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BMO, boundedness of affine operators, and frames

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Abstract

This paper addresses the construction of wavelet frames as an application of the modern theory of singular integrals. The continuous wavelet inversion formula (Calderón reproducing formula) may be viewed as the action of a Calderón–Zygmund singular integral operator. Wavelet frame operators arise as Riemann sum approximations of these singular integrals. When the analyzing and synthesizing functions are smooth and have a vanishing moment, boundedness of the approximations is a simple matter of applying, for example, the Cotlar lemma. Here we investigate the situation when only one of the analyzing/synthesizing pair has a vanishing moment. The dyadic discretizations are no longer automatically bounded. We show how the $T(\mathbf{1})$ theorem may be used to find criteria under which boundedness and invertibility are ensured. Parallels between these ideas and the frame criteria of Daubechies and Ron–Shen are also discussed.

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1. Introduction and main results

Given a function ϕ on the line and real numbers $r > 1$, $s > 0$, we define the action of the dilation operator D_r and translation operator T_s on ϕ by $D_r\phi(x) = r^{1/2}\phi(rx)$, $T_s\phi(x) = \phi(x - s)$. When $r = 2$, $s = 1$, we write $D_2 = D$, $T_1 = T$. These operators play a central role in the theory of affine frames, and also in the theory of singular integral operators. The dilation and translation operators may be iterated and composed to obtain functions $\phi_{k\ell}(x) = D_r^k T_s^\ell \phi(x) = r^{k/2}\phi(r^k x - s\ell)$. The pair (r, s) generates a *mesh* $\Lambda = \Lambda^{(r,s)}$ in the upper half-plane $H = \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$ defined by $\Lambda^{(r,s)} = \{(s\ell r^{-k}, r^{-k});\ k, \ell \in \mathbb{Z}\}$.

When constructing affine frames, it is standard practice to consider the *analysis operator* A_ϕ which takes a signal $f \in L^2(\mathbb{R})$ to the doubly-indexed sequence $(A_\phi)_{k\ell}(f) = \langle f, \phi_{k\ell}\rangle = \int_{-\infty}^{\infty} f(x) r^{k/2}\overline{\phi}(r^k x - s\ell)\,\mathrm{d}x$ $(k, \ell \in \mathbb{Z})$. When $\phi \in L^2(\mathbb{R})$, these inner products are well defined, but we will further restrict attention to the case where $A_\phi : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$ is bounded. In this case we may consider the adjoint operator $S_\phi = A_\phi^* : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$ (the *synthesis operator*), which maps a sequence $a = \{a_{k\ell}\}$ to the function $(S_\phi a)(x) = \sum_{k,\ell=-\infty}^{\infty} a_{k\ell}\phi_{k\ell}(x)$. The collection $\{\phi_{k\ell}\}_{k,\ell=-\infty}^{\infty}$ is a *frame* for $L^2(\mathbb{R})$ when A_ϕ satisfies the *frame estimates*

$$A\|f\|_{L^2(\mathbb{R})} \leqslant \|A_\phi f\|_{\ell^2(\mathbb{Z}^2)} \leqslant B\|f\|_{L^2(\mathbb{R})} \tag{1}$$

for constants $0 < A \leqslant B < \infty$. The constants A and B are the *frame bounds*. The upper bound in (1) ensures the boundedness of A_ϕ while the lower bound gives the boundedness of its inverse. In this case, the *discrete sum operator* $\mathcal{D}_\phi = S_\phi \circ A_\phi$ is bounded on $L^2(\mathbb{R})$ with bounded inverse and each $f \in L^2(\mathbb{R})$ admits the *frame expansion*

$$f = \mathcal{D}_\phi \circ \mathcal{D}_\phi^{-1} f = \sum_{k,\ell=-\infty}^{\infty} \bigl\langle f, \mathcal{D}_\phi^{-1} D_r^k T_s^\ell \phi\bigr\rangle D_r^k T_s^\ell \phi,$$

where we have used the self-adjointness of $\mathcal{D}_\phi$. The inverse $\mathcal{D}_\phi^{-1}$ is often computable via a Neumann series.

When $\phi, \psi \in L^2(\mathbb{R})$ and A_ϕ, A_ψ satisfy (1), we define

$$\mathcal{D}_{\phi\psi}^{(r,s)} f = S_\psi \circ A_\phi f = \sum_{k,\ell=-\infty}^{\infty} \bigl\langle f, D_r^k T_s^\ell \phi\bigr\rangle D_r^k T_s^\ell \psi.$$

When the mesh parameters (r, s) are fixed we will often write just $\mathcal{D}_{\phi\psi}$ for $\mathcal{D}_{\phi\psi}^{(r,s)}$. When $\mathcal{D}_{\phi\psi}$ is invertible, each $f \in L^2(\mathbb{R})$ admits the expansion

$$f = \mathcal{D}_{\phi\psi} \circ \mathcal{D}_{\phi\psi}^{-1} f = \sum_{k,\ell=-\infty}^{\infty} \bigl\langle f, \mathcal{D}_{\psi\phi}^{-1} D_r^k T_s^\ell \phi\bigr\rangle D_r^k T_s^\ell \psi, \tag{2}$$

where we have used the fact that $\mathcal{D}_{\phi\psi}^* = \mathcal{D}_{\psi\phi}$. The expansion (2) remains valid when the roles of ϕ and ψ are interchanged. This approach, requiring as it does the boundedness of A_ϕ and A_ψ, requires both ϕ and ψ to have a vanishing moment, i.e., $\int \phi = \int \psi = 0$ when φ and ψ are also integrable.

The approach taken in this paper is slightly different. We focus on the composition $\mathcal{D}_{\phi\psi} = S_\psi \circ A_\phi$ rather than the individual operators A_ϕ and S_ψ. The theory of singular integral operators and *operators of Cotlar type* can be brought to bear on the analysis of $\mathcal{D}_{\phi\psi}$ from which norm estimates are explicitly calculable. It is not our purpose here to review the role of Calderón–Zygmund theory in the analysis of $\mathcal{D}_{\phi\psi}$ (see [10]) but one comment is in order. Namely, it is well known that if ϕ and ψ have *sufficient localization and smoothness* then $\mathcal{D}_{\phi\psi}$ defines an operator with a Calderón–Zygmund kernel. Moreover, $\mathcal{D}_{\phi\psi}$ can be viewed as a sort of "change of basis" operator—from an expansion in terms of shifts and dilates of ϕ to one in terms of ψ—having an "almost diagonal" matrix in the coefficients

$$M_{kk'\ell\ell'} = \langle D_r^{k'} T_s^{\ell'} \phi, \mathcal{D}_{\psi\psi}^{-1} D_r^k T_s^\ell \psi \rangle,$$

at least when $\mathcal{D}_{\psi\psi}$ and $\mathcal{D}_{\phi\phi}$ are each invertible, as is often the case when $\int \phi = \int \psi = 0$. This almost-diagonality was exploited systematically by Frazier and Jawerth [8] and was subsequently incorporated as a standard tool in Calderón–Zygmund theory (e.g., [7,9,11,14]). The present paper seeks useful descriptions of $\mathcal{D}_{\phi\psi}$ in the absence of this *separate invertibility*. Further consequences in operator theory will be considered elsewhere.

Mild smoothness and decay will be imposed quantitatively throughout this discussion by requiring ϕ, ψ to be $\mathcal{M}_\delta$*-test functions*. One says that $\phi = \phi(x)$ is an $\mathcal{M}_\delta$*-test function*, or $\phi \in \mathcal{M}_\delta(\mathbb{R})$, $0 < \delta \leqslant 1$, when the decay condition

$$|\phi(x)| \leqslant \text{const}\frac{1}{(1+|x|)^{1+\delta}}$$

and the smoothness condition

$$|\phi(x+y) - \phi(x)| \leqslant \text{const}\frac{|y|^\delta}{(1+|x|)^{1+2\delta}}$$

hold for all $x, y \in \mathbb{R}$ with $|y| \leqslant (1/2)(1+|x|)$. An additional vanishing moment condition on an $\mathcal{M}_\delta$-test function ψ will be expressed as $\psi \in \mathcal{M}_\delta^{(0)}$.

Notice that $\mathcal{D}_{\phi\psi}$ is *dilation-invariant* in the sense that $\mathcal{D}_{\phi\psi} = D_r \mathcal{D}_{\phi\psi} D_r^{-1}$, while the 'truncation'

$$\mathcal{D}_{\phi\psi}^+(f) = \sum_{k=0}^{\infty} \sum_{\ell=-\infty}^{\infty} \langle f, D_r^k T_s^\ell \phi \rangle D_r^k T_s^\ell \psi \tag{3}$$

obtained by summing only over nonnegative k (contractive dilates) is *shift-invariant* in the sense that $\mathcal{D}_{\phi\psi}^+ = T_s \mathcal{D}_{\phi\psi}^+ T_s^{-1}$. These two properties are a principal source of many of the necessary and sufficient conditions answering the following.

Basic Problem. Under what conditions on ϕ, ψ and (r, s) is $\mathcal{D}_{\phi\psi}$ bounded on $L^2(\mathbb{R})$?

1.1. Boundedness, moment conditions, and mesh conditions

L^2-boundedness of $\mathcal{D}_{\phi\psi}$ depends intimately on some form of cancellation, the simplest being a vanishing moments condition. In fact,

$$\hat{\phi}(0) = 0 = \hat{\psi}(0) \quad \Longrightarrow \quad \mathcal{D}_{\phi\psi} \quad \text{is } L^2\text{-bounded}$$

as a Cotlar's lemma argument shows (e.g., [10]). We normalize the Fourier transform of $\phi \in \mathcal{M}_\delta(\mathbb{R})$ by $\hat{\psi}(\xi) = \int \phi(x) e^{-2\pi i x\xi}\, dx$. Thus one answer to the Basic Problem is the following result.

Theorem A. *If $\psi \in \mathcal{M}_\delta^{(0)}$, then $\mathcal{D}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta^{(0)}(\mathbb{R})$ and all mesh parameters (r, s).*

Theorem A can be proved using the same arguments employed by Chui and Shi [1], so we will not include a proof here. On the other hand, in the symmetric case ($\psi = \phi$), ψ necessarily has a vanishing moment when $\mathcal{D}_{\psi\psi}$ is bounded on $L^2(\mathbb{R})$, as Chui and Shi also proved. When $\psi \neq \phi$, a Poisson summation argument proves the following result.

Theorem B. *Let $\phi, \psi \in \mathcal{M}_\delta(\mathbb{R})$. If $\mathcal{D}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$ for some mesh size (r, s) then at least one of $\hat{\phi}(0) = 0$, $\hat{\psi}(0) = 0$ holds.*

A slightly different formulation (Theorem 1)—from which Theorem B follows directly—will be proved in Section 2.

A *unilateral* vanishing moment does not always suffice. When ϕ is a Gaussian, for instance, and f is nonnegative, all of the coefficients $(A_\phi)_{k\ell}(f)$ will be nonnegative so any cancellation needed to ensure L^2-boundedness of $\mathcal{D}_{\phi\psi}$ will have to be supplied by ψ. The vanishing moment condition on ψ always provides some cancellation, but, in general, this is not enough, as the case of a mother wavelet illustrates very clearly.

