

ENESTROM- KAKEYA THEOREM AND ITS GENERALIZATIONS

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ABSTRACT

There are some generalizations and extensions available in the literature for Enestrom-Kekeya Theorem. In this paper we give some new generalizations.

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1. INTRODUCTION AND STATEMENT OF RESULTS

In the zero distribution theory of polynomials, the Enestrom-Kekeya theorem [4] given below in theorem A is a well known result.

Theorem A: Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0$$

Then all the zeros of $P(z)$ lie in the disk $|z| \leq 1$.

Many attempts have been made to extend and generalize the Enestrom-Kekeya theorem. A. Joyal *et al* [3] extended the Enestrom-Kekeya theorem to the polynomials with general monotonic coefficients by proving that if

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0$$

Then $P(z)$ has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$$

Further Aziz and Zargar [1] generalized the result of A. Joyal *et al* [3] and the Enestrom-Kekeya theorem as given below in theorem B.

Theorem B: Suppose $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that For some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0$$

then all the zeros of $P(z)$ lie in the disk

$$|z + (\lambda - 1)| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}$$

But N.K. Govil and Rahman [2] worked over the distribution of zeros of polynomials such that the moduli of coefficients are monotonic and proved a result given below in theorem C.

Theorem C: Suppose $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that for some $a > 0$

$$|a_n| \geq a |a_{n-1}| \geq a^2 |a_{n-2}| \geq \dots \geq a^{n-1} |a_1| \geq a^n |a_0|$$

Then $P(z)$ has all its zeros in the disk $|z| \leq \frac{M_1}{a}$, where M_1 is the greatest positive root of the equation

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$$M^{n+1} - 2 M^n + 1 = 0.$$

Govil and Rahman [2] also proved that if $P(z) = \sum_{i=0}^n a_i z^i$, is a complex polynomial of degree n , then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{i=0}^{n-1} |a_i|$$

where $|\arg a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, ($i=0,1,2,\dots,n$) for some β real, and $|a_n| \geq |a_{n-1}| \geq \dots \geq |a_1| \geq |a_0|$.

But recently Shah and Liman [5] generalized theorem B and the result of Govil and Rahman [2] and thereby proved the following two theorems D and E.

Theorem D: Suppose $P(z) = \sum_{i=0}^n a_i z^i$ be a complex polynomial of degree n with $\operatorname{Re}(a_i) = \alpha_i$ and $\operatorname{Im}(a_i) = \beta_i$, $i=0,1,2,\dots,n$. If for some $\lambda \geq 1$

$$\lambda \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0, \quad \beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \beta_0 > 0$$

Then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{(\lambda-1)a_n}{a_n} \right| \leq \frac{\lambda \alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|}$$

Theorem E: Suppose $P(z) = \sum_{i=0}^n a_i z^i$ be a complex polynomial of degree n such that $|\arg a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, ($i=0,1,2,\dots,n$) for some β real and for some $\lambda \geq 1$

$$\lambda |a_n| \geq |a_{n-1}| \geq \dots \geq |a_1| \geq |a_0|$$

Then $P(z)$ has all its zeros in the disk

$$|z + (\lambda - 1)| \leq \frac{1}{|a_n|} \{ (\lambda |a_n| - |a_0|) (\cos \alpha + \sin \alpha) + |a_0| + 2 \sin \alpha \sum_{i=0}^{n-1} |a_i| \}$$

Now in this paper, we prove three generalizations of Enestrom-Kakeya Theorem.

2. THEOREMS AND PROOFS

Theorem 1: Suppose $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that for some $\lambda \neq 1$, $1 \leq k \leq n$ and $a_{n-k} \neq 0$

$$a_n \geq a_{n-1} \geq \dots \geq a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1} \geq \dots \geq a_1 \geq a_0$$

Now let $a_{n-k-1} > a_{n-k}$, then $P(z)$ has all its zeros in the disk $|z| \leq M_1$, where M_1 is the greatest positive root of the equation

$$M^{k+1} - \theta_1 M^k - |\rho_1| = 0, \text{ where } \rho_1 = \frac{(\lambda-1)a_{n-k}}{a_n} \text{ and } \theta_1 = \frac{|a_0| - a_0 + a_n + (\lambda-1)a_{n-k}}{|a_n|}$$

Again let $a_{n-k} > a_{n-k+1}$, then $P(z)$ has all its zeros in the disk $|z| \leq M_2$ where M_2 is the greatest positive root of the equation

$$M^k - \theta_2 M^{k-1} - |\rho_2| = 0, \text{ where } \rho_2 = \frac{(1-\lambda)a_{n-k}}{a_n} \text{ and } \theta_2 = \frac{|a_0| - a_0 + a_n + (1-\lambda)a_{n-k}}{|a_n|}$$

