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APPROXIMATION OF ADJOINT OF A MULTIPLIER ON BANACH ALGEBRAS*

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Abstract For a Banach algebra $\mathcal{A}$, we denote by $\mathcal{A}^*$ and $\mathcal{A}^{**}$ the first and the second duals of $\mathcal{A}$ respectively. Let T be a mapping from $\mathcal{A}^*$ to itself. In this article, we will investigate some stability results concerning the equations

$$T(\alpha f+\beta g)=\alpha T(f)+\beta T(g),\quad T(af)=aT(f)$$

and

$$T(\alpha f+\beta g)+T(\alpha f-\beta g)=2\alpha^2T(f)+2\beta^2T(g)$$

where $f,g\in\mathcal{A}^*$, $a\in\mathcal{A}$, and $\alpha,\beta\in\mathbb{Q}\setminus\{0\}$.

Key words Additive and quadratic mappings; Arens products; Banach algebra; multiplier; stability

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1 Introduction and Preliminaries

Let $\mathcal{A}$ be a Banach algebra. On the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$, there exist two natural multiplications extending that of $\mathcal{A}$, known as the first and the second Arens products. In this article, $\mathcal{A}^{**}$ will always be equipped with the first Arens product, which is defined as follows. For a, b in $\mathcal{A}$, f in $\mathcal{A}^*$, and F, G in $\mathcal{A}^{**}$, the elements fa and Gf of $\mathcal{A}^*$ and FG of $\mathcal{A}^{**}$ are defined, respectively, as follows:

$$fa(b)=f(ab),\quad Gf(a)=G(fa),\quad \text{and}\quad FG(f)=F(Gf).$$

Equipped with this multiplication, $\mathcal{A}^{**}$ is a Banach algebra and $\mathcal{A}$ is a subalgebra of it [1, 2]. Obviously, $af(b)=f(ab)$ for all $f\in\mathcal{A}^*$ and $a,b\in\mathcal{A}$. A linear operator $T:\mathcal{A}\to\mathcal{A}$ is said to be a multiplier on $\mathcal{A}$ if, for each $a,b\in\mathcal{A}$, we have $T(ab)=T(a)b$ [3]. The space of multipliers on $\mathcal{A}$ is denoted by $M(\mathcal{A})$. Each $a\in\mathcal{A}$ defines a multiplier $L_a:\mathcal{A}\to\mathcal{A}$ on $\mathcal{A}$ by $L_a(b)=ab$. Then, for $f\in\mathcal{A}^*$ and $a,b\in\mathcal{A}$, we have $L_a^*(f)(b)=f(L_a(b))=f(ab)=fa(b)$, where L_a^* is the first adjoint L_a. This shows that $L_a^*(f)=fa$. It is seen that $L_a^*(bf)=bL_a^*(f)$.

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Let $\mathcal{A}$ be a Banach algebra and $T:\mathcal{A}^*\to\mathcal{A}^*$ be a bounded linear operator. The first author found out when the equality $T(af)=aT(f)$ $(a\in\mathcal{A},\ f\in\mathcal{A}^*)$ implies that $T(Ff)=FT(f)$ $(f\in\mathcal{A}^*,\ F\in\mathcal{A}^{**})$? [4] Let G be a locally compact abelian group and $L^1(G)$ the convolution algebra of integrable functions on G. He proved that G is compact if and only if $T(af)=aT(f)$ $(a\in L^1(G),\ f\in L^1(G)^*)$ implies that $T(Ff)=FT(f)$ for every $f\in L^1(G)^*$ and $F\in L^1(G)^{**}$ (see Theorem 3.4 in [4]).

The problem of stability of functional equations was originated from a question of Ulam [5] concerning the stability of group homomorphisms: let $(G_1,*)$ be a group and $(G_2,\star,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that, if a mapping $h:G_1\longrightarrow G_2$ satisfies the inequality $d(h(x*y),h(x)\star h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y)=H(x)\star H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? Hyers [6] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f:X\longrightarrow Y$ satisfies $\|f(x+y)-f(x)-f(y)\|\le\epsilon$ for all $x,y\in X$ and some $\epsilon>0$. Then, there exists a unique additive mapping $T:X\longrightarrow Y$, such that $\|f(x)-T(x)\|\le\epsilon$ for all $x\in X$. Hyers' theorem was generalized by Aoki [7] for additive mappings. In 1978, a generalized solution for approximately linear mappings was given by Rassias [8]. He considered a mapping $f:X\to Y$ satisfying the condition

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon(\|x\|^p+\|y\|^p)$$

for all $x,y\in X$, where $\epsilon\ge0$ and $0\le p<1$. This result was later extended to all $p\neq1$ and generalized by Gajda [9], Isac and Rassias [10]. Lee and Jun [11] improved the stability problem for approximately additive mappings. The problem when $p=1$ is not true. Counterexamples for the corresponding assertion in the case $p=1$ were constructed by Gadja [9] and Rassias and Šemrl [12]. Furthermore, a generalization of Rassias theorem was obtained by Găvruta [13], who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is related to a symmetric bi-additive function [14, 15]. It is natural that this equation is called a quadratic functional equation. A number of results concerning the stability of different functional equations can be found in [16–20] and [21]. A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof for functions $f:X\longrightarrow Y$, where X is a normed space and Y is a Banach space [22]. More generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings can be found in [23–25] and [26].