Observation 1. If $\phi \in \mathcal{M}_\delta$, $\int \phi \neq 0$ and $\psi \in \mathcal{M}_\delta$ generates an orthonormal wavelet basis then, for the dyadic mesh, the operator $\mathcal{D}_{\phi\psi}$ is not L^2-bounded.

Though stated as an observation, this fact requires verification, as will be provided just after the statement of Theorem 5 in Section 3.

On the other hand, when $\hat{\psi}(0) = 0$ as we henceforth assume, then $\mathcal{D}_{\phi\psi}$ *is* bounded provided ϕ and ψ do not overlap too much in time or frequency, as the techniques of Daubechies [4] show. For example, the following 'observation' will be proved at the beginning of Section 4.

Observation 2. If $\phi, \psi \in \mathcal{M}_\delta$, $\hat{\psi}(0) = 0$ and the Fourier transforms $\hat{\phi}$, $\hat{\psi}$ have support in $[-1/(2s), 1/(2s)]$, then $\mathcal{D}^{(r,s)}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$.

This suggests the following more specific formulation of the earlier problem, emphasizing that cancellation of ψ must conspire with mesh conditions to produce L^2-boundedness when no cancellation is required of ϕ.

Basic Problem. Suppose $\psi \in \mathcal{M}^{(0)}_\delta$. Under what conditions on ψ and (r, s) is

$$\mathcal{D}_{\phi\psi} : f \mapsto \sum_{k,\ell=-\infty}^{\infty} \langle f, D_r^k T_s^\ell \phi \rangle D_r^k T_s^\ell \psi(x)$$

a bounded operator on $L^2(\mathbb{R})$ for each ϕ in $\mathcal{M}_\delta(\mathbb{R})$?

The following theorem provides an example of the sort of extra cancellation that one might require of ψ.

Theorem C. *Let $\psi \in \mathcal{M}_\delta$ be a mother wavelet and $\phi \in \mathcal{M}_\delta$. Then for the dyadic mesh (where $(r,s) = (2,1)$), $\mathcal{D}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$ if and only if $\hat{\phi}(0) = 0$. In contrast, when over-sampling,*

$$\mathcal{D}_{\phi\psi}^{(2,s)} : f \mapsto \sum_{k,\ell=-\infty}^{\infty} \langle f, D^k T_s^\ell \phi \rangle D^k T_s^\ell \psi$$

is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta(\mathbb{R})$ and each $s = 1/2, 1/4, 1/8, \ldots$.

What is important in this instance is the fact that the Fourier transform of a mother wavelet automatically has the property $\hat{\psi}(2n) = 0$ for all integers n, not only for $n = 0$. Theorem C turns out to be a special case of Theorem D, which is one of three boundedness results for $\mathcal{D}_{\phi\psi}$-operators that we discuss now.

Well-known methods for establishing L^2-boundedness of operators such as $\mathcal{D}_{\phi\psi}$ include:

- Daubechies' frame criteria using Poisson summation on $\mathcal{D}_{\phi\psi}$;
- Ron–Shen frame criteria exploiting the shift-invariance of $\mathcal{D}_{\phi\psi}^{+}$;
- the $T(\mathbf{1})$-theorem using Cotlar's lemma and BMO-functions.

We shall concentrate on the latter but the first boundedness result, Theorem D, draws out the parallels among the $T(\mathbf{1})$ approach and the Daubechies and Ron–Shen criteria.

Theorem D. *If $\hat{\psi}(nR) = 0$ for all integers n and some $R > 0$, then $\mathcal{D}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta(\mathbb{R})$ and each mesh size $(r, 1/mR)$ with $r > 1$ and m a positive integer.*

Theorem C is a special case in which $r = 2 = R$. Theorem D is an immediate corollary of Theorem 9 in Section 3. Its proof boils down to an application of Poisson summation, but the other two boundedness results lie deeper: they require grid properties of the dyadic mesh as well as fundamental properties of Hardy H^1-spaces. In the second boundedness result, Theorem E, H^1 arises through a lacunarity estimate due to Paley.

Theorem E. *The operator $\mathcal{D}_{\phi\psi}$ with dyadic mesh is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta(\mathbb{R})$ if*

$$\sum_{n \text{ odd}} \left(\sum_{q=0}^{\infty} \left| \sum_{k=0}^{q} \hat{\psi}\left(n2^k\right) \right|^2 \right)^{1/2}$$

is finite.

Theorem E will be proved in Section 5. The third boundedness result, Theorem F, exploits an inequality of Hardy. Recall the *2-adic* norm on integers: for an integer n its 2-adic norm $\|n\|_2$ is the largest integer k such that 2^k is a factor of n, i.e., if $n = 2^k m$ with m an odd integer, then $\|n\|_2 = k$. The following boundedness criterion will be proved in Section 6.

Theorem F. *The operator $\mathcal{D}_{\phi\psi}$ with dyadic mesh is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta(\mathbb{R})$ if*

$$\left| \sum_{k=0}^{\|n\|_2} \hat{\psi}\left(2^{-k} n\right) \right| = \mathrm{O}\left(\frac{1}{|n|} \right)$$

as $n \to \pm\infty$.

The strength of Theorems D, E, and F lies in the freedom to choose any ϕ in $\mathcal{M}_\delta(\mathbb{R})$, whether or not it has vanishing moment. The asymmetric roles of φ and ψ, however, make invertibility criteria more technical, compared to the criteria of Daubechies and of Ron–Shen in the symmetric case.

1.2. Invertibility and frames

In the symmetric case $\phi = \psi$, the most primitive approach to inversion—and the approach that we follow here—regards $\mathcal{D}_{\psi\psi}$ as an operator-valued Riemann approximation of a multiple of the identity operator $\mathcal{I}$ when the latter is expressed in the form of the Calderón reproducing formula

$$f = \int_0^\infty f * \psi_t^* * \psi_t \frac{\mathrm{d}t}{t} \approx s\left(\frac{r-1}{r}\right) \sum_k \sum_\ell \langle f, D_r^k T_s^\ell \psi \rangle D_r^k T_s^\ell \psi \equiv s\left(\frac{r-1}{r}\right) \mathcal{D}_{\psi\psi}^{(r,s)}(f),$$

where $\psi_t(x) = (1/t)\psi(x/t)$ and $\psi^*(x) = \overline{\psi}(-x)$. Under suitable conditions on ψ, the same methods used to estimate $\mathcal{D}_{\psi\psi}$ can also be used to show that the difference operator $\mathcal{I} - (s(r-1)/r)\mathcal{D}_{\psi\psi}$ has small operator norm in L^2 for small mesh size. Inversion then follows from perturbation methods.

What is new here—at least to frame theory—is that the techniques underlying Theorems E and F also apply to this perturbation problem when ϕ need not have a vanishing moment, provided the mesh used in defining $\mathcal{D}_{\phi\psi}$ (i) is sufficiently fine and (ii) possesses an effective grid structure. The following invertibility criterion will be proved in Section 7.

Theorem G. *Let $\phi \in \mathcal{M}_\delta(\mathbb{R})$ and $\psi \in \mathcal{M}_\delta^{(0)}(\mathbb{R})$ be chosen so that*

$$\int_0^\infty \hat{\psi}(t\xi)\overline{\hat{\phi}(t\xi)} \frac{\mathrm{d}t}{t} \equiv 1.$$

Suppose that $\hat{\psi}$ satisfies the following cancellation condition:

$$\sum_{n \text{ odd}} \left(\sum_{p=0}^\infty \left| \sum_{k=0}^p \hat{\psi}\left(n2^{k+L}\right) \right|^2 \right)^{1/2} \to 0 \quad \text{as } L \to \infty.$$

Then $\mathcal{D}_{\phi\psi}^{(r_K, s_L)}$ is bounded and is invertible when $r_K = 2^{1/K}$ and $s_L = 2^{-L}$ for sufficiently large positive integers K and L.

It makes sense to refer to the families $\{D_r^k T_s^\ell \psi\}$ and $\{D_r^k T_s^\ell \phi\}$ as a *frame pair* when $\mathcal{D}_{\phi\psi}^{(r,s)}$ is bounded and continuously invertible, although ϕ, ψ typically will not generate dual frames in this manner.

The flexibility of this scheme extends beyond the ability to choose any $\phi \in \mathcal{M}_\delta$, whether or not it has a vanishing moment. If ϕ, ψ generate a frame pair, then each $f \in L^2(\mathbb{R})$ could be expanded as a linear combination of translated dilates of, say, a Gaussian. On the other hand, expansions of the form (2) allow for coefficients to be obtained from integrals of the form $\langle f, \phi_{k\ell}\rangle$. These are weighted averages and at high scales may be approximated by point evaluations. Notice also that the sums defining $\mathcal{D}_{\phi\psi}$ are naturally thought of as the limits of partial sums of the form $\sum_{k=-N}^N \sum_{\ell=-\infty}^\infty \langle f, \phi_{k\ell}\rangle \psi_{k\ell}$ which has a geometric description as partial sums over mesh points lying within certain *strips*

$$\mathcal{L}_\varepsilon = \left\{ (v, t) \in \mathbb{R}_+^2;\ v \in \mathbb{R},\ \varepsilon < t < \frac{1}{\varepsilon} \right\}$$

in the upper-half plane. Notice that each partial sum involves the computation of infinitely many inner products. By considering appropriate maximal functions, these sums may be replaced by sums over the *frustrums of cones*

$$\Gamma_\varepsilon^\lambda = \left\{(v,t) \in \mathbb{R}_+^2;\ |v| < -t\log(\varepsilon)^\lambda\right\}$$

defined so that λ determines the aperture of the cone and $\lim_{\varepsilon\to 0}\Gamma_\varepsilon^\lambda = \mathbb{R}_+^2$, while preserving the convergence. Each approximation may then be made with only finitely many coefficients.

Proofs of the main theorems and observations will be found in the remaining sections as indicated above. A few other points regarding the organization of the rest of the paper are worth noting. In Section 3, David and Journé's $T(\mathbf{1})$-theorem is formulated in terms that pertain specifically to operators $\mathcal{D}_{\phi\psi}$ (Theorem 4). The $\mathcal{M}_\delta$ smoothness condition that is imposed on ϕ and ψ plays a fundamental role in establishing certain technical 'Calderón–Zygmund properties' required of this formulation. Once we give meaning to the distribution $\mathcal{D}_{\phi\psi}(\mathbf{1})$, the problem of L^2-boundedness of $\mathcal{D}_{\phi\psi}$ is then reduced to verifying the condition of Theorem 8. In Section 4 we review the criteria of Daubechies and of Ron and Shen for proving *boundedness* of the affine frame operator, emphasizing the case of the dyadic mesh. In this case, the decomposition of any integer $\ell = n2^k$, n-odd, provides a reduction amenable to subtle cancellation criteria. This decomposition also plays decisive roles in the proofs of Theorems E and F. The application of our techniques to affine frames, that is, the problem of invertibility of $\mathcal{D}_{\phi\psi}$ on $L^2(\mathbb{R})$ when only ψ has a vanishing moment, is found in Section 7. Theorem G hinges on some further ideas from the theory of singular integral operators that are also discussed in Section 7.