Proof: To prove the result, we consider a polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) = (1-z) \sum_{i=0}^n a_i z^i \\ &= (1-z)(a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1} + a_n z^n) \\ &= a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1} + a_n z^n - a_0 z - a_1 z^2 - a_2 z^3 \dots - a_{n-1} z^n - a_n z^{n+1} \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \end{aligned}$$

If $a_{n-k-1} > a_{n-k}$, then $a_{n-k+1} > a_{n-k}$

Therefore, we can also write F (z) as

$$F(z) = -a_n z^{n+1} - (\lambda - 1)a_{n-k} z^{n-k} + (a_n - a_{n-1})z^n + \dots + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (\lambda a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots + (a_1 - a_0)z + a_0$$

Let $|z| > 1$, then we have

$$\begin{aligned} |F(z)| &\geq |a_n z^{n+1} + (\lambda - 1)a_{n-k} z^{n-k}| - |z|^n \left\{ (a_n - a_{n-1}) + \dots + \frac{(a_{n-k+1} - a_{n-k})}{|z|^{k-1}} \right. \\ &\quad \left. + \frac{(\lambda a_{n-k} - a_{n-k-1})}{|z|^k} + \frac{(a_{n-k-1} - a_{n-k-2})}{|z|^{k+1}} + \dots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right\} \\ &\geq |z|^{n-k} |a_n z^{k+1} + (\lambda - 1)a_{n-k}| - |z|^n \{ a_n + (\lambda - 1)a_{n-k} - a_0 + |a_0| \} \\ &> 0, \end{aligned}$$

If $|z|^{k+1} + \rho_1 > \theta_1 |z|^k$

where $\rho_1 = \frac{(\lambda-1)a_{n-k}}{a_n}$ and $\theta_1 = \frac{|a_0|-a_0+a_n+(\lambda-1)a_{n-k}}{|a_n|}$

The above inequality will hold if

$|z|^{k+1} - |\rho_1| > \theta_1 |z|^k$. Thus all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_1$, where M_1 is the greatest positive root of the equation

$$M^{k+1} - \theta_1 M^k - |\rho_1| = 0$$

But the zeros of P(z) with modulus less than or equal to one are already present in the disk $|z| \leq M_1$ since $M_1 > 1$

Now we take second part of the theorem.

Let $a_{n-k} > a_{n-k+1}$, then $a_{n-k} > a_{n-k-1}$ and F(z) can be written as follows;

$$F(z) = -a_n z^{n+1} - (1 - \lambda)a_{n-k} z^{n-k+1} + (a_n - a_{n-1})z^n + \dots + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots + (a_1 - a_0)z + a_0$$

Let $|z| > 1$, then

$$\begin{aligned} |F(z)| &\geq |z|^{n-k+1} |a_n z^k + (1 - \lambda)a_{n-k}| - |z|^n \left\{ (a_n - a_{n-1}) + \dots + \frac{(a_{n-k+1} - \lambda a_{n-k})}{|z|^{k-1}} \right. \\ &\quad \left. + \frac{(a_{n-k} - a_{n-k-1})}{|z|^k} + \frac{(a_{n-k-1} - a_{n-k-2})}{|z|^{k+1}} + \dots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right\} \\ &\geq |z|^{n-k+1} |a_n z^k + (1 - \lambda)a_{n-k}| - |z|^n \{ a_n + (1 - \lambda)a_{n-k} - a_0 + |a_0| \} \\ &> 0 \end{aligned}$$

if $|z|^k + \rho_2 > \theta_2 |z|^{k-1}$

where $\rho_2 = \frac{(1-\lambda)a_{n-k}}{a_n}$ and $\theta_2 = \frac{|a_0|-a_0+a_n+(1-\lambda)a_{n-k}}{|a_n|}$

The above inequality will hold if $|z|^k - |\rho_2| > \theta_2 |z|^{k-1}$

Thus all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_2$,

where M_2 is the greatest positive root of the equation

$$M^k - \theta_2 M^{k-1} - |\rho_2| = 0$$

But the zeros of P(z) with modulus less than or equal to one are already present in the disk $|z| \leq M_2$ since $M_2 > 1$.

Therefore, the result follows.