Suppose that $\mathcal{A}$ is a Banach algebra without order. The stability of multipliers on $\mathcal{A}$ was studied by Miura, Hirasawa, and Takahasi [27]. Miura, Hirasawa, and Takahasi in [27] showed that each approximate multiplier $T:\mathcal{A}\to\mathcal{A}$ is an exact multiplier.

In this article, among the other things, we investigate the generalized Hyers-Ulam stability of the equations

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g), \quad T(af) = aT(f) \tag{1.1}$$

and

$$T(\alpha f + \beta g) + T(\alpha f - \beta g) = 2\alpha^2 T(f) + 2\beta^2 T(g) \tag{1.2}$$

on dual of a Banach algebra, where $\alpha, \beta \in \mathbb{Q} \setminus \{0\}$.

2 Results in Banach Algebras

We will investigate the generalized Hyers-Ulam type theorem of equations (1.1) and (1.2). In the following theorem, we will show that under special circumstances on the control functions ψ_1 and ψ_2, every ψ_2-almost additive mapping T can be approximated by an additive mapping A.

Theorem 2.1 Let α and β be nonzero rational numbers such that $|\alpha + \beta| > 1$. Let $j \in \{-1, 1\}$ be fixed, and p be a real number such that $pj < j$. Let $\psi_1 : \mathcal{A} \times \mathcal{A}^* \to \mathbb{R}^+$ and $\psi_2 : \mathcal{A}^* \times \mathcal{A}^* \to \mathbb{R}^+$ be functions such that

$$\psi_1(a, \gamma^{nj} f) = |\gamma|^{npj} \psi_1(a, f), \quad \psi_2(\gamma^{nj} f, \gamma^{nj} g) = |\gamma|^{npj} \psi_2(f, g) \tag{2.1}$$

for all $n \in \mathbb{N}$, $a \in \mathcal{A}$ and $f, g \in \mathcal{A}^*$, where $\gamma = \alpha + \beta$. Suppose that $T : A^* \to A^*$ satisfies the conditions

$$\|T(af) - aT(f)\| \le \psi_1(a, f) \tag{2.2}$$

and

$$\|T(\alpha f + \beta g) - \alpha T(f) - \beta T(g)\| \le \psi_2(f, g) \tag{2.3}$$

for all $f, g \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Then, there exists a unique additive mapping $A : \mathcal{A}^* \to \mathcal{A}^*$, such that $A(af) = aA(f)$, and

$$\|T(f) - A(f)\| \le \frac{j}{|\gamma| - |\gamma|^p} \psi_2(f, f) \tag{2.4}$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$.

Proof Case (1): $j = 1$. Put $g = f$ in (2.3) and divide the result by $|\gamma|$:

$$\left\| \frac{T(\gamma f)}{\gamma} - T(f) \right\| \le \frac{1}{|\gamma|} \psi_2(f, f). \tag{2.5}$$

Make the induction assumption:

$$\left\| \frac{T(\gamma^n f)}{\gamma^n} - T(f) \right\| \le \sum_{k=1}^{n} \frac{|\gamma|^{(k-1)p}}{|\gamma|^k} \psi_2(f, f) \tag{2.6}$$

is true for $n = 1$ by (2.5). In (2.6), replace f by γf, divide the result by $|\gamma|$ and use property (2.1) of ψ_2 to obtain:

$$\left\| \frac{T(\gamma^{n+1} f)}{\gamma^{n+1}} - \frac{T(\gamma f)}{\gamma} \right\| \le \sum_{k=1}^{n} \frac{|\gamma|^{kp}}{|\gamma|^{k+1}} \psi_2(f, f). \tag{2.7}$$