Examples of functions ψ satisfying the cancellation criteria of Theorems E, F, and G are provided in Sections 5, 6, and 7, respectively. The basic principle at work throughout this discussion is that, since $\sigma(x) = \sum_{\ell\in\mathbb{Z}}\psi(x-\ell)$ is one-periodic, a Fourier series inequality for the periodic Hardy space $H^1(\mathbb{T})$ provides in turn a criterion for membership of $\mathcal{D}_{\phi\psi}^+(\mathbf{1})$ in BMO. In this manner, the approach to proving Theorem G can yield broader frame criteria. We work in a single variable throughout. Extensions to $\mathbb{R}^n$ and to other function spaces will be considered elsewhere.

2. Necessary conditions

Chui and Shi [1] proved that, for an $\mathcal{M}_\delta$ test function ψ, in order that the collection $\{r^{k/2}\psi(r^kx - s\ell)\}_{k,\ell=-\infty}^\infty$ forms a Bessel sequence it is necessary (and in fact sufficient) that $\int\psi = 0$. Fix $\mathcal{M}_\delta$-test functions ϕ and ψ. The following analogue of Chui and Shi's condition says that if $\mathcal{D}_{\phi\psi}$ is L^2-bounded, then at least one of ϕ or ψ should have a vanishing moment:

Theorem 1. *If the inequality*

$$\sup_N \left| \sum_{k=-N}^{N} \sum_{\ell=-\infty}^{\infty} \langle f, \phi_{k\ell}\rangle \overline{\langle g, \psi_{k\ell}\rangle} \right| \leqslant \mathrm{const}\|f\|_2\|g\|_2$$

holds uniformly for all f, g in $L^2(\mathbb{R})$, then $\overline{\hat\phi(0)}\,\hat\psi(0) = 0$; *in particular, at least one of ϕ and ψ has vanishing moment.*

Proof. By Poisson summation,

$$\sum_{\ell=-\infty}^{\infty} \langle f, \phi_{k\ell}\rangle \overline{\langle g, \psi_{k\ell}\rangle} = \frac{1}{s} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \hat{f}\big(\xi - r^k n/2s\big) \overline{\hat{g}\big(\xi + r^k n/2s\big)} \times \overline{\hat{\phi}\big(r^{-k}\xi - n/2s\big)} \hat{\psi}\big(r^{-k}\xi + n/2s\big)\, \mathrm{d}\xi \right).$$

Now set $\hat{f} = \hat{g} = \varepsilon^{-1/2}\chi_{(-\varepsilon,\varepsilon)}$ so that $\|f\|_2\|g\|_2 = 2$ whatever the choice of ε. With N fixed, if

$$\chi_{(-\varepsilon,\varepsilon)}\big(\xi - r^k n/2s\big)\chi_{(-\varepsilon,\varepsilon)}\big(\xi + r^k n/2s\big) \neq 0$$

for all k, $|k| \leqslant N$ and all sufficiently small ε, then $n = 0$. In this case,

$$\sum_{k=-N}^{N} \sum_{\ell=-\infty}^{\infty} \langle f, \phi_{k\ell}\rangle \overline{\langle g, \psi_{k\ell}\rangle} = \sum_{k=-N}^{N} \left(\frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \overline{\hat{\phi}\big(r^{-k}\xi\big)}\, \hat{\psi}\big(r^{-k}\xi\big)\, \mathrm{d}\xi \right) \to 2(2N+1)\overline{\hat{\phi}(0)}\, \hat{\psi}(0)$$

as $\varepsilon \to 0$. By hypothesis therefore, the inequality

$$\big|2(2N+1)\overline{\hat{\phi}(0)}\, \hat{\psi}(0)\big| \leqslant \text{const}$$

holds uniformly in N, which in turn ensures that $\overline{\hat{\phi}(0)}\, \hat{\psi}(0) = 0$. This completes the proof. □

3. BMO and the $T(\mathbf{1})$-theorem

In the absence of a vanishing moment condition on ϕ, an alternative cancellation is needed. The celebrated $T(\mathbf{1})$-theorem of David and Journé [5] will be our main tool for determining conditions for L^2-boundedness of operators such as $\mathcal{D}_{\phi\psi}$ in terms of membership of $\mathcal{D}_{\phi\psi}(\mathbf{1})$ in BMO, the space of functions of bounded mean oscillation (see, for example, [13]). Here $\mathbf{1}$ denotes the constant function equal to one. We will not state the full theorem, but rather a form that applies particularly to $\mathcal{D}_{\phi\psi}$. We only mention further properties of $\mathcal{D}_{\phi\psi}$ as they are needed to fulfill the hypotheses of the $T(\mathbf{1})$-theorem and because the operator bounds depend on them. A more general discussion, of which the propositions mentioned in this section are special cases, can be found in [10].

We start by defining BMO in terms that we can apply quickly. A *type* $(2,1)$*-atom* a is a function that is supported on an interval I and satisfies $\|a\|_2 \leqslant |I|^{-1/2}$ and $\int a = 0$ (see, e.g., [13, p. 112]). The real Hardy space $H^1(\mathbb{R})$ consists of those $f \in L^1(\mathbb{R})$ that admit an *atomic decomposition* $f = \sum_{k=1}^{\infty} \lambda_k a_k$ in which a_k are type $(2,1)$-atoms and $\sum_k |\lambda_k| < \infty$. The expression $\inf \sum_k |\lambda_k|$, in which the infimum is taken over all atomic decompositions of f, defines a norm on $H^1(\mathbb{R})$. The norm of any $(2,1)$-atom then is at most one, while finite linear combinations of atoms form a dense subspace of $H^1(\mathbb{R})$. The dual space of $H^1(\mathbb{R})$ is $\mathrm{BMO}(\mathbb{R})$. Its elements can be represented (modulo constants) by functions g such that $|\int ag| \leqslant$ const for any $(2,1)$-atom a. Any such g clearly defines a linear functional on $H^1(\mathbb{R})$. The classical definition of the BMO norm as

$$\|\phi\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|} \int_I \big|\phi - m_I(\phi)\big|,$$

in which $m_I(\phi)$ is the mean value of ϕ on I and the supremum is taken over all intervals I, serves to show that BMO properly contains L^∞. A detailed account of the equivalence of these definitions of BMO can be found in Stein [13].

As norm estimates requiring both smoothness and decay will be needed, it will be convenient to work within the setting of test functions and molecules described in [10]. The space $\mathcal{M}_\delta(\mathbb{R})$ of $\mathcal{M}_\delta$-test functions defined above is a Banach space under the natural weighted supremum norm

$$\|\phi\|_{\mathcal{M}_\delta} = \sup_x \big(1+|x|\big)^{1+\delta}\big|\phi(x)\big| + \sup_{|y|\leqslant(1/2)(1+|x|)} \left\{\big(1+|x|\big)^{1+2\delta}\frac{|\phi(x+y)-\phi(x)|}{|y|^\delta}\right\}. \tag{4}$$

When an $\mathcal{M}_\delta$-test function has a vanishing moment it will be said to be an $\mathcal{M}_\delta$*-molecule*; the set of all such molecules is a closed subspace $\mathcal{M}_\delta^{(0)}(\mathbb{R})$ of $\mathcal{M}_\delta(\mathbb{R})$. Any C^∞-function having rapid decay—the Mexican hat function, for instance—will be an $\mathcal{M}_\delta$-test function for all $0<\delta\leqslant 1$, as will any Schwartz function, but minimally smooth examples having slow decay can be constructed using splines.

The Fourier transform of an $\mathcal{M}_\delta(\mathbb{R})$-test function ϕ also has smoothness and decay properties: for each $0\leqslant\alpha<\delta$,

$$\frac{|\hat\phi(\xi)-\hat\phi(\eta)|}{|\xi-\eta|^\alpha} \leqslant \text{const}\|\phi\|_{\mathcal{M}_\delta} \tag{5}$$

and

$$\big|\hat\phi(\xi)\big| \leqslant \text{const}\frac{1}{(1+|\xi|)^\alpha}\|\phi\|_{\mathcal{M}_\delta}. \tag{6}$$

Now fix a molecule $\psi\in\mathcal{M}_\delta^{(0)}$. Then the boundedness of $\mathcal{D}_{\phi\psi}$, as a mapping from $\mathcal{M}_\delta(\mathbb{R})$ to its dual space $\mathcal{M}_\delta^*(\mathbb{R})$, a space of distributions, follows from simple coefficient estimates (see [10]).

Proposition 2. *Fix ϕ in $\mathcal{M}_\delta(\mathbb{R})$ and $\psi\in\mathcal{M}_\delta^{(0)}(\mathbb{R})$. Then for all mesh parameters (r,s), the inequalities*

$$\big|\langle D_{\phi\psi}f,g\rangle\big| \leqslant \sum_{k,\ell=-\infty}^{\infty}\big|\langle f, D_r^kT_s^\ell\phi\rangle\big|\big|\langle g, D_r^kT_s^\ell\psi\rangle\big| \leqslant \text{const}\|f\|_{\mathcal{M}_\delta}\|g\|_{\mathcal{M}_\delta}$$

hold uniformly for all f,g in $\mathcal{M}_\delta(\mathbb{R})$.

The constant in Proposition 2 will depend on the mesh parameters as well as the $\mathcal{M}_\delta$ norms of ψ and ϕ, of course. The proposition says, essentially, that $\mathcal{D}_{\phi\psi}$ satisfies the *weak boundedness property* or *WBP* (cf., [10, p. 16])—a requirement for the $T(\mathbf{1})$-theorem.

Armed with Proposition 2, L^2-boundedness of $\mathcal{D}_{\phi\psi}$ reduces to that of its truncation $\mathcal{D}_{\phi\psi}^+$ to positive scales, as defined in (3). This useful technical device was established and exploited by Ron and Shen in [12] to provide a proof of the following.

Proposition 3. *Fix ϕ, ψ in $\mathcal{M}_\delta(\mathbb{R})$. Then the operator $\mathcal{D}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$ for mesh parameters (r,s) if and only if its 'truncation' $\mathcal{D}_{\phi\psi}^+$ is bounded on $L^2(\mathbb{R})$. Furthermore, $\|\mathcal{D}_{\phi\psi}\| = \|\mathcal{D}_{\phi\psi}^+\|$ as operators on $L^2(\mathbb{R})$.*

One further prerequisite for application of the $T(\mathbf{1})$-theorem is that $\mathcal{D}_{\phi\psi}^+$ should possess an integral kernel satisfying the so-called *standard CZ-kernel estimates*. That it does when $\varphi,\psi\in\mathcal{M}_\delta(\mathbb{R})$ follows from the techniques in Sections 2 and 3 of [10]. In our case, the action of $\mathcal{D}^+_{\phi\psi}$ on, say, an $\mathcal{M}_\delta$ function is given by integration in y against the kernel

$$K^+_{\phi\psi}(x,y)=\sum_{k=0}^{\infty}\sum_{\ell=-\infty}^{\infty}D_r^kT_s^\ell\overline{\phi(y)}D_r^kT_s^\ell\psi(x).$$

Since $\mathcal{D}_{\phi\psi}$ satisfies the WBP and the standard estimates, the $T(\mathbf{1})$-theorem applied to $\mathcal{D}_{\phi\psi}$ reduces to the following statement.

