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The generalized Roper–Suffridge extension operator ☆

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Abstract

Let $f(z)$ be a normalized convex (starlike) function on the unit disc D. Let $\Omega = \{z \in \mathbf{C}^n: |z_1|^2 + |z_2|^{p_2} + \cdots + |z_n|^{p_n} < 1\}$, where $z = (z_1, z_2, \ldots, z_n)$, $z_1 \in D$, $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$, $p_i \geqslant 1$, $i = 2, \ldots, n$, are real numbers. In this note, we prove that $\Phi(f)(z) = (f(z_1), f'(z_1)^{1/p_2} z_2, \ldots, f'(z_1)^{1/p_n} z_n)$ is a normalized convex (starlike) mapping on Ω, where we choose the power function such that $(f'(z_1))^{1/p_i}|_{z_1=0} = 1$, $i = 2, \ldots, n$. Some other related results are proved.

Keywords: ε starlikeness; Roper–Suffridge extension operator; Carathéodory metric

1. Introduction

In 1995, Roper and Suffridge [1] introduced an extension operator. This operator is defined for normalized locally biholomorphic function f on the unit disc D in $\mathbf{C}$ by

$$\Phi_n(f)(z) = F(z) = \left(f(z_1), \sqrt{f'(z_1)}z_0\right), \tag{1}$$

where $z = (z_1, z_0)$ belongs to the unit ball B^n in $\mathbf{C}^n$, $z_1 \in D$, $z_0 = (z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$, and we choose the branch of the square root such that $\sqrt{f'(0)} = 1$.

Roper–Suffridge extension operator has remarkable properties:

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(i) If f is a normalized convex function on D, then F is a normalized convex mapping on B^n;
(ii) If f is a normalized starlike function on D, then F is a normalized starlike mapping on B^n;
(iii) If f is a normalized Bloch function on D, then F is a normalized Bloch mapping on B^n.

These results were proved by Roper and Suffridge [1], Graham and Kohr [2]. Until now, we only know a few concrete examples about the convex mappings, starlike mappings and Bloch mappings on B^n. By Roper–Suffridge extension operator, we may construct a lots of concrete examples about these mappings on B^n. This is one reason why people are interested in this extension operator.

After that there are many papers to discuss this operator (for example, [3–6], etc.). They generalized the Roper–Suffridge extension operators and discussed their properties.

In [3], Graham et al. generalized the operator (1) as

$$\Phi_{n,\alpha}(f)(z) = F_\alpha(z) = \bigl(f(z_1), \bigl(f'(z_1)\bigr)^\alpha z_0\bigr), \tag{2}$$

where $\alpha \in [0, 1/2]$, f, z_1, z_0, z are defined as above, and we choose the branch of the power function such that $(f'(z_1))^\alpha|_{z_1=0} = 1$. They proved that this operator maps the normalized starlike function on D to the normalized starlike mapping on B^n, and maps the normalized Bloch function on D to the normalized Bloch mapping on B^n, but it does not preserve convexity on B^n when $\alpha \in [0, 1/2)$. In [2], Graham and Kohr proposed the following open problem: consider the Reinhardt domain

$$\Omega_{2,p} = \bigl\{z = (z_1, z_2) \in \mathbf{C}^2\colon |z_1|^2 + |z_2|^p < 1\bigr\},$$

where $p \geqslant 1$. Does the operator

$$\Phi_{2,1/p}(f)(z) = F_{1/p}(z) = \bigl(f(z_1), \bigl(f'(z_1)\bigr)^{1/p} z_2\bigr)$$

extend convex functions on D to the convex mappings on $\Omega_{2,p}$?

In [7], we defined the ε starlike mappings on a domain in $\mathbf{C}^n$.

