General Patterns for Nonmonotonic Reasoning: From Basic Entailments to Plausible Relations

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Abstract

This paper has two goals. First, we develop frameworks for logical systems which are able to reflect not only nonmonotonic patterns of reasoning, but also paraconsistent reasoning. Our second goal is to have a better understanding of the conditions that a useful relation for nonmonotonic reasoning should satisfy. For this we consider a sequence of generalizations of the pioneering works of Gabbay, Kraus, Lehmann, Magidor and Makinson. These generalizations allow the use of monotonic nonclassical logics as the underlying logic upon which nonmonotonic reasoning may be based. Our sequence of frameworks culminates in what we call (following Lehmann) plausible, nonmonotonic, multiple-conclusion consequence relations (which are based on a given monotonic one). Our study yields intuitive justifications for conditions that have been proposed in previous frameworks and also clarifies the connections among some of these systems. In addition, we present a general method for constructing plausible nonmonotonic relations. This method is based on a multiple-valued semantics, and on Shoham's idea of preferential models. [1]

1 Introduction

Nonmonotonicity is generally considered as a desirable property in commonsense reasoning; Many approaches to basic problems in artificial intelligence such as belief revision, database updating, and action planning, rely in one way or another on some form of nonmonotonic reasoning. This led to a wide study of general patterns of nonmonotonic reasoning (see, e.g., [16, 18, 19, 24, 25, 26, 27, 28, 29, 39]). The basic idea behind most of these works is to classify nonmonotonic formalisms and to recognize several logical properties that nonmonotonic systems should satisfy.

The logic behind most of the systems which were proposed so far is *supraclassical*, i.e.: every first-degree inference rule that is classically sound remains valid in the resulting logics. As a result, the consequence relations introduced in these works are *not paraconsistent* [11], that is: they are not capable of drawing conclusions from inconsistent theories in a nontrivial way. Moreover, the basic idea behind most of the nonmonotonic approaches is significantly different from the idea of paraconsistent reasoning: While the usual approaches to nonmonotonic reasoning rule out contradictions when a new data arrives in order to maintain the consistency of a knowledge-base, the paraconsistent approach to reasoning accepts knowledge-bases as they are, and tolerates contradictions in them, if such exist.

Our goal in this paper is twofold. First, we want to develop frameworks for logical systems which will be able to reflect not only nonmonotonic patterns of reasoning, but also paraconsistent reasoning. Such systems will be useful also for reasoning with uncertainty, conflicts,

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[1] A preliminary version of this paper appeared in [4].

and contradictions. Our second goal is to have a better understanding of the conditions that a plausible relation for nonmonotonic reasoning should satisfy. The choice of the various conditions that have been proposed in previous works seem to us to be a little bit ad-hoc, making one wonder why certain conditions were adopted while others (that might seem not less plausible) have been rejected. We would like to remedy this.

To achieve these goals, we consider a sequence of generalizations of the pioneering works of Gabbay [18], Kraus, Lehmann, Magidor [24], and Makinson [28]. These generalizations are based on the following ideas:

- Each nonmonotonic logical system is based on some underlying monotonic one.
- The underlying monotonic logic should not necessarily be classical logic, but should be chosen according to the intended application. If, for example, inconsistent data is not to be totally rejected, then an underlying paraconsistent logic might be a better choice than classical logic.
- The more significant logical properties of the main connectives of the underlying monotonic logic, especially conjunction and disjunction (which have crucial roles in monotonic consequence relations), should be preserved as far as possible.
- On the other hand, the conditions that define a certain class of nonmonotonic systems should not assume anything concerning the language of the system (in particular, the existence of appropriate conjunction or disjunction should *not* be assumed).

The rest of this work is divided into two main sections. Section 2, the major one, is a study of nonmonotonic reasoning on the syntactical level. First we review the basic theory introduced in [24] (Section 2.1), which is based on a classical entailment relation and assumes the classical language. Then we consider nonmonotonic relations that are based on *arbitrary* entailment relations (Section 2.3). The next generalization (Section 2.4) uses Tarskian consequence relations [44] instead of just entailment relations. Finally, we consider multiple-conclusion relations that are based on Scott consequence relations [37, 38] (Section 2.5). For defining the latter relations we indeed need not assume the availability of any specific connective in the underlying language. However, the hierarchy of relations which we consider is based first of all on the question: What properties of the conjunction and disjunction of the underlying monotonic logic are preserved in the nonmonotonic logic which is based on it. Our sequence of frameworks culminates in what we call (following [25]) plausible nonmonotonic consequence relations. We believe that this notion captures the intuitive idea of "correct" nonmonotonic reasoning.

Section 3 provides a general semantical method for constructing plausible nonmonotonic consequence relations. This method is based on a combination of a lattice-valued semantics[2] with Shoham's idea of using only certain *preferential models* for drawing conclusions ([40, 41]). We show that some well-known plausible nonmonotonic logics can be constructed using this method. Most of these logics are paraconsistent as well (these include some logics that we have considered in previous works [2, 3, 5]).

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[2] This is a common method for dealing with inconsistent theories — see, e.g., [13, 14, 15, 20, 21, 23, 34, 35, 39, 42, 43].