Theorem 2: Suppose $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that for some $a > 0$, $\lambda (\neq 1) > 0$, $1 \leq k \leq n$ and $a_{n-k} \neq 0$

$$|a_n| \geq a |a_{n-1}| \geq a^2 |a_{n-2}| \geq \dots \geq a^{k-1} |a_{n-k+1}| \geq \lambda a^k$$

$$|a_{n-k}| \geq a^{k+1} |a_{n-k-1}| \geq \dots \geq a^n |a_0|$$

If $|a_{n-k}| < a |a_{n-k-1}|$ (i.e., $\lambda > 1$)

Then $P(z)$ has all its zeros in the disk $|z| \leq \frac{M_1}{a}$,

where M_1 is the greatest positive root of the equation

$$M^{n+1} - 2M^n + 1 = 0$$

If $a |a_{n-k}| > |a_{n-k+1}|$ (i.e., $0 < \lambda < 1$)

Then $P(z)$ has all its zeros in the disk $|z| \leq \frac{M_2}{a}$, where M_2 is the greatest positive root of the equation

$$M^{n+1} - 2M^n + \left(\frac{\lambda-1}{\lambda}\right)M^{n-k+1} + \frac{1}{\lambda} = 0$$

Proof: Let $|a_{n-k}| < a |a_{n-k-1}|$ then $a^{k-1} |a_{n-k+1}| \geq a^k |a_{n-k}|$ and thereby we obtain the same result as theorem B following the proof of theorem1 in [2]. Now let us take $a |a_{n-k}| > |a_{n-k+1}|$ and suppose $F(z) = \sum_{i=0}^{n-1} a_i z^i$ be a polynomial of degree $(n-1)$.

Then for $|z| = R (> \frac{1}{a})$, we have

$$\begin{aligned} |F(z)| &\leq |a_{n-1}| R^{n-1} \left\{ 1 + \frac{1}{aR} + \frac{1}{(aR)^2} + \dots + \frac{1}{(aR)^{k-2}} + \frac{1}{\lambda(aR)^{k-1}} + \frac{1}{(aR)^k} + \dots + \frac{1}{(aR)^{n-1}} \right\} \\ &\leq |a_{n-1}| R^{n-1} \left\{ 1 + \frac{1}{aR} + \frac{1}{(aR)^2} + \dots + \frac{1}{(aR)^{k-2}} + \frac{1}{\lambda(aR)^{k-1}} + \frac{1}{\lambda(aR)^k} + \dots + \frac{1}{\lambda(aR)^{n-1}} \right\} \\ &= |a_{n-1}| R^{n-1} \left\{ \frac{(aR)^{k-1} - 1}{(aR)^{k-2} (aR - 1)} + \frac{(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1} (aR - 1)} \right\} \\ &= |a_{n-1}| R^{n-1} \left\{ \frac{\lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1} (aR - 1)} \right\} \end{aligned}$$

Since $P(z) = \sum_{i=0}^n a_i z^i$, therefore, we have

$$|P(z)| \geq |a_n| R^n - |a_{n-1}| R^{n-1} \left\{ \frac{\lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1} (aR - 1)} \right\} > 0$$

$$\text{If } \frac{|a_n|}{a |a_{n-1}|} > \frac{\lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1} (aR - 1)}$$

Because by hypothesis $\frac{|a_n|}{a |a_{n-1}|} > 1$, the above inequality holds

$$\text{If } \lambda(aR)^{n-1} (aR - 1) > \lambda(aR)^n + (1 - \lambda)(aR)^{n-k+1} - 1$$

Replacing aR by M , we obtain the desired result.

Theorem 3: Let $P(z) = \sum_{i=0}^n a_i z^i$ be a complex polynomial of degree n with $\text{Re}(a_i) = \alpha_i$ and $\text{Im}(a_i) = \beta_i$, $i=0,1,2,\dots,n$. Let for some $\lambda \neq 1$, $1 \leq k \leq n$ and $a_{n-k} \neq 0$,

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_{n-k+1} \geq \lambda \alpha_{n-k} \geq \alpha_{n-k-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \beta_0 .$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then $P(z)$ has all its zeros in the disk $|z| \leq M_1$ where M_1 is the greatest positive root of the equation $M^{K+1} - \theta_1 M^K - |\rho_1| = 0$, where

$$\rho_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n} \quad \text{and} \quad \theta_1 = \frac{|\alpha_0| - \alpha_0 + \alpha_n + (\lambda-1)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|}{|a_n|}$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then $P(z)$ has all its zeros in the disk $|z| \leq M_2$,

where M_2 is the greatest positive root of the equation

$$M^K - \theta_2 M^{K-1} - |\rho_2| = 0$$

where $\rho_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ and $\theta_2 = \frac{|\alpha_0| - \alpha_0 + \alpha_n + (1-\lambda)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|}{|a_n|}$