Definition 1. Let Ω be a domain in $\mathbf{C}^n$, and let $f : \Omega \to \mathbf{C}^n$ be a locally biholomorphic mapping and $0 \in f(\Omega)$. We say f is ε starlike mapping on Ω if there exists a positive number ε, $0 \leqslant \varepsilon \leqslant 1$, such that $f(\Omega)$ is starlike with respect to every point in $\varepsilon f(\Omega)$. All ε starlike mappings on Ω form the family of ε starlike mappings on Ω.

When $\varepsilon = 0$, it is exactly the family of starlike mappings, and when $\varepsilon = 1$, it is exactly the family of convex mappings.

In [7], we proved the following result.

Theorem A. *Let $f(z_1)$ be a normalized biholomorphic ε starlike function on the unit disk $D = \{z_1 \in \mathbf{C}\colon |z_1| < 1\}$ in $\mathbf{C}$, $0 \leqslant \varepsilon \leqslant 1$, then*

$$\Phi_{n,1/p}(f)(z) = F_{1/p}(z) = \bigl(f(z_1), \bigl(f'(z_1)\bigr)^{1/p} z_0\bigr), \quad p \geqslant 1, \tag{3}$$

is a normalized biholomorphic ε *starlike mapping on*

$$\Omega_{n,p} = \left\{z \in \mathbf{C}^n\colon\ |z_1|^2 + \|z_0\|_p^p < 1\right\}, \tag{4}$$

where $z_0 = (z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$, $z = (z_1, z_0) \in \Omega_{n,p}$, *we choose the branch of the power function in* (3) *such that* $(f'(z_1))^{1/p}|_{z_1=0} = 1$, *and*

$$\|z\|_p = \begin{cases} \left(\sum_{j=1}^n |z_j|^p\right)^{1/p}, & 1 \leqslant p < \infty; \\ \max_{j=1,\ldots,n} |z_j|, & p = \infty. \end{cases}$$

When $n = 2$, $\varepsilon = 1$, Theorem A solved the open problem of Graham and Kohr [2]. The answer is affirmative, and it holds true for any $n \geqslant 2$. When $p = 2$, $\varepsilon = 1$, Theorem A is the result of Roper and Suffridge [1]. When $p = 2$, $\varepsilon = 0$, Theorem A is the result of Graham and Kohr [2].

Theorem A told how to construct concrete examples of convex mappings and starlike mappings on a class of Reinhardt domains (4). No doubt, it is an important class of Reinhardt domains in several complex variables, especially, it is a class of weak pseudoconvex domains when $p > 2$.

In Section 2, we will introduce some generalized Roper–Suffridge extension operator in purpose to construct the concrete convex mappings and starlike mappings on some class of more general Reinhardt domains. In Section 3, we will extend the Roper–Suffridge extension operator from on complex variable to several complex variables.

2. Generalized Roper–Suffridge operator on a class of Reinhardt domain

We have already known that for the class of Reinhardt domains (4), we may generalize the Roper–Suffridge extension operator as (3) such that we can use it to construct the convex mappings and the starlike mappings on (4). Now we consider the more general class of Reinhardt domains. Let

$$\Omega = \left\{z \in \mathbf{C}^n\colon\ |z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1\right\}, \tag{5}$$

where $p_i \geqslant 1$, $i = 1, 2, \ldots, n$, $z = (z_1, \ldots, z_n)$. How to generalize the Roper–Suffridge extension operator such that we can use it to construct the convex mappings and the starlike mappings on it? In general, we do not know how to do it. But we have the following result.

