Gödel Logics: Foundations and Applications to Computer Science

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Research Proposal

1 Introduction

Gödel logics are a family of many-valued logics which have recently received significant attention in Computer Science. They are one of the families of logics which have been used as a basis of fuzzy logic; they have been used to give characterizations of the stable model semantics in logic programming; and they have been put forward as candidates for a logical analysis of parallel computation. The aim of my project is to investigate applications of these logics in computer science, in particular, in the areas of automated deduction for reasoning with uncertainty and in the theoretical foundations of parallel programming. The connection with parallel programs is perhaps the most significant aspect: the so-called proofs-as-programs paradigm identifies proofs in intuitionistic logic with programs (specifically, typed lambda-terms). Gödel logics are closely related to intuitionistic logic, but have certain proof-theoretical advantages. One aim is to exploit these advantages to extend the proofs-as-programs paradigm to Gödel logics and extensions of lambda calculus with operators for parallel execution of subprograms, and to the extraction of parallel programs from proofs.

The research will rely on an investigation of the foundational framework to be carried out initially; this requires a study of the formal characteristics of families of quantified Gödel logics and the development of deductive systems for them. In this part of the project, I expand upon my previous research in many-valued and other logics of relevance to computer science (including Gödel logics), as well as automated deduction and proof theory more generally.

The most important specific goals of the project are: (1) characterization of first-order Gödel logics as regards axiomatizability and decidability (of fragments); (2) a characterization of complexity and expressive power of quantified propositional Gödel logics; (3) development of deductive systems (both for theoretical purposes and for automated reasoning with uncertainty); (4) the development of extensions of first-order Gödel logics to include both first-order and propositional quantifiers; (5) suitable interpretations of derivations in various formalizations of Gödel logics as parallel programs under the proofs-as-programs paradigm; (6) investigation of the connections between Gödel logics and other formalisms of interest to computer science, such as temporal logics.

2 Recent Progress in Related Research Activities

Propositional Gödel Logics. Gödel logics are a family of many-valued logics, each based on a set of truth values $V \subseteq [0,1]$ and the following truth functions:

$$\begin{aligned} v(\bot) &= 0 \\ v(\top) &= 1 \\ v(A \land B) &= \min(v(A), v(B)) \end{aligned} \qquad \begin{aligned} v(A \lor B) &= \max(v(A), v(B)) \\ v(A \to B) &= \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ \text{v(B)} & \text{otherwise} \end{cases} \end{aligned}$$

Any mapping $v\colon \mathrm{Var} \to V$ can then be extended to a mapping from all formulas to V. A formula A is a *tautology* if $v(A) = 1$ for all valuations v. We denote the set of tautologies of the Gödel logic based on V by $\mathbf{G}_V$.

The propositional Gödel logics are well understood: Any infinite set of truth-values characterizes the same set of tautologies; they are axiomatized by **LC**, i.e., intuitionistic logic plus the linearity principle $(A \to B) \vee (B \to A)$. **LC** is also characterized as the intersection of all finite-valued Gödel logics as well as the logic determined either by linearly ordered Kripke structures [12] or linearly ordered Heyting algebras [15].

In previous research [9], I have obtained related results about entailment relations in Gödel logics. By contrast with the set of tautologies, different infinite sets of truth-values give rise to different entailment relations in Gödel logics. We define $\Gamma \models A$ as: for all valuations $v : Var \to V$, $\inf\{v(B) \mid B \in \Gamma\} \leq v(A)$. My result is that the entailment relation on $V = [0,1]$ is the same as on any uncountable subset of $[0,1]$; it is the only infinite-valued entailment relation which is (a) compact; (b) interpolates; or (c) is recursively enumerable.

First-order Extension To define first-order versions of Gödel logics, let $\mathrm{Distr}_v(A(x))$, the *distribution* of the formula $A(x)$, be $\mathrm{Distr}_{\mathbf{M}}(A(x)) = \{\mathbf{M}(A(d)) : d \in |\mathbf{M}|\}$ ($\mathbf{M}(A(d))$ denotes the truth-value of the formula $A(d)$ if d is an element of the domain $|\mathbf{M}|$). Then the satisfaction clauses for the universal and existential quantifier are given by $\mathbf{M}((\forall x)A(x)) = \inf \mathrm{Distr}_{\mathbf{M}}(A(x))$ and $\mathbf{M}((\exists x)A(x)) = \sup \mathrm{Distr}_{\mathbf{M}}(A(x))$. Let $\mathbf{G}_V^{\mathrm{fo}}$ be the set of validities over truth-value set V.