Combine the last inequality with (2.5) to find that (2.6) holds with n replaced by $n+1$, so the induction proof of (2.6) is established for all f in $\mathcal{A}^*$ and all positive integers n. To prove convergence of the sequence $\{\frac{T(\gamma^n f)}{\gamma^n}\}$, we divide inequality (2.6) by $|\gamma|^m$ and also replace f by $\gamma^m f$ to find that

$$\begin{aligned}\left\|\frac{T(\gamma^{n+m}f)}{\gamma^{n+m}}-\frac{T(\gamma^m f)}{\gamma^m}\right\| &\le \sum_{k=1}^{n}\frac{|\gamma|^{(k-1)p}}{|\gamma|^{k+m}}\psi_2(\gamma^m f,\gamma^m f)\\ &= |\gamma|^{m(p-1)}\sum_{k=1}^{n}|\gamma|^{(k-1)p-k}\psi_2(f,f)\\ &\le \frac{|\gamma|^{m(p-1)}}{|\gamma|^p}\left(\frac{|\gamma|^{(p-1)}-|\gamma|^{(p-1)n+1}}{1-|\gamma|^{p-1}}\right)\psi_2(f,f)\end{aligned}$$

for $m>0$. Because $|\gamma|>1$ and $p-1<0$, the right-hand side of the above sequence of inequalities tends to zero as m tends to infinity. Therefore, $\{\frac{T(\gamma^n f)}{\gamma^n}\}$ is a Cauchy sequence. But $\mathcal{A}^*$, as a Banach space, is complete, and thus the sequence converges. Define $A(f)=\lim\limits_{n\to\infty}\frac{T(\gamma^n f)}{\gamma^n}$ for all $f\in\mathcal{A}^*$. Inequality (2.6) implies

$$\|A(f)-T(f)\|\le\frac{1}{|\gamma|-|\gamma|^p}\psi_2(f,f).$$

To show that A is additive, replacing f by $\gamma^n f$ and g by $\gamma^n g$ in (2.3) and divide by γ^n, we have

$$\left\|\frac{T(\gamma^n\alpha f+\gamma^n\beta g)}{\gamma^n}-\alpha\frac{T(\gamma^n f)}{\gamma^n}-\beta\frac{T(\gamma^n g)}{\gamma^n}\right\|\le\frac{|\gamma|^{np}}{|\gamma|^n}\psi_2(f,g)=|\gamma|^{n(p-1)}\psi_2(f,g).$$

Taking the limit as $n\to\infty$, we find that

$$A(\alpha f+\beta g)-\alpha A(f)-\beta A(g)=0, \tag{2.8}$$

because the sequence $|\gamma|^{n(p-1)}$ converges to zero when n tends to infinity. Now, we prove that A is additive. In (2.8), set $f=g=0$ to see that $A(0)=(\alpha+\beta)A(0)$, and so $A(0)=0$. Putting 0 in place of g in inequality (2.8), we obtain $A(\alpha f)=\alpha A(f)$ for all $f\in\mathcal{A}^*$. Similarly, $A(\beta g)=\beta A(g)$ for all $g\in\mathcal{A}^*$. For every $f,g\in\mathcal{A}^*$, we have

$$\begin{aligned}\alpha\beta A(f+g)&=A(\alpha\beta(f+g))=A(\alpha\beta f+\alpha\beta g)\\ &=\alpha A(\beta f)+\alpha A(\beta g)=\alpha\beta(A(f)+A(g)).\end{aligned}$$

This shows that A is additive.

Now, let $f\in\mathcal{A}^*$ and $a\in\mathcal{A}$. To show that $A(af)=aA(f)$, replacing f by $\gamma^n f$ in (2.2) and divide by $|\gamma|^n$, we have

$$\left\|\frac{T(\gamma^n af)}{\gamma^n}-a\frac{T(\gamma^n f)}{\gamma^n}\right\|\le\frac{1}{|\gamma|^n}\psi_1(a,\gamma^n f)=\frac{|\gamma|^{np}}{|\gamma|^n}\psi_1(a,f)\le|\gamma|^{n(p-1)}\psi_1(a,f).$$

Taking the limit as $n\to\infty$, we find that $A(af)=aA(f)$ for every $a\in\mathcal{A}$ and $f\in\mathcal{A}^*$.

To demonstrate the uniqueness of the additive mapping A subject to (2.4), let us assume that there is another additive mapping $A':\mathcal{A}^*\to\mathcal{A}^*$ satisfying (2.4). Clearly, $A(\gamma^n f)=$

$\gamma^n A(f)$ and $A'(\gamma^n f) = \gamma^n A'(f)$ for all $n \in \mathbb{N}$ and $f \in \mathcal{A}^*$. We have

$$\begin{aligned}\|A(f) - A'(f)\| &= \left\| \frac{A(\gamma^n f)}{\gamma^n} - \frac{A'(\gamma^n f)}{\gamma^n} \right\| \\ &\leq \left\| \frac{A(\gamma^n f)}{\gamma^n} - \frac{T(\gamma^n f)}{\gamma^n} \right\| + \left\| \frac{T(\gamma^n f)}{\gamma^n} - \frac{A'(\gamma^n f)}{\gamma^n} \right\| \\ &\leq \frac{2}{|\gamma|^{n+1} - |\gamma|^{n+p}} \psi_2(\gamma^n f, \gamma^n f) \\ &\leq \frac{2|\gamma|^{n(p-1)}}{|\gamma| - |\gamma|^p} \psi_2(f, f).\end{aligned}$$

As the right-hand side in the above inequality tends to zero as n tends to infinity, $A = A'$.