Theorem 1. *Let* $f(z_1)$ *be a normalized biholomorphic* ε *starlike function on the unit disk* D *in* $\mathbf{C}$, $0 \leqslant \varepsilon \leqslant 1$. *Then*

$$\begin{aligned} \Phi_{n,1/p_2,\ldots,1/p_n}(f)(z) &= F_{1/p_2,\ldots,1/p_n}(z) \\ &= \left(f(z_1), \left(f'(z_1)\right)^{1/p_2} z_2, \ldots, \left(f'(z_1)\right)^{1/p_n} z_n\right) \end{aligned} \tag{6}$$

is a normalized biholomorphic ε *starlike mapping on the Reinhardt domain*

$$\Omega_{n,p_2,\ldots,p_n} = \left\{z \in \mathbf{C}^n\colon\ |z_1|^2 + |z_2|^{p_2} + \cdots + |z_n|^{p_n} < 1\right\}, \tag{7}$$

where $z = (z_1, \ldots, z_n)$, $p_i \geqslant 1$, $i = 2, \ldots, n$, *and we choose the branch of the power functions in* (6) *such that* $(f'(z_1))^{1/p_i}|_{z_1=0} = 1$, $i = 2, \ldots, n$.

Theorem 1 is a special case of the following result.

Theorem 2. *Let $\|\cdot\|_i$ be the Banach norms of $\mathbf{C}^{n_i}$, $i=1,2,\dots,k$, where n_i are positive integers. Let*

$$\Omega_N=\left\{(z_1,z,\dots,w)\in\mathbf{C}\times\mathbf{C}^{n_1}\times\cdots\times\mathbf{C}^{n_k}\colon |z_1|^2+\|z\|_1^{p_1}+\cdots+\|w\|_k^{p_k}<1\right\}, \tag{8}$$

where $p_i\geqslant 1$, $i=1,2,\dots,k$, $N=1+n_1+\cdots+n_k$, $z_1\in\mathbf{C}$, $z\in\mathbf{C}^{n_1},\dots,w\in\mathbf{C}^{n_k}$.

If $f(z_1)$ is a normalized biholomorphic ε starlike function on the unit disk D in $\mathbf{C}$ then

$$\begin{aligned}\Phi_{N,1/p_1,\dots,1/p_k}(f)(z)&=F_{1/p_1,\dots,1/p_k}(z)\\&=\left(f(z_1),\left(f'(z_1)\right)^{1/p_1}z,\dots,\left(f'(z_1)\right)^{1/p_k}w\right)\end{aligned} \tag{9}$$

is a normalized biholomorphic ε starlike mapping on Ω_N, where we choose the branch of the power function in (9) such that $(f'(z_1))^{1/p_i}|_{z_1=0}=1$, $i=1,\dots,k$.

When $k=1$, $\|\cdot\|_1$ is the p-norm, it is Theorem A. When $n_i=1$, $\|\cdot\|_i$ is p-norm, $i=1,\dots,k$, it is Theorem 1.

Proof of Theorem 2. For any $\lambda\in[0,1]$, $(z_1,z,\dots,w)\in\Omega_N$ and $(a_1,a,\dots,b)\in\Omega_N$, where $z_1\in D$, $a_1\in D$, $z\in\mathbf{C}^{n_1}$, $a\in\mathbf{C}^{n_1}$, $\dots$, and $w\in\mathbf{C}^{n_k}$, $b\in\mathbf{C}^{n_k}$, if we can find $(u_1,u,\dots,v)\in\Omega_N$, where $u_1\in D$, $u\in\mathbf{C}^{n_1},\dots,v\in\mathbf{C}^{n_k}$, such that

$$\begin{aligned}&\left(f(u_1),\left(f'(u_1)\right)^{1/p_1}u,\dots,\left(f'(u_1)\right)^{1/p_k}v\right)\\&\quad=(1-\lambda)\left(f(z_1),\left(f'(z_1)\right)^{1/p_1}z,\dots,\left(f'(z_1)\right)^{1/p_k}w\right)\\&\qquad+\lambda\varepsilon\left(f(a_1),\left(f'(a_1)\right)^{1/p_1}a,\dots,\left(f'(a_1)\right)^{1/p_k}b\right),\end{aligned} \tag{10}$$

then Theorem 2 has been proved.