It turns out that, as in the case of entailment relations and in contrast to propositional tautologies, different truth value sets V result in different logics. *Intuitionistic fuzzy logic*, based on $V = [0,1]$ and discussed by Takeuti and Titani [18] is perhaps the most interesting. $\mathbf{G}_{\downarrow}^{\mathrm{fo}}$, where $V_{\downarrow} = \{1/k : k \geq 1\} \cup \{0\}$, is the logic of well-founded linearly ordered Kripke semantics with constant domains. This logic is closely connected to Kröger's temporal logic of programs [16]. In [4, 5], Baaz, Leitsch, and I have obtained a non-axiomatizability result which transfers to temporal logic and which significantly strengthens previous results.

Propositional Quantified Extensions In *classical* propositional logic one defines $(\exists p)A(p)$ by $A(\bot) \vee A(\top)$ and $(\forall p)A(p)$ by $A(\bot) \wedge A(\top)$. This can be extended to Gödel logic by using fuzzy quantifiers. The semantics of propositional quantifiers is defined analogously to that of first-order quantifiers as the infimum and supremum of the corresponding distribution. In this context a distribution of a formula A and free propositional variable a with respect to a valuation v is defined as $\mathrm{Distr}_v(A[p])) = \{v'(A[p])) \mid v' \sim_p v\}$, where $v' \sim_p v$ means that v' is exactly as v with the possible exception of the truth-value assigned to p. We write $\mathbf{G}_V^{\mathrm{qp}}$ for the set of tautologies in the extended language over V.

Already in classical logic, propositional quantification allows for succinct expression of complex properties; this is witnessed by the fact that satisfiability of quantified boolean formulas is PSPACE complete. In Gödel logic, it in fact increases the expressive power of the language, e.g., using propositional quantifiers it is possible to express certain topological properties of the underlying set of truth values. This enabled Baaz and Veith [8] to show that there are uncountably many different propositional quantified Gödel logics. I have investigated some of these logics in detail, e.g., $\mathbf{G}_{\uparrow}^{\mathrm{qp}}$ has been shown to be decidable and given an axiomatization in [3].

Proof Systems In [10], Baaz and I have introduced and studied an analytic calculus for intuitionistic fuzzy logic $\mathbf{G}_{[0,1]}^{\mathrm{fo}}$. This calculus uses hypersequents, a simple and natural generalization of Gentzen sequents. **HIF** is based on Avron's hypersequent calculus **GLC** for **LC** [1, 2].

The most significant feature of **HIF** is its close relation to Gentzen's sequent calculus **LJ** for intuitionistic logic. In [10], Baaz and I showed that **HIF** admits cut elimination. Moreover, in contrast to intuitionistic logic, it also satisfies the "mid-hypersequent theorem," establishing that, in analogy to classical logic, any cut-free proof of a prenex hypersequent can be transformed into a cut-free proof which contains a hypersequent separating propositional from quantifier inferences. This theorem corresponds to Herbrand's theorem for classical logic, which is the basis of automated deduction by resolution.

3 Research Objectives

3.1 Application to Parallel Programs

Natural deduction systems for intuitionistic logic stand in an interesting and important correspondence with functional programs in the following sense: Every proof in natural deduction corresponds to a typed λ term (functional program); and normalization of the proof corresponds to reduction of the corresponding term to a normal form (execution of the program). This correspondence is known as the Curry-Howard isomorphism. Avron [1] has suggested that a similar relationship may hold between Gödel logic and parallel programs. One of my main aims is to investigate to which extent this idea can be carried out. Building on the development of proof systems (see below), I propose to extend the Curry-Howard isomorphism to derivations in natural deduction systems for Gödel logics on the one hand, and terms in a suitable typed λ-calculus on the other.

Another application to be investigated is that of extraction of programs from proofs. It is in general not possible to extract a term π from intuitionistic proofs (in a theory T) of $\forall\bar{x}A(\bar{x}) \vdash \forall y \exists z B(y,z)$ such that $\forall\bar{x}A(\bar{x}) \vdash \forall y \exists z B(y,\pi(y))$ holds. This is connected to the failure of the midsequent theorem in intuitionistic logic (and extractions are possible where the midsequent theorem holds, e.g., for derivations without any applications of the $\vee$-left rule).

In the hypersequent calculus **HIF** for $\mathbf{G}^{\mathrm{fo}}_{[0,1]}$ the mid-hypersequent theorem holds, i.e., several terms π_i are exhibited by the proof. I aim to show how parallel programs can be extracted from such proofs by interpreting the terms π_i as subroutines which will be executed in parallel; the program returns the result of the first subroutine which terminates successfully.

3.2 First-order Gödel Logics

Axiomatizability. As pointed out above, it is known that the first-order Gödel logic based on $[0,1]$ is axiomatizable. I will give a complete characterization of all axiomatizable first-order Gödel logics, and investigate the structure of the lattice of all Gödel logics. Another important problem I aim to solve is that of the correspondence between the finite-valued logics and the infinite valued ones; in particular, I want to characterize the intersection of all finite-valued logics by a particular infinite set of truth values.