Proof: To prove the result, we consider a polynomial $F(z)$ defined by

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \dots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\quad + i \{ (\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \dots + (\beta_1 - \beta_0)z + \beta_0 \} \end{aligned}$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then $\alpha_{n-k+1} > \alpha_{n-k}$ and $F(z)$ can be written as

$$\begin{aligned} F(z) &= -a_n z^{n+1} - (\lambda - 1)\alpha_{n-k} z^{n-k} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{n-k+1} - \alpha_{n-k}) z^{n-k+1} + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} \\ &\quad + \dots + (\alpha_1 - \alpha_0)z + \alpha_0 + i \{ (\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \dots + (\beta_1 - \beta_0)z + \beta_0 \} \end{aligned}$$

Now if $|z| > 1$, then

$$\begin{aligned} |F(z)| &\geq |a_n z^{n+1} + (\lambda - 1)\alpha_{n-k} z^{n-k}| - |z|^n \{ (\alpha_n - \alpha_{n-1}) + \dots + \frac{(\alpha_{n-k+1} - \alpha_{n-k})}{|z|^{k-1}} \\ &\quad + \frac{(\lambda\alpha_{n-k} - \alpha_{n-k-1})}{|z|^k} + \dots + \frac{(\alpha_1 - \alpha_0)}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} \} - |z|^n \{ (\beta_n - \beta_{n-1}) + \dots + \frac{(\beta_1 - \beta_0)}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \} \\ &\geq |a_n z^{n+1} + (\lambda - 1)\alpha_{n-k} z^{n-k}| - |z|^n \{ |\alpha_0| - \alpha_0 + \alpha_n + (\lambda - 1)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0| \} \\ &> 0, \end{aligned}$$

If $|z^{k+1} + \rho_1| > \theta_1 |z|^k$ where $\rho_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n}$ and $\theta_1 = \frac{|\alpha_0| - \alpha_0 + \alpha_n + (\lambda-1)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|}{|a_n|}$

Now the above inequality holds if

$$|z|^{k+1} - |\rho_1| > \theta_1 |z|^k$$

Therefore, all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq M_1$

where M_1 is the greatest positive root of the equation

$$M^{K+1} - \theta_1 M^K - |\rho_1| = 0$$

As in theorem 1, $M_1 > 1$. Therefore, all the zeros of $P(z)$ with modulus less than or equal to one are already lying in the disk $|z| \leq M_1$ and hence the proof of the first part follows.

To prove the second part of the theorem, we take $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$

And therefore, $F(z)$ can be written as

$$\begin{aligned} F(z) &= -a_n z^{n+1} - (1 - \lambda)\alpha_{n-k} z^{n-k+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + \\ &\quad + (\alpha_{n-k+1} - \lambda\alpha_{n-k}) z^{n-k+1} + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + \dots + \\ &\quad + (\alpha_1 - \alpha_0)z + \alpha_0 + i \{ (\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \dots + (\beta_1 - \beta_0)z + \beta_0 \} \end{aligned}$$

Now if $|z| > 1$, then

$$|F(z)| \geq |a_n z^{n+1} + (1 - \lambda)\alpha_{n-k} z^{n-k+1}| - |z|^n \{ (\alpha_n - \alpha_{n-1}) + \dots + \frac{(\alpha_{n-k+1} - \lambda \alpha_{n-k})}{|z|^{k-1}} \\ + \frac{(\alpha_{n-k} - \alpha_{n-k-1})}{|z|^k} + \dots + \frac{(\alpha_1 - \alpha_0)}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} \} - |z|^n \{ (\beta_n - \beta_{n-1}) + \dots + \frac{(\beta_1 - \beta_0)}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \} \\ \geq |a_n z^{n+1} + (1 - \lambda)\alpha_{n-k} z^{n-k+1}| - |z|^n \{ |\alpha_0| - \alpha_0 + \alpha_n + (1 - \lambda)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0| \} \\ > 0,$$

If $|z^k + \rho_2| > \theta_2 |z|^{k-1}$ where $\rho_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ and $\theta_2 = \frac{|\alpha_0| - \alpha_0 + \alpha_n + (1-\lambda)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|}{|a_n|}$

The above inequality holds if $|z|^k - |\rho_2| > \theta_2 |z|^{k-1}$

Therefore, all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_2$,

where M_2 is the greatest positive root of the equation

$$M^k - \theta_2 M^{k-1} - |\rho_2| = 0$$

But the zeros of P(z) with modulus less than or equal to one are already present in the disk $|z| \leq M_2$ since $M_2 > 1$.

Hence the result follows.

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