Theorem 4. *Fix ϕ in $\mathcal{M}_\delta(\mathbb{R})$ and ψ in $\mathcal{M}_\delta^{(0)}(\mathbb{R})$. The operator $\mathcal{D}_{\phi\psi}$ is bounded on $L^2(\mathbb{R})$ for mesh parameters (r,s) if and only if $\mathcal{D}_{\phi\psi}(\mathbf{1})$ (alternatively, $(\mathcal{D}^+_{\phi\psi})(\mathbf{1})$) belongs to* BMO$(\mathbb{R})$. *The operator norm of $\mathcal{D}_{\phi\psi}$ then depends on the mesh parameters, the constants appearing in the WBP and the standard kernel estimates, and on the BMO norm of $\mathcal{D}_{\phi\psi}(\mathbf{1})$.*

It is worth observing that, in the $T(\mathbf{1})$-theorem, one also imposes the condition $\mathcal{D}^*_{\phi\psi}(\mathbf{1})\in$ BMO on the adjoint of $\mathcal{D}_{\phi\psi}$. Under the vanishing moment condition on ψ one has, in fact, $\mathcal{D}^*_{\phi\psi}(\mathbf{1})=0$. However, as Observation 1 asserts, $\int\psi=0$ is not always sufficient for boundedness of $\mathcal{D}_{\phi\psi}$. To establish Observation 1, we make use of the following Carleson-type characterization of BMO (see [11]).

Theorem 5. *Let ψ be an orthonormal mother wavelet in $\mathcal{M}_\delta$ for some $\delta>0$. Then $\sum_{k,\ell}c_{k\ell}\psi_{k\ell}\in$ BMO if and only if there is a $C>0$ such that, for any interval I,*

$$\frac{1}{|I|}\sum_{I(k,\ell)\subset I}|c_{k\ell}|^2\leqslant C.$$

Here $I(k,\ell)=[\ell/2^k,(\ell+1)/2^k)$.

Now let $\mathcal{D}_{\phi\psi}$ be the operator in which ψ is such a mother wavelet and ϕ is a Gaussian, or really any $\mathcal{M}_\delta$ function with $\int\phi\neq 0$. Then, at least formally, $\langle\mathbf{1},\phi_{k\ell}\rangle=c2^{-k/2}=c_{k\ell}$ independent of ℓ. Therefore, $\sum_{I(k,\ell)\subset[0,1)}|c_{k\ell}|^2=\sum_{k=0}^{\infty}c=\infty$. Hence $\mathcal{D}_{\phi\psi}(\mathbf{1})\notin$ BMO and $\mathcal{D}_{\phi\psi}$ is not L^2-bounded. This establishes Observation 1, provided we are interpreting $\mathcal{D}_{\phi\psi}(\mathbf{1})$ correctly.

Since the conditions on $\mathcal{D}_{\phi\psi}(\mathbf{1})$ and $\mathcal{D}^+_{\phi\psi}(\mathbf{1})$ are pivotal, one should take care in making sense of them. One can define $\mathcal{D}_{\phi\psi}(\mathbf{1})$ formally as

$$\begin{aligned}\mathcal{D}_{\phi\psi}(\mathbf{1})&=\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\langle\mathbf{1},D_r^kT_s^\ell\phi\rangle D_r^kT_s^\ell\psi=\overline{\hat{\phi}(0)}\sum_{k=-\infty}^{\infty}r^{-k/2}\sum_{\ell=-\infty}^{\infty}D_r^kT_s^\ell\psi\\&=\overline{\hat{\phi}(0)}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\psi(r^kx-s\ell).\end{aligned}\tag{7}$$

In certain cases it will be easier to make sense of $\mathcal{D}^+_{\phi\psi}(\mathbf{1})$ but for the moment we develop the meanings of $\mathcal{D}_{\phi\psi}(\mathbf{1})$ and $\mathcal{D}^+_{\phi\psi}(\mathbf{1})$ together.

Since a constant function is obviously dilation and shift-invariant, the invariance properties of $\mathcal{D}_{\phi\psi}$ and $\mathcal{D}^+_{\phi\psi}$ are reflected in those of $\mathcal{D}_{\phi\psi}(\mathbf{1})$ and $\mathcal{D}^+_{\phi\psi}(\mathbf{1})$. More explicitly, let

$$\sigma(x)=\sum_{\ell=-\infty}^{\infty}\psi(x-s\ell)=\sum_{n=-\infty}^{\infty}\hat{\psi}\left(\frac{n}{s}\right)e^{2\pi isnx}.\tag{8}$$

Obviously, σ has period s. Moreover,

$$\mathcal{D}_{\phi\psi}(\mathbf{1})(x) = \overline{\hat{\phi}(0)} \sum_{k=-\infty}^{\infty} \sigma\left(r^k x\right), \qquad \mathcal{D}^+_{\phi\psi}(\mathbf{1})(x) = \overline{\hat{\phi}(0)} \sum_{k=0}^{\infty} \sigma\left(r^k x\right).$$

Then by invariance or direct evaluation,

$$\mathcal{D}_{\phi\psi}(\mathbf{1})(rx) = \mathcal{D}_{\phi\psi}(\mathbf{1})(x)$$

whereas

$$\mathcal{D}^+_{\phi\psi}(\mathbf{1})(x - s) = \mathcal{D}^+_{\phi\psi}(\mathbf{1})(x),$$

provided $r \in \mathbb{N}$.

The first of these identities suggests that $\mathcal{D}_{\phi\psi}(\mathbf{1})(x)$ might have bizarre behavior if it exists as a *pointwise* function. The following proposition gives a precise meaning to $\mathcal{D}_{\phi\psi}(\mathbf{1})(x)$ and its companion $\mathcal{D}^+_{\phi\psi}(\mathbf{1})(x)$ as linear functionals on $\mathcal{M}^{(0)}_\delta$. Its proof follows the same lines as Proposition 2 (see [10]) so we will not include it here.

Proposition 6. *The inequalities*

$$\left|\langle \mathcal{D}_{\phi\psi}(\mathbf{1}), g\rangle\right| \leqslant \left|\hat{\phi}(0)\right| \sum_{k=-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \overline{g(x)} \sigma\left(r^k x\right) \mathrm{d}x \right| \leqslant \mathrm{const}\|g\|_{\mathcal{M}_\delta}$$

hold uniformly for all g in $\mathcal{M}^{(0)}_\delta(\mathbb{R})$ and each set of mesh parameters (r, s). A corresponding inequality applies to $\mathcal{D}^+_{\phi\psi}(\mathbf{1})(x)$.

This ensures that the series $\sum_k \sigma(r^k x)$ converges weakly in the dual space of $\mathcal{M}_\delta$-molecules. It may diverge pointwise yet converge weakly if ‘floating’ constants are added as necessary, reminiscent of BMO-functions. Surprisingly, the conditions needed to secure convergence of $\sum_{k=M}^{N} \sigma(r^k x)$ as $M \to -\infty$ are quite different from those needed when $N \to \infty$. For the former it is only a question of smoothness.

Proposition 7. *Fix φ in $\mathcal{M}_\delta(\mathbb{R})$, If a is a $(2, 1)$-atom supported in $[-R, R]$, then*

$$\sum_{k=-\infty}^{N} \left| \int_{-\infty}^{\infty} \left(\sum_{\ell=-\infty}^{\infty} \varphi\left(r^k x - s\ell\right)\right) \overline{a(x)} \, \mathrm{d}x \right| \leqslant \mathrm{const} R^\delta r^{N\delta}$$

uniformly in a, R, and N.

The proof of Proposition 7 is a straightforward consequence of the Cauchy–Schwarz inequality and the fact that σ as in (8) is Hölder continuous of order δ. For convergence as $N \to \infty$ one needs the ability to take extra advantage of cancellation. The following theorem is a reformulation of the BMO criterion of Theorem 4 for boundedness of $\mathcal{D}^+_{\phi\psi}$. Specifically, it is a statement of what it means for the partial sums $\sum_{k=0}^{N} \sigma(r^k x)$ to converge weakly in BMO (to $\mathcal{D}^+_{\phi\psi}(\mathbf{1})$).

Theorem 8. *Let $\psi \in \mathcal{M}_\delta^{(0)}$ and $\sigma(x)$ be defined as in* (8). *Then the operator $\mathcal{D}_{\phi\psi}$ with mesh parameters (r,s) is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta(\mathbb{R})$ if and only if the inequality*

$$\left| \int_{-\infty}^{\infty} \left(\sum_{k=0}^{N} \sigma\left(r^k x\right) \right) \overline{a(x)}\, \mathrm{d}x \right| \leqslant \text{const}$$

holds uniformly in N for all $(2,1)$-atoms a.

To prove Theorems D, E, and F, it suffices to show that the conditions of those theorems ensure that the inequality of Theorem 8 is satisfied.

By applying Poisson summation as in (8) the integrals in the condition for L^2-boundedness in Theorem 8 can be written

$$\int_{-\infty}^{\infty} \sigma\left(r^k x\right)\overline{a(x)}\, \mathrm{d}x = \sum_{\ell=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi\left(r^k x - s\ell\right)\overline{a(x)}\, \mathrm{d}x \right)$$
$$= \sum_{\ell=-\infty}^{\infty} \left(r^{-k} \int_{-\infty}^{\infty} \psi(x - s\ell)\overline{a\left(r^{-k}x\right)}\, \mathrm{d}x \right) = \sum_{n=-\infty}^{\infty} \hat{\psi}\left(\frac{n}{s}\right)\overline{\hat{a}\left(\frac{r^k n}{s}\right)}.$$

Thus from Theorem 8 a general criterion for L^2-boundedness follows.

Theorem 9. *The operator $\mathcal{D}_{\phi\psi}$ with mesh parameters (r,s) is bounded on $L^2(\mathbb{R})$ for all ϕ in $\mathcal{M}_\delta(\mathbb{R})$ if the inequality*

$$\left| \sum_{k,n=-\infty}^{\infty} \hat{\psi}\left(\frac{n}{s}\right)\overline{\hat{a}\left(\frac{r^k n}{s}\right)} \right| \leqslant \text{const}$$

holds uniformly for all $(2,1)$-atoms a.