Case (2): $j = -1$. We can state the proof in the same pattern as we did in the first case. Let f be any fixed element in $\mathcal{A}^*$. The relation (2.3) for $g = f$ yields $\|T(\gamma f) - \gamma T(f)\| \leq \psi_2(f, f)$. Replacing f by $\frac{f}{\gamma}$ in the last inequality, we obtain

$$\left\| T(f) - \gamma T\left(\frac{f}{\gamma}\right) \right\| \leq \psi_2\left(\frac{f}{\gamma}, \frac{f}{\gamma}\right) = \frac{1}{|\gamma|^p} \psi_2(f, f). \tag{2.9}$$

Replacing f by $\frac{f}{\gamma^n}$ in (2.9) and multiplying both sides of the result by $|\gamma|^n$, we obtain

$$\left\| \gamma^n T\left(\frac{f}{\gamma^n}\right) - \gamma^{n+1} T\left(\frac{f}{\gamma^{n+1}}\right) \right\| \leq |\gamma|^n \psi_2\left(\frac{f}{\gamma^n}, \frac{f}{\gamma^n}\right) = |\gamma|^{n(1-p)} \psi_2(f, f)$$

where $n \in \mathbb{N}$. Applying the triangular inequality, we obtain

$$\left\| T(f) - \gamma^n T\left(\frac{f}{\gamma^n}\right) \right\| \leq \sum_{k=1}^{n} \frac{1}{|\gamma|^{kp-(k-1)}} \psi_2(f, f). \tag{2.10}$$

It is seen that

$$\left\| \gamma^n T\left(\frac{f}{\gamma^n}\right) - \gamma^m T\left(\frac{f}{\gamma^m}\right) \right\| \leq \sum_{k=n+1}^{m} \frac{1}{|\gamma|^{kp-(k-1)}} \psi_2(f, f) \tag{2.11}$$

for all $f \in \mathcal{A}^*$ and $m > n$. As the right-hand of (2.11) converges to zero by assumption of $p > 1$ as $m, n \to \infty$, this shows that $\left\{\gamma^n T(\frac{f}{\gamma^n})\right\}$ is a Cauchy sequence in $\mathcal{A}^*$. Therefore, there is a mapping $A : \mathcal{A}^* \to \mathcal{A}^*$ defined by $A(f) = \lim\limits_{n \to \infty} \gamma^n T(\frac{f}{\gamma^n})$. Employing (2.10), we obtain

$$\|T(f) - A(f)\| \leq \frac{-1}{|\gamma| - |\gamma|^p} \psi_2(f, f)$$

for all $f \in \mathcal{A}^*$. The rest of the proof for this case proceeds similarly to that in the previous case, hence is omitted.

Now, we present a result similar to Theorem 2.1 for the case where $|\gamma| < 1$.

Theorem 2.2 Let α and β be nonzero rational numbers such that $0 < |\alpha + \beta| < 1$. Let $j \in \{-1, 1\}$, and p be a real number such that $pj > j$. Let $\psi_1 : \mathcal{A} \times \mathcal{A}^* \to \mathbb{R}^+$ and $\psi_2 : \mathcal{A}^* \times \mathcal{A}^* \to \mathbb{R}^+$ be functions such that

$$\psi_1(a, \gamma^{nj} f) = |\gamma|^{npj} \psi_1(a, f), \quad \psi_2(\gamma^{nj} f, \gamma^{nj} g) = |\gamma|^{npj} \psi_2(f, g),$$

where $\gamma = \alpha + \beta$. Suppose that $T : \mathcal{A}^* \to \mathcal{A}^*$ satisfies

$$\|T(af) - aT(f)\| \leq \psi_1(a, f)$$

and

$$\|T(\alpha f+\beta g)-\alpha T(f)-\beta T(g)\|\leq \psi_2(f,g)$$

for all $f,g\in\mathcal{A}^*$ and $a\in\mathcal{A}$. Then, there exists a unique additive mapping $A:\mathcal{A}^*\to\mathcal{A}^*$, such that $A(af)=aA(f)$, and the inequality

$$\|T(f)-A(f)\|\leq \frac{j}{|\gamma|-|\gamma|^p}\psi_2(f,f)$$

for all $a\in\mathcal{A}$ and $f\in\mathcal{A}^*$.

Proof The techniques are completely similar to that of Theorem 2.1. Hence, it is omitted.