Since f is a ε starlike function on D, for any λ, $\lambda\in[0,1]$, and $z_1\in D$, $a_1\in D$, there exists $u_1\in D$, such that

$$f(u_1)=(1-\lambda)f(z_1)+\lambda\varepsilon f(a_1). \tag{11}$$

Thus the right hand side of (10) is

$$\begin{aligned}&\Big(f(u_1),(1-\lambda)\left(f'(z_1)\right)^{1/p_1}z+\lambda\varepsilon\left(f'(a_1)\right)^{1/p_1}a,\dots,\\&\quad(1-\lambda)\left(f'(z_1)\right)^{1/p_k}w+\lambda\varepsilon\left(f'(a_1)\right)^{1/p_k}b\Big).\end{aligned}$$

Let

$$\begin{aligned}(u,\dots,v)=\bigg(&\frac{(1-\lambda)(f'(z_1))^{1/p_1}z+\lambda\varepsilon(f'(a_1))^{1/p_1}a}{(f'(u_1))^{1/p_1}},\dots,\\&\frac{(1-\lambda)(f'(z_1))^{1/p_k}w+\lambda\varepsilon(f'(a_1))^{1/p_k}b}{(f'(u_1))^{1/p_k}}\bigg).\end{aligned} \tag{12}$$

We need to prove the following inequality

$$\|u\|_1^{p_1} + \cdots + \|v\|_k^{p_k} < 1 - |u_1|^2 \tag{13}$$

holds. From (11), we have

$$u_1(z_1, a_1) = f^{-1}\big[(1-\lambda) f(z_1) + \lambda\varepsilon f(a_1)\big]. \tag{14}$$

Regarding (14) as a mapping from $D \times D$ to D, we have already proved [7] that

$$\frac{\big|\frac{\partial u_1}{\partial z_1}\big||\xi| + \big|\frac{\partial u_1}{\partial a_1}\big||\eta|}{1 - |u_1|^2} \leqslant \max\left(\frac{|\xi|}{1-|z_1|^2}, \frac{|\eta|}{1-|a_1|^2}\right), \tag{15}$$

where ξ, η are any two arbitrary complex numbers. Moreover, by (14) we know that

$$\frac{\partial u_1}{\partial z_1} = \frac{1}{f'(u_1)}(1-\lambda) f'(z_1), \qquad \frac{\partial u_1}{\partial a_1} = \frac{1}{f'(u_1)}\lambda\varepsilon f'(a_1).$$

Substituting it into (12), we have

$$(u, \ldots, v) = \left((1-\lambda)^{1/q_1}\left(\frac{\partial u_1}{\partial z_1}\right)^{1/p_1} z + (\lambda\varepsilon)^{1/q_1}\left(\frac{\partial u_1}{\partial a_1}\right)^{1/p_1} a, \ldots,\right.$$
$$\left.(1-\lambda)^{1/q_k}\left(\frac{\partial u_1}{\partial z_1}\right)^{1/p_k} w + (\lambda\varepsilon)^{1/q_k}\left(\frac{\partial u_1}{\partial a_1}\right)^{1/p_k} b\right),$$

where $1/p_1 + 1/q_1 = 1, \ldots, 1/p_k + 1/q_k = 1$. By the triangle inequality of Banach norm and Hölder inequality, we have

$$\|u\|_1 \leqslant (1-\lambda)^{1/q_1}\left|\frac{\partial u_1}{\partial z_1}\right|^{1/p_1}\|z\|_1 + (\lambda\varepsilon)^{1/q_1}\left|\frac{\partial u_1}{\partial a_1}\right|^{1/p_1}\|a\|_1$$
$$\leqslant (1-\lambda+\lambda\varepsilon)^{1/q_1}\left(\left|\frac{\partial u_1}{\partial z_1}\right|\|z\|_1^{p_1} + \left|\frac{\partial u_1}{\partial a_1}\right|\|a\|_1^{p_1}\right)^{1/p_1}.$$

Thus

$$\|u\|_1^{p_1} \leqslant \left|\frac{\partial u_1}{\partial z_1}\right|\|z\|_1^{p_1} + \left|\frac{\partial u_1}{\partial a_1}\right|\|a\|_1^{p_1}.$$