Theorem D is an immediate corollary of Theorem 9 applied to the case $R = 1/s$: the hypothesis of Theorem D then implies that $\hat{\psi}(n/s) = 0$ for each n so the sum in Theorem 9 is zero. This is a rather remarkable fact: it says that besides the moment condition $\hat{\phi}(0) = 0$, the conditions $\hat{\psi}(n/s) = 0$, $n \in \mathbb{Z}$, also yield $\mathcal{D}_{\phi\psi}(\mathbf{1}) = 0$. The vanishing of $\hat{\psi}$ on a grid is a rather strong cancellation condition. Thus, in what follows, we seek criteria weaker than $\mathcal{D}_{\phi\psi}(\mathbf{1}) = 0$ that still imply $\mathcal{D}_{\phi\psi}(\mathbf{1}) \in \mathrm{BMO}$.

4. Daubechies and Ron–Shen criteria

Daubechies [4] used Poisson summation to write

$$\langle \mathcal{D}_{\phi\psi} f, g\rangle = \sum_{k,n=-\infty}^{\infty} \left(\frac{1}{s} \int_{-\infty}^{\infty} \hat{f}\left(\xi - \frac{r^k n}{2s}\right)\overline{\hat{g}\left(\xi + \frac{r^k n}{2s}\right)}\,\overline{\hat{\phi}\left(r^{-k}\xi - \frac{n}{2s}\right)}\,\hat{\psi}\left(r^{-k}\xi + \frac{n}{2s}\right) \mathrm{d}\xi \right). \tag{9}$$

Consider now Observation 2. Suppose that $\hat{\psi}$ and $\hat{\phi}$ are continuous and supported in $[-1/(2s), 1/(2s)]$. This implies that $\overline{\hat{\phi}(r^{-k}\xi - n/2s)}\, \hat{\psi}(r^{-k}\xi + n/2s)$ vanishes unless both $r^{-k}\xi - n/2s$ and $r^{-k}\xi + n/2s$ lie in $(-1/(2s),\ 1/(2s))$ which, in turn, implies $n = 0$. Consequently, the Fourier support hypothesis gives

$$\langle \mathcal{D}_{\phi\psi} f, g\rangle = \sum_{k,\ell=-\infty}^{\infty} \langle f, \phi_{k\ell}\rangle \overline{\langle g, \psi_{k\ell}\rangle} = \frac{1}{s}\left(\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)} \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}\big(r^{-k}\xi\big)}\hat{\psi}\big(r^{-k}\xi\big)\, \mathrm{d}\xi \right)$$

and, by Cauchy–Schwarz,

$$\big|\langle \mathcal{D}_{\phi\psi} f, g\rangle\big| \leqslant \frac{1}{s} M_{\phi\psi} \|f\|_2 \|g\|_2$$

in which

$$M_{\phi\psi} = \sup_{1\leqslant|\xi|\leqslant r} \left| \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}\big(r^{-k}\xi\big)}\hat{\psi}\big(r^{-k}\xi\big) \right|.$$

This proves Observation 2, since $M_{\phi\psi}$ is finite when $\phi \in \mathcal{M}_\delta$ and $\psi \in M_\delta^{(0)}$. When $\hat{\phi}$ and $\hat{\psi}$ are supported in $[-1/(2s), 1/(2s)]$, one also has

$$\|\mathcal{D}_{\phi\psi} f\|_2^2 \geqslant \frac{1}{s} m_{\phi\psi} \|f\|_2^2,$$

where

$$m_{\phi\psi} = \inf_{1\leqslant|\xi|\leqslant r} \left| \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}\big(r^{-k}\xi\big)}\hat{\psi}\big(r^{-k}\xi\big) \right|.$$

Without the Fourier support condition on ϕ, ψ one still has

$$\left\{ m_{\phi\psi} - \sum_{n\neq 0} \beta_{\phi\psi}(n)\beta_{\psi\phi}(-n) \right\} \leqslant s \frac{\|\mathcal{D}_{\phi\psi} f\|_2^2}{\|f\|_2^2} \leqslant \left\{ M_{\phi\psi} + \sum_{n\neq 0} \beta_{\phi\psi}(n)\beta_{\psi\phi}(-n) \right\} \tag{10}$$

in which

$$\beta_{\phi\psi}(\eta) = \sup_{1\leqslant|\xi|\leqslant r} \left(\sum_{k=-\infty}^{\infty} \left| \hat{\phi}\left(r^{-k}\xi - \frac{\eta}{s} \right) \hat{\psi}\big(r^{-k}\xi\big) \right| \right)^{1/2}$$

and $\beta_{\psi\phi}$ is defined by interchanging ϕ and ψ. It is particularly difficult, though, to verify that $m_{\phi\psi}$ dominates $\sum_{n\neq 0} \beta_{\phi\psi}(n)\beta_{\psi\phi}(-n)$.

In the special case $r = 2$ the function $\beta_{\phi\psi}$ can be replaced by

$$\gamma_{\phi\psi}(\eta) = \sup_{1\leqslant|\xi|\leqslant 2} \left(\sum_{k=-\infty}^{\infty} \left| \sum_{\nu\geqslant 0} \hat{\phi}\left(2^\nu \left(2^k\xi - \frac{\eta}{s} \right) \right) \overline{\hat{\psi}\big(2^{k+\nu}\xi\big)} \right| \right)^{1/2}$$

and, importantly, the sum in (10) over all $n \neq 0$ by the sum over all odd n. Additionally, when $s = 1$ and $\phi = \psi \in \mathcal{M}_\delta^{(0)}$ generates an orthonormal wavelet basis, $\gamma_{\phi\psi}(2n + 1) = 0$ (see [4]). In that case, the analogue of (10) identifies $\mathcal{D}_{\psi\psi}$ as being unitary.

Ron and Shen [12] make cancellation attributable to the dyadic mesh even more explicit in exploiting the reduction in Proposition 3. Then

$$\langle \mathcal{D}^+_{\phi\psi} f, g\rangle = \sum_{k=0}^{\infty}\sum_{\ell=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat f(\xi)\overline{\hat g(\eta)}\overline{\hat\phi\big(2^{-k}\xi\big)}\hat\psi\big(2^{-k}\eta\big)e^{2\pi i 2^{-k}\ell(\xi-\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\right).$$

For fixed k one then computes the coset decomposition $\ell = 2^k m + d$ of ℓ with $0 \leqslant d < 2^k$. Then, after periodization, the double integral becomes

$$\frac{1}{2^k}\sum_{d=0}^{2^k-1}\left\{\int_0^1\int_0^1\left(\sum_{n,n'=-\infty}^{\infty} f(\xi+n)\overline{\hat g(\eta+n')}e^{2\pi i 2^{-k}(\xi+n-\eta-n')d}\right.\right.$$
$$\left.\left.\times\,\overline{\hat\phi\big(2^{-k}(\xi+n)\big)}\,\hat\psi\big(2^{-k}(\eta+n')\big)\right)e^{2\pi i\ell(\xi-\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}.$$

Summing this double integral over ℓ yields a single integral

$$\int_0^1\left(\sum_{n,n'=-\infty}^{\infty} f(\xi+n)\overline{\hat g(\xi+n')}\,\overline{\hat\phi\big(2^{-k}(\xi+n)\big)}\,\hat\psi\big(2^{-k}(\xi+n')\big)\frac{1}{2^k}\left\{\sum_{d=0}^{2^k-1}e^{2\pi i 2^{-k}(n-n')d}\right\}\right)\mathrm{d}\xi\,\mathrm{d}\eta.$$

However, the sum in the integrand is easily shown to be

$$\sum_{d=0}^{2^k-1}e^{2\pi i 2^{-k}(n-n')d} = \begin{cases} 2^k & \text{if } \|n-n'\|_2 \geqslant k,\\ 0 & \text{otherwise},\end{cases} \tag{11}$$

where $\|n\|_2$ denotes the 2-adic norm on $\mathbb{Z}$. Consequently, if we follow Ron–Shen and define a matrix $\Phi(\xi) = (\Phi(\xi)_{mn})$ by setting

$$\Phi(\xi)_{mn} = \sum_{k=0}^{\|m-n\|_2}\overline{\hat\phi\big(2^{-k}(\xi+m)\big)}\,\hat\psi\big(2^{-k}(\xi+n)\big),$$

then

$$\langle \mathcal{D}^+_{\phi\psi} f, g\rangle = \int_0^1\left(\sum_{m,n=-\infty}^{\infty} f(\xi+m)\overline{\hat g(\xi+n)}\,\Phi_{mn}(\xi)\right)\mathrm{d}\xi.$$

The L^2-operator norm of $\mathcal{D}^+_{\phi\psi}$ and hence $\mathcal{D}_{\phi\psi}$ can now be estimated in terms of the ℓ^2-operator norm of the matrix $\Phi(\xi) = (\Phi(\xi)_{mn})$:

$$\|\mathcal{D}_{\phi\psi}\|_{\mathcal{L}(L^2)} = \sup_{0\leqslant\xi\leqslant 1}\big\|\Phi(\xi)\big\|_{\mathcal{L}(\ell^2)}.$$

This is reminiscent of the boundedness condition stemming from the use of Hardy's inequality in the $T(\mathbf{1})$-theorem (see [5]).

5. Paley's inequality and L^2-boundedness of $\mathcal{D}_{\phi\psi}$: Theorem E

Theorem E makes use of Paley's criterion for membership in the periodic Hardy space $H^1(\mathbb{T})$ (e.g., [2]).

Theorem 10 (Paley's theorem). *There is a constant C such that the sequence of Fourier coefficients $\{\hat{f}(\ell)\}$ of any $f \in H^1(\mathbb{T})$ satisfies, for any integer n,*

$$\left(\sum_{k=0}^{\infty}\left|\hat{f}\left(n2^k\right)\right|^2\right)^{1/2} \leqslant C\|f\|_{H^1}.$$

What is important for us is that if $f \in H^1(\mathbb{R})$ then its periodization $\pi(f) = \sum_{\ell\in\mathbb{Z}} f(x+k)$ belongs to $H^1(\mathbb{T})$ and $\|\pi(f)\|_{H^1(\mathbb{T})} \leqslant C\|f\|_{H^1(\mathbb{R})}$ (e.g., [2]). Since the ℓth Fourier coefficient of $\pi(f)$ is $\hat{f}(\ell)$, Paley's inequality extends to $H^1(\mathbb{R})$ as well. By the H^1-BMO duality, one then has:

Theorem 11. *If $\{c_k\} \in \ell^2(\mathbb{Z})$ then $b(x) = \sum_k c_k e^{2\pi i n 2^k x}$ converges in* BMO$(\mathbb{R})$ *and*

$$\|b\|_{\mathrm{BMO}} \leqslant C\left\|\{c_k\}\right\|_{\ell^2}.$$

We will make use of this estimate to prove Theorem E, taking advantage of the dyadic mesh setting to exploit one-periodicity of the 'truncation' $\mathcal{D}^+_{\phi\psi}(1)$. Parseval's equation for Fourier series gives, formally,

$$\left\langle\mathcal{D}^+_{\phi\psi}(\mathbf{1}), a\right\rangle = \int_0^1\left(\sum_{k=0}^{\infty}\sigma\left(2^k x\right)\right)\pi(\bar{a})(x)\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\left(\int_0^1\left(\sum_{k=0}^{\infty}\sigma\left(2^k x\right)\right)e^{2\pi i n x}\,\mathrm{d}x\right)\overline{\hat{a}(n)}.$$

This is where things get interesting arithmetically, for as in (11),

$$\int_0^1\left(\sum_{k=0}^{\infty}\sigma\left(2^k x\right)\right)e^{2\pi i n x}\,\mathrm{d}x = \sum_{k=0}^{\|n\|_2}\hat{\sigma}\left(\frac{n}{2^k}\right).$$

Proof of Theorem E. For each odd integer n, denote by $\sigma^{(n)}$ the 1-periodic function $\sigma^{(n)}(x) = \sum_{k=0}^{\infty}\hat{\sigma}(n2^k)e^{2\pi i 2^k x}$. Let $F_N^+(a) = \sum_{k=0}^{N}\langle\sigma(2^k x), a\rangle$ denote the truncated action of the positive scales of $\mathcal{D}_{\phi\psi}(\mathbf{1})$ on an atom a.