Corollary 2.3 Let α and β be nonzero rational numbers such that $|\alpha+\beta|>1$. Let $j\in\{-1,1\}$, and p be a real number such that $pj<j$. Let $T:\mathcal{A}^*\to\mathcal{A}^*$ be a mapping for which there exists $\epsilon>0$ such that

$$\|T(af)-aT(f)\|\leq\epsilon\|a\|\|f\|^p$$

and

$$\|T(\alpha f+\beta g)-\alpha T(f)-\beta T(g)\|\leq\epsilon(\|f\|^p+\|g\|^p)$$

for all $f,g\in\mathcal{A}^*$ and $a\in A$. Then, there exists a unique additive mapping $A:\mathcal{A}^*\to\mathcal{A}^*$, such that $A(af)=aA(f)$ and

$$\|T(f)-A(f)\|\leq\frac{2j\epsilon}{|\gamma|-|\gamma|^p}\|f\|^p$$

for all $a\in\mathcal{A}$ and $f\in\mathcal{A}^*$.

Proof According to Theorem 2.1, if we select $\psi_1(a,f)=\epsilon\|a\|\|f\|$ and $\psi_2(f,g)=\epsilon(\|f\|^p+\|g\|^p)$, then, the conditions of Theorem 2.1 are fulfilled. Consequently, there exist a unique additive function $A:\mathcal{A}^*\to\mathcal{A}^*$, such that $A(af)=aA(f)$ and

$$\|T(f)-A(f)\|\leq\frac{2j\epsilon}{|\gamma|-|\gamma|^p}\|f\|^p$$

for all $a\in\mathcal{A}$ and $f\in\mathcal{A}^*$, as desired.

Remark 2.4 Let G be a noncompact locally compact group. Theorem 3.4 in [4] implies the existence of a bounded linear operator $A:L^\infty(G)\to L^\infty(G)$ satisfying $A(af)=aA(f)$ for every $a\in L^1(G)$ and $f\in L^\infty(G)$, and also $A(Ff)\neq FA(f)$ for some $F\in L^1(G)^{**}$ and $f\in L^\infty(G)$. This shows that the additive mapping A obtained in Theorem 2.1 can not commute with every $F\in L^1(G)^{**}$.

Consider the following equation:

$$T(f+g)+T(f-g)=2T(f)+2T(g). \tag{2.12}$$

We define any solution of (2.12) to be a quadratic mapping. In the next theorem, we prove the generalized Hyers-Ulam stability of the quadratic equation. The following lemma is needed for the sequel.

Lemma 2.5 Let α and β be nonzero rational numbers with $\alpha^2+\beta^2\neq 1$. A mapping $T:\mathcal{A}^*\to\mathcal{A}^*$ satisfies the equation (2.12), if and only if

$$T(\alpha f+\beta g)+T(\alpha f-\beta g)=2\alpha^2T(f)+2\beta^2T(g) \tag{2.13}$$

for all $f, g \in \mathcal{A}^*$.

Proof Necessity. From (2.12), putting $f = g = 0$ yields $T(0) = 0$. Setting $f = 0$ in (2.12), we get $T(g) = T(-g)$ for all $g \in \mathcal{A}^*$. Thus, T is even. One can verify that $T(2f) = 4T(f)$ and $T(3f) = 9T(f)$. By induction, we infer that

$$T(kf) = k^2 T(f)$$

for any nonzero integer k. Replacing f by $\frac{f}{k}$ in the last equality, we have

$$T\left(\frac{f}{k}\right) = \frac{T(f)}{k^2}. \tag{2.14}$$

It follows that $T\big(\frac{m}{k}f\big) = \big(\frac{m}{k}\big)^2 T(f)$ for any $k, m \in \mathbb{N}$. Hence, for every rational number r, $T(rx) = r^2 T(x)$. The substitutions $f = \alpha f$ and $g = \beta g$ in (2.12) give the relation $T(\alpha f + \beta g) + T(\alpha f - \beta g) = 2\alpha^2 T(f) + 2\beta^2 T(g)$.

Sufficiency. In (2.13), set $f = g = 0$ to see $T(0) = 0$. Setting $g = 0$ in (2.13), we get $T(\alpha f) = \alpha^2 T(f)$ for any $f \in A^*$. In (2.13), replace g by $-g$ to get

$$T(\alpha f - \beta g) + T(\alpha f + \beta g) = 2\alpha^2 T(f) + 2\beta^2 T(-g). \tag{2.15}$$

Combine (2.13) with (2.15) to see that $T(g) = T(-g)$. Thus, T is even. Replacing f by 0 in (2.13) and using the evenness of T, we have $T(\beta g) = \beta^2 T(g)$. Consequently,

$$\begin{aligned}
\alpha^2\beta^2\big(T(f+g) + T(f-g)\big) &= \alpha^2\beta^2 T(f+g) + \alpha^2\beta^2 T(f-g) \\
&= T(\alpha\beta f + \alpha\beta g) + T(\alpha\beta f - \alpha\beta g) \\
&= 2\alpha^2 T(\beta f) + 2\beta^2 T(\alpha g) \\
&= 2\alpha^2\beta^2\big(T(f) + T(g)\big).
\end{aligned}$$

Thus, $T : \mathcal{A}^* \to \mathcal{A}^*$ is a quadratic mapping.