Using the same process, we may obtain the estimations of the other terms, for example, the estimations of the last term is the following inequality:

$$\|v\|_k^{p_k} \leqslant \left|\frac{\partial u_1}{\partial z_1}\right|\|w\|_k^{p_k} + \left|\frac{\partial u_1}{\partial a_1}\right|\|b\|_k^{p_k}.$$

Thus,

$$\|u\|_1^{p_1} + \cdots + \|v\|_k^{p_k} \leqslant \left|\frac{\partial u_1}{\partial z_1}\right|\big(\|z\|_1^{p_1} + \cdots + \|w\|_k^{p_k}\big)$$
$$+ \left|\frac{\partial u_1}{\partial a_1}\right|\big(\|a\|_1^{p_1} + \cdots + \|b\|_k^{p_k}\big). \tag{16}$$

Let $\xi = \|z\|_1^{p_1} + \cdots + \|w\|_k^{p_k}$, $\eta = \|a\|_1^{p_1} + \cdots + \|b\|_k^{p_k}$ in (15); then

$$\begin{aligned}&\|u\|_1^{p_1}+\cdots+\|v\|_k^{p_k}\\&\quad\leqslant\left(1-|u_1|^2\right)\max\left(\frac{\|z\|_1^{p_1}+\cdots+\|w\|_k^{p_k}}{1-|z_1|^2},\frac{\|a\|_1^{p_1}+\cdots+\|b\|_k^{p_k}}{1-|a_1|^2}\right)\\&\quad<1-|u_1|^2\end{aligned}$$

by (16).

We have proved (13), and hence we have proved Theorem 2. □

3. Roper–Suffridge extension operator for several complex variables

The Roper–Suffridge extension operator and its generalizations which we mentioned above start from a locally biholomorphic function f of one complex variable on the unit disk in **C**, by the Roper–Suffridge extension operator or its generalizations Φ, we get a locally biholomorphic mapping $\Phi(f)=F$ on some domain in $\mathbf{C}^n$, then we discussed the properties of F. Now we try to extend the Roper–Suffridge extension operator and its generalizations from one variable to several complex variables.

We start with a locally biholomorphic mapping $f:R\to\mathbf{C}^n$, where R is a domain in $\mathbf{C}^n$ and

$$f=\begin{pmatrix}f_1\\\vdots\\f_n\end{pmatrix},$$

then we construct a generalized Roper–Suffridge extension operator, using it we may get a locally biholomorphic mappings on some domain in $\mathbf{C}^m$ $(m>n)$. In this section, we give one example of such kind generalized Roper–Suffridge extension operator.

Let

$$D^n=\left\{z=\begin{pmatrix}z_1\\\vdots\\z_n\end{pmatrix}\in\mathbf{C}^n\colon\ |z_i|<1,\ i=1,\dots,n\right\}$$

be the unit polydisk in $\mathbf{C}^n$. Let $f:D^n\to\mathbf{C}^n$ be a normalized biholomorphic convex mapping on D^n; then by Suffridge theorem [8],

$$f(z)=\begin{pmatrix}f_1(z_1)\\\vdots\\f_n(z_n)\end{pmatrix},$$

so

$$J_f(z)=\begin{pmatrix}f_1'(z_1)&&0\\&\ddots&\\0&&f_n'(z_n)\end{pmatrix},$$

$J_f(z)$ is the Jacobi matrix of f at z, where $f_i(z_i)$, $i=1,2,\dots,n$, are normalized biholomorphic convex functions on the unit disk. Let

$$\Omega_{2,p}^n=\left\{\begin{pmatrix}z\\w\end{pmatrix}\in\mathbf{C}^{2n}\colon\ |z_i|^2+|w_i|^p<1,\ i=1,2,\dots,n\right\},\quad p\geqslant 1,\tag{17}$$

where

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbf{C}^n.$$

As a consequence of Theorem A, we have the following result.

