re^{i\theta_j}) + \Im \log \left(\sum_{j=1}^{n} \frac{\alpha_j}{1 - re^{i\theta_j}} - 1 \right) \\ &= \text{arc tan} \frac{\sum_{j=1}^{n} \alpha_j v_j}{\left(\sum_{j=1}^{n} \alpha_j u_j \right) - 1} - \sum_{j=1}^{n} \alpha_j \text{ arc tan } \frac{v_j}{u_j} = B - A, \end{aligned} \tag{3.6}$$

where the points $w_j = u_j + iv_j$ lie on the boundary of the same circle K as in equation (2.11). Since

$$|R_2(\rho)| \leqq |B| + |A|, \tag{3.7}$$

we have immediately, from the proof of Theorem 1, the following result.

THEOREM 4. *If* $\Phi(\zeta) \in \Sigma$, *and* $|\zeta| = \rho = 1/r > 1$, *then*

$$(3.8)\quad \left| \arg \Phi'(\zeta) \right| \leqq \arc\tan \frac{\sin \theta_0}{\rho + \cos \theta_0} + 3 \arc\tan \frac{\sin \theta_0}{\rho - \cos \theta_0} < \frac{3\pi}{2},$$

where $0<\theta_0<\pi/2$ *is given by* (2.15).

Obviously this result is not sharp for any value of $\rho>1$, since equality can occur in (3.7) only if $n=1$, and then both A and B will have the same sign. Indeed for the particular function $\Phi_M(\zeta)=\zeta-2+\zeta^{-1}$

$$(3.9)\qquad \left| \arg \Phi_M'(\zeta) \right| \leqq \arc\sin \rho^{-2} \leqq \pi/2.$$

Despite the attractive form of (3.9) it can easily be shown that *the constant* $3\pi/2$ *in* (3.8) *cannot be replaced by any smaller constant valid for all* $\rho>1$.

To prove this assertion, it is sufficient to consider the special function $\Phi(\zeta)$ which maps $|\zeta|>1$ onto the exterior of an arrow. By an arrow we mean the figure consisting of three line segments meeting at a common end point. For simplicity we suppose the common end point set at the origin, the other end points of the segments at $s_1e^{\pm i(1-\gamma)\pi}$, and $-s_2$, $s_j>0$. Thus, the arrow is symmetrical about its shaft and the angle between the barbs and the shaft is $\gamma\pi$. The argument could now be concluded on the basis of known existence theorems, and the theory of functions which map onto variable domains, but we can avoid appealing to these results, since the formula is explicitly known [**4**; **6**] for such a special function $\Phi(\zeta)$. Indeed it is not difficult to see that if $0<\gamma<1$, and $0<\theta_1<\pi/2$, then

$$(3.10)\qquad \Phi(\zeta) = \zeta\left(1 - \frac{1}{\zeta}\right)^{2-2\gamma}\left(1 - \frac{2}{\zeta}\cos\theta_1 + \frac{1}{\zeta^2}\right)^{\gamma}$$

maps $|\zeta|>1$ onto the exterior of the arrow described above. The critical points corresponding to the free end points of the line segment are $\zeta=-1$, and $\zeta=e^{\pm i\beta}$ where

$$(3.11)\qquad \cos\beta = \gamma + (1-\gamma)\cos\theta_1 > 0.$$

The lengths of the barb and the shaft are given by

$$(3.12)\qquad s_1 = (1-\cos\theta_1)(2-2\gamma)^{1-\gamma}\gamma^{\gamma}$$

and

$$(3.13)\qquad s_2 = 2^{2-\gamma}(1+\cos\theta_1)^{\gamma}.$$

As $\theta_1\to 0$ it is clear that for each fixed $\gamma>0$ $s_1\to 0$ and $s_2\to 4$, and since $\theta_1 > \beta > 0$, it follows that $\beta \to 0$. The smaller arc on $|\zeta| = 1$ bounded by $e^{i\beta}$ and $e^{i\theta_1}$ is transformed by the function (3.10) into the *under* side of the upper barb, the mapping being conformal on the interior of the arc. Any radial line in the closed region $|\zeta| \geqq 1$ ending at a point of this arc is transformed by $\Phi(\zeta)$ into a curve ending on the *under* side of the upper barb and perpendicular to it. If $\beta < \theta < \theta_1$, then $\arg \Phi'(e^{i\theta}) = 3\pi/2 - \gamma - \theta$. But γ and θ_1 can be taken arbitrarily small.

4. **Starlike functions with symmetry about the origin.** We denote by S_k the subclass of S of functions of the form

$$F(z) = z + \sum_{n=1}^{\infty} a_n z^{nk+1}, \tag{4.1}$$

k a positive integer. For this subclass equation (2.8) can be replaced by

$$F(z) = \frac{z}{\prod_{j=1}^{n} (1 - e^{i\theta_j} z^k)^{\alpha_j}}, \tag{4.2}$$

where now

$$\sum_{j=1}^{n} \alpha_j = \frac{2}{k}. \tag{4.3}$$

Equation (2.11) becomes

$$R_1(r) = \sum_{j=1}^{n} \alpha_j \arctan \frac{v_j}{u_j} + \arctan \frac{k \sum_{j=1}^{n} \alpha_j v_j}{\left(k \sum_{j=1}^{n} \alpha_j u_j\right) - 1} \tag{4.4}$$

where $w_j = u_j + iv_j$ lies on the boundary of the circle Γ which is the image of $|z| \leqq 1$ under

$$w = (1 - r^k z)^{-1}.$$

The same proof as before shows this is a maximum when all the w_j's are coincident.

THEOREM 5. *If $F(z) \in S_k$, and $0 < |z| = r < 1$, then*

$$|\arg F(z) - \arg z| \leqq \frac{2}{k} \arcsin r^k \tag{4.5}$$

and

$$(4.6)\quad \left| \arg F'(z) \right| \leqq \arc\tan \frac{r^k \sin \theta_0}{1 + r^k \cos \theta_0} + \frac{k+2}{k} \arc\tan \frac{r^k \cos \theta_0}{1 - r^k \cos \theta_0},$$

where $0 < \theta_0 < \pi/2$ *is given by*

$$(4.7)\quad \cos \theta_0 = \frac{(k+1)(r^{2k}-1) + ((k+1)^2(1-r^{2k})^2 + 8r^{2k}(1+r^{2k}))^{1/2}}{4r^k},$$

and both the bounds (4.5) *and* (4.6) *are sharp for all* $r < 1$.

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