The preceding Lemma allows us to prove the next result. The following Theorem is one of the main results of this article.

Theorem 2.6 Let α and β be nonzero rational numbers with $\alpha^2 > 1$. Let $j \in \{-1, 1\}$, and let p be a real number such that $pj < j$. Let $\psi_1 : \mathcal{A} \times \mathcal{A}^* \to \mathbb{R}^+$ and $\psi_2 : \mathcal{A}^* \times \mathcal{A}^* \to \mathbb{R}^+$ be functions such that

$$\psi_1(a, \alpha^{nj} f) = \alpha^{2npj}\psi_1(a, f), \quad \psi_2(\alpha^{nj} f, \alpha^{nj} g) = \alpha^{2npj}\psi_2(f, g). \tag{2.16}$$

Suppose that $T : \mathcal{A}^* \to \mathcal{A}^*$ satisfies

$$\|T(af) - aT(f)\| \le \psi_1(a, f), \quad T(0) = 0 \tag{2.17}$$

and

$$\|T(\alpha f + \beta g) + T(\alpha f - \beta g) - 2\alpha^2 T(f) - 2\beta^2 T(g)\| \le \psi_2(f, g) \tag{2.18}$$

for all $f, g \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Then, there exists a unique quadratic mapping $Q : \mathcal{A}^* \to \mathcal{A}^*$ such that $Q(af) = aQ(f)$,

$$\|T(f) - Q(f)\| \le \frac{j}{2(\alpha^2 - \alpha^{2p})}\psi_2(f, 0) \tag{2.19}$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$.

Proof Case 1: $j = 1$. Putting $g = 0$ in (2.18) and dividing by $2\alpha^2$, we have

$$\left\|\frac{T(\alpha f)}{\alpha^2} - T(f)\right\| \le \frac{1}{2\alpha^2}\psi_2(f,0). \tag{2.20}$$

Replacing f by αf in (2.20) and dividing by α^2 and summing the resulting inequality and (2.20), we obtain

$$\left\|\frac{T(\alpha^2 f)}{\alpha^4} - T(f)\right\| \le \frac{1+\alpha^{2p-2}}{2\alpha^2}\psi_2(f,0). \tag{2.21}$$

By the induction on n, we have

$$\left\|\frac{T(\alpha^n f)}{\alpha^{2n}} - T(f)\right\| \le \sum_{k=1}^{n} \frac{\alpha^{2(k-1)(p-1)}}{2\alpha^2}\psi_2(f,0). \tag{2.22}$$

To prove the convergence of sequence $\left\{\frac{T(\alpha^n f)}{\alpha^{2n}}\right\}$, we divide inequality (2.22) by α^{2m} and also replace f by $\alpha^m f$ to find that, for $n, m \in \mathbb{N}$,

$$\left\|\frac{T(\alpha^{n+m} f)}{\alpha^{2(n+m)}} - \frac{T(\alpha^m f)}{\alpha^{2m}}\right\| \le \sum_{k=1}^{n} \frac{\alpha^{2(k-1)(p-1)+2mp}}{2\alpha^{2(m+1)}}\psi_2(f,0) \le \sum_{k=1}^{\infty} \frac{\alpha^{2(k-1)(p-1)+2mp}}{2\alpha^{2(m+1)}}\psi_2(f,0).$$

As the right-hand side of the inequality tends to 0 as $m \to \infty$, the sequence $\left\{\frac{T(\alpha^n f)}{\alpha^{2n}}\right\}$ is a Cauchy sequence. Therefore, we may define $Q(f) = \lim_{n\to\infty} \frac{T(\alpha^n f)}{\alpha^{2n}}$ for all $f \in \mathcal{A}^*$. By letting $n \to \infty$ in (2.22), we arrive at (2.19). To show that Q satisfies (2.12), let us replace f and g by $\alpha^n f$ and $\alpha^n g$ in (2.18), respectively, and divide by α^{2n}. Then, it follows that

$$\left\|\frac{T\big(\alpha^n(\alpha f+\beta g)\big)}{\alpha^{2n}} + \frac{T\big(\alpha^n(\alpha f-\beta g)\big)}{\alpha^{2n}} - 2\alpha^2\frac{T(\alpha^n f)}{\alpha^{2n}} - 2\beta^2\frac{T(\alpha^n g)}{\alpha^{2n}}\right\| \le \alpha^{2n(p-1)}\psi_2(f,g)$$

and taking the limit as $n \to \infty$, we see that

$$Q(\alpha f+\beta g) + Q(\alpha f-\beta g) = 2\alpha^2 Q(f) + 2\beta^2 Q(g)$$

for all $f, g \in \mathcal{A}^*$. By Lemma 2.5, the mapping Q is quadratic.