Corollary 1. *Let $f(z)$ be a normalized biholomorphic convex mapping on D^n, where $f(z): D^n \to \mathbf{C}^n$ is a column vector. Then*

$$\Phi_{2,1/p}(f)(z,w) = \begin{pmatrix} f(z) \\ (J_f(z))^{1/p} w \end{pmatrix} \tag{18}$$

is a normalized biholomorphic convex mapping on $\Omega^n_{2,p}$, where

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbf{C}^n, \qquad \big(J_f(z)\big)^{1/p} = \begin{pmatrix} (f_1'(z_1))^{1/p} & & O \\ & \ddots & \\ O & & (f_n'(z_n))^{1/p} \end{pmatrix},$$

and we choose the branch of the power functions in (18) *such that $(f_i'(z_i))^{1/p}|_{z_i=0} = 1$, $i = 1, 2, \ldots, n$.*

Proof. From Theorem A,

$$\begin{pmatrix} f_i(z_i) \\ (f_i'(z_i))^{1/p} w_i \end{pmatrix} \quad (i = 1, 2, \ldots, n)$$

is a normalized biholomorphic convex mapping on $\Omega_{2,p}$, so

$$\begin{pmatrix} f(z) \\ (J_f(z))^{1/p} w \end{pmatrix} = \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \\ (f_1'(z_1))^{1/p} w_1 \\ \vdots \\ (f_n'(z_n))^{1/p} w_n \end{pmatrix}$$

is a normalized biholomorphic convex mapping on $\Omega^n_{2,p}$. □

When $\varepsilon \neq 1$, we cannot define $(J_f(z))^{1/p}$ because $J_f(z)$ is not a diagonal matrix. But we have the following result.

Theorem 3. *Let $f(z): D^n \to \mathbf{C}^n$ be a normalized biholomorphic ε starlike mapping on D^n. Then $\Phi_{2,1}(f)(z,w)$ is a normalized biholomorphic ε starlike mapping on $\Omega^n_{2,1}$.*

In purpose to prove Theorem 3, we need the following lemma.

Lemma 1. *The infinitesimal form of the Carathéodory metric of D^n is*

$$F_C^{D^n}(z,\zeta) = \max\left(\frac{|\zeta_1|}{1-|z_1|^2}, \ldots, \frac{|\zeta_n|}{1-|z_n|^2}\right),$$

where

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in D^n, \qquad \zeta = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix} \in \mathbf{C}^n.$$

Proof. Fix

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in D^n.$$

Let

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \in D^n, \qquad \varphi(s) = \begin{pmatrix} \frac{z_1 - s_1}{1-\bar{z}_1 s_1} \\ \vdots \\ \frac{z_n - s_n}{1-\bar{z}_n s_n} \end{pmatrix};$$

then $\varphi(s) \in \mathrm{Aut}(D^n)$, $\varphi(z) = 0$,

$$J_\varphi(z) = \begin{pmatrix} \frac{-1}{1-|z_1|^2} & & O \\ & \ddots & \\ O & & \frac{-1}{1-|z_n|^2} \end{pmatrix} \quad \text{and} \quad J_\varphi(z)\zeta = \begin{pmatrix} \frac{-\zeta_1}{1-|z_1|^2} \\ \vdots \\ \frac{-\zeta_n}{1-|z_n|^2} \end{pmatrix}.$$

Let $F_C^{D^n}(\cdot,\cdot)$ be the infinitesimal form of Carathéodory metric, then $F_C^{D^n}(z,\zeta) = F_C^{D^n}(0, J_\varphi(z)\zeta)$. Since D^n is a bounded convex circular domain, $F_C^{D^n}(0,\zeta) = \rho(\zeta)$ [9, 10], where $\rho(\zeta)$ is the Minkowski functional of D^n. We already know that the Minkowski functional of D^n is $\max_{1 \leqslant i \leqslant n} |z_i|$. Hence

$$F_C^{D^n}\big(0, J_\varphi(z)\zeta\big) = \max\left(\frac{|\zeta_1|}{1-|z_1|^2}, \ldots, \frac{|\zeta_n|}{1-|z_n|^2}\right).$$

We have proved Lemma 1. □

By Lempert theorem [11], we know the infinitesimal form of Carathéodory metric of D^n and the infinitesimal form of Kobayashi metric of D^n are the same.