By Theorem 8 it is enough to show that F_N^+ is uniformly bounded and converges weakly to a corresponding limit F^+ as N tends to infinity. It suffices to prove bounds on atoms. In estimating $F_N^+(a)$, a simple application of Fubini's theorem gives

$$F_N^+(a) = \sum_{n\ \mathrm{odd}}\sum_{k=0}^{N}\sum_{p=k}^{\infty}\left(\sigma^{(n)}\right)^{\wedge}\left(2^{p-k}\right)\bar{\hat{a}}\left(n2^p\right).$$

We have

$$\sum_{k=0}^{N}\sum_{p=k}^{\infty}\hat{\sigma}\left(n2^{p-k}\right)\bar{\hat{a}}\left(n2^p\right) = \sum_{p=0}^{\infty}\sum_{k=0}^{\min(N,p)}\hat{\sigma}\left(n2^{p-k}\right)\bar{\hat{a}}\left(n2^p\right)$$

$$= \left\{ \sum_{p=0}^{N} \sum_{k=0}^{p} + \sum_{p=N+1}^{\infty} \sum_{k=0}^{N} \right\} \hat{\sigma}\big(n2^{p-k}\big)\bar{\hat{a}}\big(n2^{p}\big) = \mathrm{I} + \mathrm{II}.$$

The Cauchy–Schwarz inequality and Paley's theorem give

$$|\mathrm{I}| \leqslant \sum_{p=0}^{N} \left| \sum_{k=0}^{p} \hat{\sigma}\big(n2^{k}\big) \right| \big|\hat{a}\big(n2^{p}\big)\big| \leqslant \left(\sum_{p=0}^{N} \left| \sum_{k=0}^{p} \hat{\sigma}\big(n2^{k}\big) \right|^{2} \right)^{1/2} \left(\sum_{p=0}^{\infty} \big|(a_n)\hat{}\,\big(2^{p}\big)\big|^{2} \right)^{1/2}$$
$$\leqslant c \left(\sum_{p=0}^{\infty} \left| \sum_{k=0}^{p} \hat{\sigma}\big(n2^{k}\big) \right|^{2} \right)^{1/2} \equiv G(n).$$

Here $a_n(x) = na(nx)$. Note that since a is a $(2,1)$-atom, so too is a_n. The sum II may be split further

$$\mathrm{II} = \sum_{p=N+1}^{\infty} \sum_{k=0}^{p} \hat{\sigma}\big(n2^{p-k}\big)\bar{\hat{a}}\big(n2^{p}\big) - \sum_{p=N+1}^{\infty} \sum_{k=N+1}^{p} \hat{\sigma}\big(n2^{p-k}\big)\bar{\hat{a}}\big(n2^{p}\big) = \mathrm{III} + \mathrm{IV}.$$

The sum III is bounded in terms of $G(n)$. To estimate IV one changes variable and applies Cauchy–Schwarz to obtain

$$|\mathrm{IV}| = \left| \sum_{p=N+1}^{\infty} \sum_{k=0}^{p-N-1} \hat{\sigma}\big(n2^{k}\big)\bar{\hat{a}}\big(n2^{p}\big) \right| = \left| \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \hat{\sigma}\big(n2^{k}\big)\bar{\hat{a}}\big(n2^{N+1+\ell}\big) \right|$$
$$\leqslant \left(\sum_{\ell=0}^{\infty} \left| \sum_{k=0}^{\ell} \hat{\sigma}\big(n2^{k}\big) \right|^{2} \right)^{1/2} \left(\sum_{\ell=0}^{\infty} \big|\hat{a}\big(n2^{N+1+\ell}\big)\big|^{2} \right)^{1/2} \leqslant c \left(\sum_{\ell=0}^{\infty} \left| \sum_{k=0}^{\ell} \hat{\sigma}\big(n2^{k}\big) \right|^{2} \right)^{1/2} = G(n)$$

where we have applied Paley's inequality to dilates of a. Thus we have

$$\left| \sum_{k=0}^{N} \sum_{p=k}^{\infty} \big(\sigma^{(n)}\big)\hat{}\,\big(2^{p-k}\big)\hat{a}\big(n2^{p}\big) \right| \leqslant G(n)$$

independently of N. Moreover, one has $\sum_n G(n) < \infty$ by the hypothesis on σ, so $F_N^+(a)$ is uniformly bounded.

Next we claim that F_N^+ converges weakly to $F^+(a) = \sum_{k=0}^{\infty} \sigma(2^k x)$. In fact, the estimates above show that, for any $(2,1)$-atom a,

$$\big|\langle F_N^+ - F_M^+, a\rangle\big| \leqslant C \sum_n \left(\sum_{p=N+1}^{M} \left| \sum_{k=0}^{p} \hat{\sigma}\big(n2^{k}\big) \right|^{2} \right)^{1/2}.$$

Since $\sum_n G(n) < \infty$ it follows that the sequence $\underline{F_N^+}$ is weakly Cauchy and hence converges weakly to F^+ with $\|F^+\| \leqslant \limsup \|F_N^+\|$. Since $\mathcal{D}_{\phi\psi}^{+}(\mathbf{1}) = \hat{\bar{\phi}}(0) F^+$, Theorem E follows. □

Corollary 12. *Suppose $\phi \in \mathcal{M}_\delta(\mathbb{R})$, $\psi \in \mathcal{M}_\delta^{(0)}(\mathbb{R})$ and ψ satisfies*:

- *For each integer q, $\hat{\psi}(q) = 0$ unless $q = \pm 2^m$ for some nonnegative integer m, and*

- *the sequences* $\{b_q^{\pm}\}$ *defined by*

$$b_q^+ = \sum_{k=0}^{q} \hat{\psi}(2^k), \qquad b_q^- = \sum_{k=0}^{q} \hat{\psi}(-2^k)$$

are in $\ell^2(\mathbb{Z}^+)$.

Then $\mathcal{D}_{\phi\psi}(\mathbf{1}) \in \mathrm{BMO}$.

The corollary follows immediately from Theorem E since the condition on $\hat{\psi}$ implies that

$$\sum_{n \text{ odd}} \left(\sum_{k=0}^{\infty} \left| \sum_{k=0}^{q} \hat{\psi}(2^k n) \right|^2 \right)^{1/2} = \sum_{n=\pm 1} \left(\sum_{k=0}^{\infty} \left| \sum_{k=0}^{q} \hat{\psi}(2^k n) \right|^2 \right)^{1/2} = \left\| \{b_q^+\} \right\|_{\ell^2} + \left\| \{b_q^-\} \right\|_{\ell^2} < \infty.$$

5.1. *Examples of boundedness*

Examples of ψ satisfying the condition of Theorem E are easy to generate. For example, let c_ℓ $(\ell \in \mathbb{Z})$ be given by the following rule: if n is odd or zero and k is a nonnegative integer, then

$$c_{n2^k} = \begin{cases} 0 & \text{if } n = 0, \\ \frac{1}{|n|^{1+\beta}(2^\gamma - 1)} & \text{if } k = 0,\ n \neq 0, \\ \frac{-1}{|n|^{1+\beta} 2^{k\gamma}} & \text{if } k, n \neq 0, \end{cases}$$

where $\gamma, \beta > 0$ and let $\sigma(x) = \sum_\ell c_\ell e^{2\pi i \ell x}$. The Fourier coefficients of σ then satisfy

$$\sum_\ell |\ell|^\delta |c_\ell| = \left\{ \frac{1}{2^\gamma - 1} - \frac{1}{2^{\gamma - \delta} - 1} \right\} \sum_{n \text{ odd}} |n|^{-1-\beta+\delta} < \infty$$

provided $\delta < \min(\beta, \gamma)$. Then σ is Hölder continuous of order δ and one easily checks that

$$\sum_{k=0}^{p} c_{n2^k} = \frac{2^{-p\gamma}}{|n|^{1+\beta}(2^\gamma - 1)}.$$

One constructs $\psi \in \mathcal{M}_\delta^{(0)}(\mathbb{R})$ such that, for each $\ell \in \mathbb{Z}$, $\hat{\psi}(\ell) = \hat{\sigma}(\ell)$ by setting

$$\hat{\psi}(\xi) = \sum_{\ell=-\infty}^{\infty} \hat{\sigma}(\ell) \hat{B}_m(\xi - \ell).$$

Here B_m is a high order B-spline so that $\hat{B}_m$ is a high power of the cardinal sine function. Then $\hat{\psi}$ agrees with $\hat{\sigma}$ at integers. We can write $\psi(x) = \sigma(x) B_m(x)$, from which we see that ψ is continuous, bounded and compactly supported. The continuity of ψ is derived from that of σ.

6. Hardy's inequality and L^2-boundedness of $\mathcal{D}_{\phi\psi}$: Theorem F

Hardy's inequality

$$\sum_{n \neq 0} \frac{|\hat{f}(n)|}{|n|} \leqslant \mathrm{const} \|f\|_{H^1} \tag{12}$$

holds uniformly for all f in $H^1(\mathbb{T})$ (e.g., [2]). The H^1-BMO duality then provides a criterion for convergence of a trigonometric series in periodic BMO, as C. Fefferman [6] observed.

Corollary 13. *If* $b(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$, *where* $c_k = \mathrm{O}(1/|k|)$ *then* $b \in \mathrm{BMO}(\mathbb{R})$.