To show that $Q(af) = aQ(f)$, replace f by $\alpha^n f$ in (2.17) and divide by α^{2n}, then it follows that

$$\left\|\frac{T(\alpha^n af)}{\alpha^{2n}} - a\frac{T(\alpha^n f)}{\alpha^{2n}}\right\| \le \frac{\psi_1(a,\alpha^n f)}{\alpha^{2n}} = \alpha^{2n(p-1)}\psi_1(a,f).$$

The right-hand side of the last inequality converges to zero because of $\alpha^2 > 1$ and $p < 1$. Hence, $Q(af) = aQ(f)$.

To prove that the quadratic function Q is unique under the inequality (2.19), if we assume that there exists a quadratic function Q' which satisfies (2.17) and (2.19), then we have $Q(\alpha^n f) = \alpha^{2n}Q(f)$ and $Q'(\alpha^n f) = \alpha^{2n}Q'(f)$ for all $f \in \mathcal{A}^*$ and $n \in \mathbb{N}$. Hence, it follows from (2.17) that

$$\begin{aligned}\|Q(f) - Q'(f)\| &= \left\|\frac{Q(\alpha^n f)}{\alpha^{2n}} - \frac{Q'(\alpha^n f)}{\alpha^{2n}}\right\| \\ &\le \left\|\frac{Q(\alpha^n f)}{\alpha^{2n}} - \frac{T(\alpha^n f)}{\alpha^{2n}}\right\| + \left\|\frac{T(\alpha^n f)}{\alpha^{2n}} - \frac{Q'(\alpha^n f)}{\alpha^{2n}}\right\| \\ &\le \frac{\psi_2(\alpha^n f,0)}{\alpha^{2(n+1)} - \alpha^{2(n+p)}} = \frac{\alpha^{2n(p-1)}}{\alpha^2 - \alpha^{2p}}\psi_2(f,0).\end{aligned}$$

By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of Q.

Case 2. Let $j = -1$. We can prove the theorem by a similar technique.

Remark 2.7 Using the above technique, one can get another version of our result when $\alpha^2 > 1$ and $pj > j$. In contrast, we can formulate statements similar to Theorem 2.1 and Theorem 2.6, in which we can define the sequences

$$A(f) := \lim_{n\to\infty} \frac{T(\gamma^n f)}{\gamma^n} \quad \text{and} \quad Q(f) := \lim_{n\to\infty} \frac{T(\alpha^n f)}{\alpha^{2n}},$$

respectively, under suitable conditions on the functions $\psi_1 : \mathcal{A}^{**} \times \mathcal{A}^* \to \mathbb{R}^+$ and $\psi_2 : \mathcal{A}^* \times \mathcal{A}^* \to \mathbb{R}^+$ and then obtain results similar to Theorem 2.1 and Theorem 2.6. In this case, $A(Ff) = FA(f)$ and $Q(Ff) = Q(F)f$ for every $F \in \mathcal{A}^{**}$ and $f \in \mathcal{A}^*$.

Example 2.8 Let G be a locally compact group and $L_0^\infty(G)$ be the Banach space of all essentially bounded measurable functions on G, vanishing at infinity (see [28]). For each $a \in L^1(G)$, let a also denote the functional in $L_0^\infty(G)^*$ defined by

$$f(a) := \int f(x)a(x)\mathrm{d}\lambda(x) \quad (f \in L_0^\infty(G)).$$

Note that this duality defines a linear isometric embedding of $L^1(G)$ into $L_0^\infty(G)^*$. It is well known that $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$ [28]. Now, let α and β be nonzero rational numbers with $\alpha^2 > 1$. Let $j \in \{-1, 1\}$, and let p be a real number such that $pj < j$. Let $\psi_1 : L^1(G) \times L_0^\infty(G) \to \mathbb{R}^+$ and $\psi_2 : L_0^\infty(G) \times L_0^\infty(G) \to \mathbb{R}^+$ be mappings such that

$$\psi_1(a, \alpha^{nj} f) = \alpha^{2npj}\psi_1(a, f), \quad \psi_2(\alpha^{nj} f, \alpha^{nj} g) = \alpha^{2npj}\psi_2(f, g).$$