Proof of Theorem 3. For any $\lambda \in [0,1]$, $\binom{z}{w} \in \Omega^n_{2,1}$, $\binom{\xi}{\eta} \in \Omega^n_{2,1}$, if we can fine $\binom{u}{v} \in \Omega^n_{2,1}$ such that

$$\begin{pmatrix} f(u) \\ J_f(u)v \end{pmatrix} = (1-\lambda)\begin{pmatrix} f(z) \\ J_f(z)w \end{pmatrix} + \lambda\varepsilon\begin{pmatrix} f(\xi) \\ J_f(\xi)\eta \end{pmatrix},$$

then Theorem 3 have been proved.

Since f is a ε starlike mapping on D^n, for any $\lambda \in [0,1]$ and $z \in D^n$, $\xi \in D^n$, there exists $u \in D^n$, such that

$$f(u) = (1-\lambda)f(z) + \lambda\varepsilon f(\xi). \tag{19}$$

Let

$$v = (1-\lambda)J_f^{-1}(u)J_f(z)w + \lambda\varepsilon J_f^{-1}(u)J_f(\xi)\eta. \tag{20}$$

We need to prove $\binom{u}{v} \in \Omega^n_{2,1}$, i.e., the following inequality

$$\max\left(\frac{|v_1|}{1-|u_1|^2}, \ldots, \frac{|v_n|}{1-|u_n|^2}\right) < 1 \tag{21}$$

holds. From (19), we have

$$u(z,\xi) = f^{-1}\big[(1-\lambda)f(z) + \lambda\varepsilon f(\xi)\big]. \tag{22}$$

We regard it as a mapping from $D^n \times D^n$ to D^n, then by the contraction property of Carathéodory metric, the following inequality

$$F_C^{D^n}\left(u(z,\xi), J_u(z,\xi)\begin{pmatrix} w \\ \eta \end{pmatrix}\right) \leqslant F_C^{D^n\times D^n}\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, \begin{pmatrix} w \\ \eta \end{pmatrix}\right) \tag{23}$$

holds for any column vector $\binom{w}{\eta} \in \mathbf{C}^{2n}$, where J_u is the Jacobi matrix of u. From (22),

$$J_u(z,\xi) = \big((1-\lambda)J_f^{-1}(u)J_f(z),\ \lambda\varepsilon J_f^{-1}(u)J_f(\xi)\big).$$

Hence

$$J_u(z,\xi)\begin{pmatrix} w \\ \eta \end{pmatrix} = (1-\lambda)J_f^{-1}(u)J_f(z)w + \lambda\varepsilon J_f^{-1}(u)J_f(\xi)\eta = v$$

by (20). Thus (23) becomes

$$F_C^{D^n}(u,v) \leqslant F_C^{D^n\times D^n}\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, \begin{pmatrix} w \\ \eta \end{pmatrix}\right).$$

By Lemma 1, it is exactly the following inequality

$$\max\left(\frac{|v_1|}{1-|u_1|^2}, \ldots, \frac{|v_n|}{1-|u_n|^2}\right) \leqslant \max\left(\frac{|w_1|}{1-|z_1|^2}, \ldots, \frac{|w_n|}{1-|z_n|^2}, \frac{|\eta_1|}{1-|\xi_1|^2}, \ldots, \frac{|\eta_n|}{1-|\xi_n|^2}\right) < 1.$$

Hence (21) holds true. We have proved the Theorem 3. □

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