Proof of Theorem F. Set $b_n = \sum_{k=0}^{\|n\|_2} \hat{\psi}(n/2^k)$. The hypothesis of the theorem says that $|b_n| \leqslant c/|n|$ $(n \in \mathbb{Z})$. Now let $b(x) = \sum_n b_n e^{2\pi inx}$. Then $b \in \mathrm{BMO}$ by Corollary 13. Thus, to prove Theorem F it suffices to prove that, for any (2, 1)-atom a, $\langle b, a\rangle = \langle \mathcal{D}^+_{\phi\psi}(\mathbf{1}), a\rangle$. Fix a (2, 1)-atom a. Since b is 1-periodic,

$$\langle b, a\rangle = \sum_{\ell\in\mathbb{Z}} \int_0^1 b(x)\overline{a(x-\ell)}\,\mathrm{d}x = \int_0^1 b(x)\pi(\bar{a})(x)\,\mathrm{d}x,$$

where, as before, $\pi(a)$ is the periodization of a. Since a is an atom, $a * a^*$ is continuous, bounded and compactly supported, so $|\hat{a}|^2 = (a * a^*)\hat{\ }$ is continuous. By the Poisson summation formula,

$$\sum_\ell \bigl|\widehat{\pi(a)}(\ell)\bigr|^2 = \sum_\ell \bigl|\hat{a}(\ell)\bigr|^2 = \sum_k a * a^*(k) \leqslant c\max\bigl(1, |B|^{-1}\bigr),$$

where B is the ball supporting a. Hence $\pi(a) \in L^2(\mathbb{T})$ and, by Plancherel's theorem,

$$\langle b, a\rangle = \sum_\ell b_\ell \bar{\hat{a}}(\ell). \tag{13}$$

Let $\sigma(x) = \sum_k \psi(x-k)$ and the linear functionals F_N^+ be defined as in the proof of Theorem E. By Plancherel's theorem and (13), the action of F_N^+ on an atom is

$$\begin{aligned} F_N^+(a) &= \sum_{k=0}^{N} \sum_\ell \bigl(\sigma\bigl(2^k\cdot\bigr)\bigr)\hat{\ }(\ell)\bar{\hat{a}}(\ell) = \sum_\ell \sum_{k=0}^{\min(N,\|\ell\|_2)} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell) \\ &= \sum_{\|\ell\|_2\leqslant N} \sum_{k=0}^{\|\ell\|_2} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell) + \sum_{\|\ell\|_2\geqslant N+1} \sum_{k=0}^{N} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell) \\ &= \sum_{\|\ell\|_2\leqslant N} b_\ell \bar{\hat{a}}(\ell) + \sum_{\|\ell\|_2\geqslant N+1} \sum_{k=0}^{N} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell). \end{aligned}$$

But

$$\begin{aligned} \left|\sum_{\|\ell\|_2\geqslant N+1} \sum_{k=0}^{N} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell)\right| &= \left|\sum_{\|\ell\|_2\geqslant N+1} \left(\sum_{k=0}^{\|\ell\|_2} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell) - \sum_{k=N+1}^{\|\ell\|_2} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell)\right)\right| \\ &= \left|\sum_{\|\ell\|_2\geqslant N+1} \sum_{k=0}^{\|\ell\|_2} \hat{\sigma}\left(\frac{\ell}{2^k}\right)\bar{\hat{a}}(\ell) - \sum_{m=1}^{\infty} \sum_{k=0}^{\|m\|_2} \hat{\sigma}\left(\frac{m}{2^k}\right)\bar{\hat{a}}\bigl(m2^{N+1}\bigr)\right| \\ &\leqslant c \sum_{\|\ell\|_2\geqslant N+1} \frac{|\hat{a}(\ell)|}{|\ell|} + c\sum_{m=1}^{\infty} \frac{|\hat{a}(m2^{N+1})|}{|m|}. \end{aligned}$$

Since $\pi(a) \in L^2(\mathbb{T})$, the Cauchy–Schwarz inequality implies that this last term tends to zero as $N \to \infty$. We conclude $F_N^+(a)$ converges to $\langle b, a\rangle$ as $N \to \infty$, that is, F_N^+ converges weakly in BMO to $\mathcal{D}_{\phi\psi}^+(\mathbf{1})$ and Theorem F follows. □

The example of $\psi \in \mathcal{M}_\delta^{(0)}$ given in Section 5 for which the conditions of Theorem E hold also satisfies the conditions of Theorem F when $\gamma \geqslant 1$.

7. Invertibility: Theorem G

When suitably normalized, the discrete sum operator $\mathcal{D}_{\phi\psi}$ can be thought of as a Riemann approximation of a corresponding *continuous* operator $\mathcal{T}_{\phi\psi}$ defined below. When $\mathcal{T}_{\phi\psi}$ is invertible, it is convenient to think of $\mathcal{D}_{\phi\psi}$ as a small perturbation of $\mathcal{T}_{\phi\psi}$. 'Small' here depends on the mesh parameters (r, s). The upper half plane $H = \mathbb{R}_+^2 = \mathbb{R} \times \mathbb{R}_+$ is equipped with the structure of the 'AX + B' group with multiplication $(u, \tau) \cdot (v, t) = (tu + v, \tau t)$. The group can be represented on $L^2(\mathbb{R})$ by means of the unitary action $U(v, t) f(x) = t^{1/2} f(tx + v)$. Its adjoint or inverse action is $U^*(v, t)g(x) = U(-v/t, 1/t)g(x) = t^{-1/2}g((x - v)/t)$. Let $\Delta = \Delta^{(r,s)} = [0, s) \times [1, r)$ be the *basic tile* in $\mathbb{R}_+^2$ and define tiles $\Delta_{k\ell} = \Delta_{k\ell}^{(r,s)}$ to be translates of Δ by elements of the mesh, i.e., $\Delta_{k\ell} = \Delta \cdot (s\ell r^{-k}, r^{-k})$. Then the collection $\{\Delta_{k\ell}\}_{k,\ell=-\infty}^{\infty}$ forms a disjoint (a.e.) covering of $\mathbb{R}_+^2$. Each $\Delta_{k\ell}$ has the same right Haar measure $|\Delta_{k\ell}| = |\Delta| = s(1 - 1/r)$.

Henceforth we denote $\tilde{\mathcal{D}}_{\phi\psi} = \tilde{\mathcal{D}}_{\phi\psi}^{(r,s)} = |\Delta|\mathcal{D}_{\phi\psi}^{(r,s)}$. The use of the dyadic mesh as in the Paley's inequality approach of Theorem E is flexible enough to generate examples of invertible $\mathcal{D}_{\phi\psi}$ when $\hat{\phi}(0) \neq 0$ and a *grid structure* is present. This is the case when the shift parameter $s = 2^{-L}$ and scale parameter $r = 2^{1/K}$ are as in Theorem G: then every unit interval $[\ell, \ell + 1)$ is the KLth dilate by the factor r of $[s\ell, s(\ell + 1))$.

To estimate $\mathcal{T}_{\phi\psi} - \tilde{\mathcal{D}}_{\phi\psi}^{(r,s)}$ we will need to apply a result from [10]. For this sole purpose we recall briefly a formal class of operators said to be of *Cotlar type* in analogy to a family of singular integral operators first introduced by Cotlar [3].

We say that $a = a(v, t; x) \in L^\infty(H, \mathcal{M}_\delta)$ if $a(v, t; \cdot) \in \mathcal{M}_\delta$ for each $(v, t) \in H$, if a is Haar measurable in H, and if $\|a\|_{\infty,\delta} = \sup_{(v,t)\in H} \|a(v, t; \cdot)\|_{\mathcal{M}_\delta} < \infty$. The notation for a here no longer refers to H^1-atoms. To a pair $a, b \in L^\infty(H, \mathcal{M}_\delta)$ one formally assigns the *operator of Cotlar type*

$$\mathcal{T}_{ab} : f \mapsto \int_0^\infty \int_{\mathbb{R}} \bigl\langle f, U^*(v, t)a(v, t)\bigr\rangle U^*(v, t)b(v, t)\, \mathrm{d}v \frac{\mathrm{d}t}{t^2}.$$

The *homogeneous* case

$$\mathcal{T}_{\phi\psi} f(x) = \int_0^\infty f * \phi_t^* * \psi_t(x) \frac{\mathrm{d}t}{t} = \int_0^\infty t \int_{\mathbb{R}} f * \phi_t^*(u)\psi_t(x - u)\, \mathrm{d}u \frac{\mathrm{d}t}{t^2}$$

(as before, $\phi_t(x) = (1/t)\phi(x/t)$ while $\phi^*(x) = \overline{\phi(-x)}$) conforms to the special case in which a takes the constant value ϕ and b the constant value $\psi \in \mathcal{M}_\delta$ independently of $(v, t) \in H$. Importantly, $\mathcal{T}_{\phi\psi}$ is a singular integral operator (see [10]) that commutes with all translations and dilations and, therefore, has the form $\mathcal{T}_{\phi\psi} = \alpha\mathcal{I} + i\beta\mathcal{H}$, where $\mathcal{H}$ denotes the Hilbert transform. Choosing ϕ, ψ as in Theorem G guarantees that $\mathcal{T}_{\phi\psi} = \mathcal{I}$.

Then $\tilde{\mathcal{D}}_{\phi\psi}$ is an operator of Cotlar type $\mathcal{T}_{ab}$ in which $a(v,t;\cdot)$ takes the value

$$a(v,t;\cdot) = |\Delta|^{1/2} U\big((v,t)\cdot\big(-s\ell r^{-k}, r^{-k}\big)\big)\phi = |\Delta|^{1/2} U\big((v-sl)r^{-k}, tr^{-k}\big)\phi \in \mathcal{M}_\delta$$

whenever $(v,t) \in \Delta_{\ell k}$ while b is defined similarly with ϕ replaced by ψ.

There are three principal observations to be made. The first is that the difference operator $\mathcal{T}_{\phi\psi} - \tilde{\mathcal{D}}_{\phi\psi}$ can be expressed as a sum of two operators of Cotlar type, namely

$$\mathcal{T}_{\phi\psi} - \tilde{\mathcal{D}}^{(r,s)}_{\phi\psi} = (\mathcal{T}_{\phi\psi} - \mathcal{T}_{a\psi}) + \big(\mathcal{T}_{a\psi} - \tilde{\mathcal{D}}^{(r,s)}_{\phi\psi}\big) = \mathcal{T}_{\tilde{a}\psi} + \mathcal{T}_{a,\tilde{b}}.$$

Here $\mathcal{T}_{\tilde{a}\psi}$ is the operator of Cotlar type having first argument $\tilde{a}(v,t;\cdot) = \phi - a(v,t;\cdot)$, where a is as in the definition of $\tilde{\mathcal{D}}_{\phi\psi}$ and second argument identically equal to $\psi(\cdot)$. The operator $\mathcal{T}_{a\tilde{b}}$ has first argument a just as in $\tilde{\mathcal{D}}_{\phi\psi}$ and second argument $\tilde{b} = b(v,t;\cdot) - \psi(\cdot)$ with b the same as in $\tilde{\mathcal{D}}_{\phi\psi}$.

The second principal observation boils down to continuous dependence of $U(v,t)$ on $(v,t) \in H$—see [10, Corollary 4.1, p. 66]—which states that $\|(I - U(v,t))\phi\|_{\mathcal{M}_\gamma} \leqslant c((r-1)^\delta + s^\delta)^{1-\gamma/\delta}\|\phi\|_{\mathcal{M}_\delta}$ whenever $(v,t) \in \Delta^{(r,s)}$ and $0 < \gamma < \delta$. This allows us to make the $L^\infty(H, \mathcal{M}_\gamma)$ norms of $\tilde{a}$ and $\tilde{b}$ as small as we like by taking a fine enough mesh.