In connection with these classification problems, I will then proceed to a finer-grained analysis, investigating questions such as: What is the exact recursion-theoretic complexity of the logics in question? Which monadic fragments are axiomatizable or decidable? If decidable, what is the complexity?

Proof systems. In previous work with Baaz (see Section 2), I proposed a hypersequent calculus for an infinite-valued Gödel logic. Hypersequent calculi, first introduced by Avron, provide a suitable framework for the study of many non-classical logics besides Gödel logic; see e.g., [11] for hypersequent calculi for Urquhart's **C** and monoidal t-norm logic. They are close to standard systems for classical and intuitionistic logic, share many of their properties (e.g., cut elimination), and as such provide a suitable framework for the study of proof-theoretic properties of first-order Gödel logics. More work has to be done, however, both as regards the development of systems for Gödel logics other than $\mathbf{G}^{\mathrm{fo}}_{[0,1]}$, as well as in the investigation of the systems that are available. For instance, I aim to obtain results such as interpolation theorems. With a view to applications, it will also be necessary to give complete systems suitable for automated deduction, such as resolution and tableaux, as well as natural deduction systems (needed as a basis for the proposed research discussed above on the connections between Gödel logics and parallel computation).

3.3 Quantified Propositional Gödel logics

Classification. I intend to classify the propositional quantified Gödel logics $\mathbf{G}^{\mathrm{qp}}_V$ according to topological and order-theoretic properties of the truth-value sets V. I am especially interested in V where $\mathbf{G}^{\mathrm{qp}}_V$ is well-behaved, i.e., admits recursive axiomatization, is decidable, or has quantifier elimination. As a particular case, I hope to establishing a finite axiomatization for the logic of linearly ordered well-founded Kripke structures $\mathbf{G}^{\mathrm{qp}}_{\downarrow}$. This logic has the same relation to the temporal logic of "always" as quantified intuitionistic logic has to quantified propositional S4. We know that $\mathbf{G}^{\mathrm{qp}}_{\downarrow}$ is decidable. The next step will be to establish a suitable axiomatization by using quantifier elimination.

Proof systems. I intend to construct adequate analytic proof systems for the axiomatizable quantified propositional and first-order Gödel logics. These calculi will shed light on the non-axiomatizable first-order variants (e.g., in the case of $V_{\downarrow}$).

Hypersequent calculi yield a suitable formalism to cope with the above task. As shown in [11], they are particularly suited to deal with logics containing the linearity principle. Thus hypersequents are a natural candidate for formalizing quantified propositional Gödel logics.

3.4 Combining First-order and Quantified Propositional Gödel logics

First-order Gödel logics (with exception of $\mathbf{G}^{\mathrm{fo}}_{\uparrow}$) do not admit provably equivalent prenex normal forms for all formulas. Such normal forms, however, are needed for clause-based automated deduction systems such as resolution, and they can be obtained by suitably combining first-order and propositional quantifiers.

I will thus investigate first-order Gödel logics extended by propositional quantifiers, in particular, the Gödel logic based on the truth-value set $[0,1]$. I aim at

- Investigating the expressive power of the combined logics with particular attention to the one based on the set of truth-values $[0,1]$.
- Extending the quantifier elimination procedure for propositional quantifiers to these combined logics. This will provide a sound and complete axiomatization for the logic based on $[0,1]$.
- Defining suitable hypersequent calculi and natural deduction systems. In particular, the calculus for the logic based on $[0,1]$ consist of a suitable combination of the calculi **HIF** and **HQG**. It is intended to prove a mid-hypersequent theorem for this calculus. This theorem might provide foundations for sound and complete skolemizations, and therefore enlarged possibilities to extract information from proofs, as will be explained in the next section.

4 Anticipated Significance of the Work

The work carried out in the project will represent a significant contribution to the theory of first-order fuzzy logics. This area has only recently received attention. The main reason for this is the fact that Łukasiewiz logic, one of the most important first-order infinite-valued logics, is known not to be axiomatizable. Since some first-order Gödel logics are not only axiomatizable, but have "nice" properties, they represent a promising alternative to traditional first-order fuzzy logic. The project will contribute to the foundational investigations of Gödel logics and will be significant for further applications thereof, e.g., for reasoning with uncertainty. The specific project goal of developing proof systems suitable for automated deduction will further this long-term aim by laying the groundwork for an implementation of inference mechanisms for Gödel logics. The close relationship of Gödel logic with intuitionistic logic on the one hand and with temporal logics on the other in itself will provide a better understanding of these systems. In particular, the exploitation of the similarities between intuitionistic logic and Gödel logic will shed additional light on those areas of theoretical computer science and mathematics where intuitionistic logic is applied, e.g., in the theory of computation, extraction of programs from proofs, and in constructive mathematics.

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