Suppose that $T : L_0^\infty(G) \to L_0^\infty(G)$ satisfies the following conditions:

$$\|T(af) - aT(f)\| \le \psi_1(a, f)$$

and

$$\|T(\alpha f + \beta g) + T(\alpha f - \beta g) - 2\alpha^2 T(f) - 2\beta^2 T(g)\| \le \psi_2(f, g)$$

for all $f, g \in L_0^\infty(G)$ and $a \in L^1(G)$. By Theorem 2.6, there exists a unique quadratic mapping $Q : L_0^\infty(G) \to L_0^\infty(G)$, such that $Q(af) = aQ(f)$, and the inequality

$$\|T(f) - Q(f)\| \le \frac{j}{\alpha^2 - \alpha^{2p}}\psi_2(f, 0)$$

holds for all $a \in L^1(G)$ and $f \in L_0^\infty(G)$. By Example 3.3 in [4], $Q(Ff) = FQ(f)$ for every $F \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$.

References

[1] Baker J, Lau A T, Pym J S. Module homomorphisms and topological centers associated with weakly sequentially complete Banach algebras. J Funct Anal, 1998, **158**: 186–208

[2] Ghaffari A. Strongly and weakly almost periodic linear maps on semigroup algebras. Semigroup Forum, 2008, **76**: 95–106

[3] Larsen R. An introduction to the theory of multipliers. Berlin, Heidelberg, New York: Springer Verlag, 1971

[4] Ghaffari A. Module homomorphisms associated with Banach algebras. Taiwaness Journal of Mathematics, 2011, **15**: 1078–1088

[5] Ulam S M. Problems in Modern Mathematics, Chapter VI. Science Editions. New York: Wiley, 1964

[6] Hyers D H. On the stability of the linear functional equation. Proc Nat Acad Sci U S A, 1941, **27**: 222–224

[7] Aoki T. On the stability of the linear transformation in Banach spaces. J Math Soc Japan, 1950, **2**: 64–66

[8] Rassias Th M. On the stability of the linear mapping in Banach spaces. Proc Amer Math Soc, 1978, **72**: 297–300

[9] Gajda Z. On stability of additive mappings. Int J Math Math Sci, 1991, **14**: 431–434

[10] Isac G, Rassias Th M. On the Hyers-Ulam stability of ψ-additive mappings. J Approx Theory, 1993,**72**: 131–137

[11] Lee Y H, Jun K W. On the stability of approximately additive mappings. Proc Amer Math Soc, 2000, **128**: 1361–1369

[12] Rassias Th M, Šemrl P. On the behavior of mappings which do not satisfy Hyers-Ulam stability. Proc Amer Math Soc, 1992, **114**: 989–993

[13] Găvruta P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J Math Anal Appl, 1994, **184**: 431–436

[14] Aczel J, Dhombres J. Functional Equations in Several Variables. Cambridge University Press, 1989

[15] Kannappan P. Quadratic functional equation and inner product spaces. Results Math, 1995, **27**: 368–372

[16] Czerwik S. On the stability of the quadratic mapping in normed spaces. Abh Math Sem Univ Hamburg, 1992, **62**: 59–64

[17] Eshaghi Gordji M, Khodaei H. Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces. Nonlinear Analysis-TMA, 2009, **71**: 5629–5643

[18] Jung S M. On the Hyers-Ulam stability of the functional equations that have the quadratic property. J Math Anal Appl, 1998, **222**: 126–137

[19] Khodaei H, Rassias Th M. Approximately generalized additive functions in several variables. Int J Nonlinear Anal Appl, 2010, **1**: 22–41

[20] Park C. Generalized quadratic mappings in several variables. Nonlinear Analysis-TMA, 2004, **57**: 713–722

[21] Sahoo P K. A generalized cubic functional equation. Acta Math Sin (Engl Ser), 2005, **21**: 1159–1166

[22] Skof F. Proprieta locali e approssimazione di operatori. Rend Sem Mat Fis Milano, 1983, **53**: 113–129

[23] Ghaffari A, Alinejad A, Eshaghi Gordji M. On the stability of general cubic-quartic functional equations in Menger probabilistic normed spaces. J Math Phys, 2009, **50**: 123301

[24] Hyers D H, Rassias Th M. Approximate homomorphisms. Aequationes Math, 1992, **44**: 125–153

[25] Jun K W, Kim H M. Stability problem for Jensen-type functional equations of cubic mappings. Acta Math Sin (Engl Ser), 2006, **22**: 1781–1788

[26] Rassias Th M. On the stability of functional equations and a problem of Ulam. Acta Appl Math, 2000, **62**: 23–130

[27] Miura T, Hirasawa G, Takahasi S. Stability of multipliers on Banach algebras. Int J Math Math Sci, 2004, **45**: 2377–2381

[28] Lau A T, Pym J. Concerning the second dual of the group algebra of a locally compact group. J London Math Soc, 1990, **41**: 445–460