The third observation is that $\mathcal{T}_{ab}$ is controlled in terms of $\|a\|_{\infty,\delta}\|b\|_{\infty,\delta}$. An immediate consequence of [10, Corollary 3.4, p. 27]—noting that the moment condition here is the one on the adjoint there—together with the $T(\mathbf{1})$-theorem is:

Proposition 14. *Let $a: H \to \mathcal{M}_\delta(\mathbb{R})$ and $b: H \to \mathcal{M}^{(0)}_\delta(\mathbb{R})$ be L^∞ bounded functions on H. If, in addition, $\mathcal{T}_{ab}(\mathbf{1}) \in \mathrm{BMO}$ then $\mathcal{T}_{ab}$ is bounded on $L^2(\mathbb{R})$ and*

$$\|\mathcal{T}_{ab}\|_{L^2\to L^2} \leqslant \mathrm{const}\big(\|a\|_{\infty,\delta}\|b\|_{\infty,\delta} + \big\|\mathcal{T}^*_{ab}(\mathbf{1})\big\|_{\mathrm{BMO}}\big).$$

The upshot of this lengthy preamble is that, in order to prove continuous invertibility of $\tilde{\mathcal{D}}^{(r,s)}_{\phi\psi}$ when it holds for $\mathcal{T}_{\phi\psi}$, one only needs to show that $(\mathcal{T}_{\phi\psi} - \tilde{\mathcal{D}}^{(r,s)}_{\phi\psi})(\mathbf{1})$ is small in BMO when the mesh size is small.

Proof of Theorem G. As before, let $\sigma(x) = \sum_\ell \psi(x - \ell s)$ and $F_N^+(a) = \sum_{k=0}^N \langle \sigma(r^k x), a\rangle$ where again we use a to denote a $(2,1)$-atom. Now σ is s-periodic and Hölder continuous of order $\delta > 0$. For each odd integer n, let $\sigma^{(n)}(x) = (1/\sqrt{s}\,)\sum_{k=0}^\infty \hat{\sigma}(n2^k)e^{2\pi i 2^k x/s}$, where $\hat{\sigma}(m) = (1/\sqrt{s}\,)\int_0^s \sigma(x)e^{2\pi i m x/s}$. One takes advantage of the grid structure induced by the tiles $\Delta_{\ell k}$ with mesh parameters $(r,s) = (2^{1/K}, 2^{-L})$ to express the action $F_N^+(a)$ on a $(2,1)$-atom as

$$\begin{aligned}
F_N^+(a) &= \sum_{k=0}^N \int_{-\infty}^{\infty} \sigma\big(2^{k/K}x\big)\bar{a}(x)\,\mathrm{d}x = \sum_{k=0}^N \sum_{m=0}^{K-1} \int_{-\infty}^{\infty} \sigma\big(2^k x\big)\bar{a}_{2^{m/K}}(x)\,\mathrm{d}x \\
&= \sum_{k=0}^N \sum_{m=0}^{K-1} \int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \big(\sigma\big(2^k\cdot\big)\big)^{\wedge}(\ell)\frac{1}{\sqrt{s}}e^{2\pi i \ell x/s}\bar{a}_{2^{m/K}}(x)\,\mathrm{d}x \\
&= \sum_{k=0}^N \sum_{m=0}^{K-1} \int_{-\infty}^{\infty} \sum_{n\ \mathrm{odd}} \sum_{p=k}^{\infty} \hat{\sigma}\big(n2^{p-k}\big)\frac{1}{\sqrt{s}}e^{2\pi i n 2^p x/s}\bar{a}_{2^{m/K}}(x)\,\mathrm{d}x
\end{aligned}$$

$$=\frac{1}{\sqrt{s}}\lim_{N\to\infty}\sum_{k=0}^{N}\sum_{m=0}^{K-1}\sum_{n\ \text{odd}}\sum_{p=k}^{\infty}\left(\sigma^{(n)}\right)^{\wedge}\left(2^{p-k}\right)\overline{\left(a_{2^{m/K}n/s}\right)^{\wedge}\left(2^{p}\right)}.$$

Following the pattern of the proof of Theorem E, we have

$$\begin{aligned}\left|F_N^+(a)\right| &\leqslant \frac{1}{\sqrt{s}}\sum_{n\ \text{odd}}\sum_{m=0}^{K-1}\left|\sum_{k=0}^{N}\sum_{p=k}^{\infty}\left(\sigma^{(n)}\right)^{\wedge}\left(2^{p-k}\right)\overline{\left(a_{2^{m/K}n/s}\right)^{\wedge}\left(2^{p}\right)}\right| \\ &\leqslant \frac{1}{\sqrt{s}}\sum_{n\ \text{odd}}\sum_{m=0}^{K-1}\left(\sum_{p=0}^{\infty}\left|\sum_{k=0}^{p}\hat{\sigma}\left(n2^k\right)\right|^2\right)^{1/2}\|a_{2^{m/K}n/s}\|_{H^1} \\ &= \frac{K}{\sqrt{s}}\sum_{n\ \text{odd}}\left(\sum_{p=0}^{\infty}\left|\sum_{k=0}^{p}\hat{\sigma}\left(n2^k\right)\right|^2\right)^{1/2} = \frac{K}{s}\sum_{n\ \text{odd}}\left(\sum_{p=0}^{\infty}\left|\sum_{k=0}^{p}\hat{\psi}\left(n2^{k+L}\right)\right|^2\right)^{1/2}\end{aligned}$$

when $s=2^{-L}$ and a is a $(2,1)$-atom. Using the Hölder continuity of σ we conclude as in Theorem E that $\|F_N^+\|_{\mathrm{BMO}}$ is bounded and that F_N^+ converges weakly to $F^+=\sum_{k=0}^{\infty}\sigma(r^k x)$ in BMO.

In view of Proposition 14, in order to show that the operator norm of $\mathcal{T}_{\phi\psi}-\tilde{\mathcal{D}}_{\phi\psi}^{(r,s)}$ can be made arbitrarily small by allowing $(r,s)\to(1,0)$, it is enough to show that $(\mathcal{T}_{\phi\psi}-\tilde{\mathcal{D}}_{\phi\psi}^{(r,s)})(\mathbf{1})$ can thus be made small in BMO (as before, $(\mathcal{T}_{\phi\psi}-\tilde{\mathcal{D}}_{\phi\psi}^{(r,s)})^*(\mathbf{1})=0$). Actually, $\mathcal{T}_{\phi\psi}(\mathbf{1})=0$. To see this, observe that $\phi_t^* * \psi_t=(\phi^* * \psi)_t$ and the latter has integral zero. Consequently, with weak convergence as before,

$$\mathcal{T}_{\phi\psi}(\mathbf{1})=\lim_{\varepsilon\to 0,\ N\to\infty}\int_{\varepsilon}^{N}\mathbf{1}*(\phi^* * \psi)_t\,\frac{\mathrm{d}t}{t}=\lim_{\varepsilon\to 0,\ N\to\infty}\int_{\varepsilon}^{N}\int_{-\infty}^{\infty}(\phi^* * \psi)_t(x)\,\mathrm{d}x\,\frac{\mathrm{d}t}{t}=0$$

since the inside integral is zero for each $t>0$. Therefore, we just need to show that $\mathcal{D}_{\phi\psi}^{(r,s)}(\mathbf{1})$ can be made small. But by the estimate for $\|F^+\|_{\mathrm{BMO}}$ above, and the fact that $\|\mathcal{D}_{\phi\psi}^{(r,s)}(\mathbf{1})\|_{\mathrm{BMO}}=s(1-1/r)\times\|F^+(\mathbf{1})\|_{\mathrm{BMO}}$,

$$\left\|\mathcal{D}_{\phi\psi}^{(r_K,s_L)}\mathbf{1}\right\|_{\mathrm{BMO}}\leqslant \text{const}\sum_{n\ \text{odd}}\left(\sum_{p=0}^{\infty}\left|\sum_{k=0}^{p}\hat{\psi}\left(n2^{k+L}\right)\right|^2\right)^{1/2}$$

which tends to zero as L increases. This proves Theorem G. □

7.1. *Examples of invertibility*

To construct ψ satisfying the cancellation condition of Theorem G, as before, we construct ψ via its Fourier transform. Fix an integer $q\geqslant 2$ and $\varepsilon,\beta>0$. We wish to regard L in the theorem as $L=tq$, where $q\geqslant 2$ is a fixed integer and t is a large integer to be chosen. Each positive integer p may be written $p=kq+m$ with $k\in\mathbb{Z}$ and $0\leqslant m\leqslant q-1$. Hence we may write each nonzero integer ℓ as $\ell=n2^{kq+m}$ with n an odd integer, $k\in\mathbb{Z}$, and $0\leqslant m\leqslant q-1$. Define a sequence c_ℓ $(\ell\in\mathbb{Z})$ by

$$c_{n2^{kq+m}}=\begin{cases}0 & \text{if } n=0,\\ \frac{1}{|n|^{1+\varepsilon}2^{\beta kq}} & \text{if } m=0,\ n\neq 0,\ n\ \text{odd},\\ -\frac{2^{\beta(q-1)}(2^{\beta}-1)}{(2^{\beta(q-1)}-1)}\frac{1}{|n|^{1+\varepsilon}2^{\beta(kq+m)}} & \text{if } m,n\neq 0,\ n\ \text{odd}.\end{cases}$$

Then an easy calculation gives us that $\sum_{\ell=0}^{q-1} c_{n2^{kq+\ell}} = 0$, so if $p = kq + m$ ($\ell \in \mathbb{Z}$, $0 \leqslant m \leqslant q-1$) then

$$\sum_{\ell=0}^{p} c_{n2^{tq+\ell}} = c_{n2^{tq}} + \cdots + c_{n2^{(t+k)q}} + \cdots + c_{n2^{(t+k)q+m}} = c_{n2^{(t+k)q}} + \cdots + c_{n2^{(t+k)q+m}}$$
$$= \left(\frac{2^{\beta(q-1)} - 2^{\beta m}}{2^{\beta(q-1)} - 1}\right) \frac{1}{|n|^{1+\varepsilon} 2^{\beta p}} \leqslant \frac{1}{|n|^{1+\varepsilon} 2^{\beta p}}.$$

Put $\sigma(x) = \sum_{n \text{ odd}} \sum_{k \in \mathbb{Z}} c_{n2^k} e^{2\pi i n 2^k x}$. Then σ is Hölder continuous of order δ for $0 < \delta < \min(\varepsilon, \beta)$. Finally $\psi(x) = \sigma(x) B_m(x)$, where, once again, B_m is an mth order B-spline, has the desired properties. In particular, $\psi \in \mathcal{M}_\delta^{(0)}$. Choosing any $\phi \in \mathcal{M}_\delta$ such that $\int_0^\infty \hat{\psi}(t\xi)\overline{\hat{\phi}(t\xi)}\, \mathrm{d}t/t \equiv 1$ has thus fulfilled all the criteria needed for invertibility of $\mathcal{D}_{\phi\psi}$.

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