2 Preferential systems from an abstract point of view

In this section we investigate preferential reasoning from an abstract point of view. First we briefly review the original treatments of Makinson [28] and Kraus, Lehmann, and Magidor [24]. Then we consider several generalizations of this framework.

2.1 The standard basic theory – A general overview

The language that is considered in [24, 28] is based on the standard propositional one. Here, $\hookrightarrow$ denotes the material implication (i.e., $\psi \hookrightarrow \phi = \neg\psi \vee \phi$) and $\sim$ denotes the corresponding equivalence operator (i.e., $\psi \sim \phi = (\psi \hookrightarrow \phi) \wedge (\phi \hookrightarrow \psi)$). The classical propositional language, with the connectives $\neg$, $\vee$, $\wedge$, $\hookrightarrow$, $\sim$, and with a propositional constant t, is denoted here by $\Sigma_{\rm cl}$. An arbitrary language is denoted by Σ. Given a set of formulae Γ in a language Σ, we denote by $\mathcal{A}(\Gamma)$ the set of the atomic formulae that occur in Γ, and by $\mathcal{L}(\Gamma)$ the corresponding set of literals.

Definition 2.1 *[24] Let* $\vdash_{\rm cl}$ *be the classical consequence relation. A binary relation*[3] $\mathrel{|\!\sim}'$ *between formulae in* $\Sigma_{\rm cl}$ *is called* cumulative *if it is closed under the following inference rules:*

reflexivity:	$\psi \mathrel{|\!\sim}' \psi$.
cautious monotonicity:	*if* $\psi \mathrel{|\!\sim}' \phi$ *and* $\psi \mathrel{|\!\sim}' \tau$, *then* $\psi \wedge \phi \mathrel{|\!\sim}' \tau$.
cautious cut:	*if* $\psi \mathrel{|\!\sim}' \phi$ *and* $\psi \wedge \phi \mathrel{|\!\sim}' \tau$, *then* $\psi \mathrel{|\!\sim}' \tau$.
left logical equivalence:	*if* $\vdash_{\rm cl} \psi \sim \phi$ *and* $\psi \mathrel{|\!\sim}' \tau$, *then* $\phi \mathrel{|\!\sim}' \tau$.
right weakening:	*if* $\vdash_{\rm cl} \psi \hookrightarrow \phi$ *and* $\tau \mathrel{|\!\sim}' \psi$, *then* $\tau \mathrel{|\!\sim}' \phi$.

Definition 2.2 *[24] A cumulative relation* $\mathrel{|\!\sim}'$ *is called* preferential *if it is closed under the following rule:*

$\vee$-introduction *(Or):*	*if* $\psi \mathrel{|\!\sim}' \tau$ *and* $\phi \mathrel{|\!\sim}' \tau$, *then* $\psi \vee \phi \mathrel{|\!\sim}' \tau$.

Note In order to distinguish between the rules of Definitions 2.1, 2.2, and their generalized versions that will be considered in the sequel, the condition above will usually be preceded by the string "KLM". Also, a relation that satisfies the rules of Definition 2.1 [Definition 2.2] will sometimes be called KLM-cumulative [KLM-preferential].

The conditions above might look a little-bit ad-hoc. For example, one might ask why $\hookrightarrow$ is used on the right, while the stronger $\sim$ is on the left. A discussion and some justification appears in [24, 27].[4] A stronger intuitive justification will be given below, using more general frameworks.

2.2 Generalizations

In the sequel we will consider several generalizations of the basic theory presented above:

1. In their formulation, [23, 24, 28, 29] consider the classical setting, i.e.: the basic language is that of the classical propositional calculus ($\Sigma_{\rm cl}$), and the basic entailment relation is the

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[3] A "conditional assertion" in terms of [24].

[4] Systems that satisfy the conditions of Definitions 2.1, 2.2, as well as other related systems, are also considered in [16, 26, 29, 39].

classical one ($\vdash_{\mathrm{cl}}$). Our first generalization concerns with an abstraction of the syntactic components and the entailment relations involved: Instead of using the classical entailment relation $\vdash_{\mathrm{cl}}$ as the basis for definitions of cumulative nonmonotonic entailment relations, we allow the use of any entailment relation which satisfies certain minimal conditions.

2. The next generalization is to use Tarskian consequence relations instead of entailment relations (i.e. we consider the use of a set of premises rather than a single one). These consequence relations should satisfy some minimal conditions concerning the availability of certain connectives in their language. Accordingly, we consider cumulative and preferential nonmonotonic *consequence relations* that are based on those Tarskian consequence relations.
3. We further extend the class of Tarskian consequence relations on which nonmonotonic relations can be based by removing almost all the conditions on the language. The definition of the corresponding notions of a cumulative and a preferential nonmonotonic consequence relation is generalized accordingly.
4. Our final generalization is to allow relations with *multiple conclusions* rather than the single conclusion ones. Within this framework *all* the conditions on the language can be removed.

2.3 Entailment relations and cautious entailment relations

In what follows ψ, ϕ, τ denote arbitrary formulae in a language Σ, and Γ, Δ denote *finite* sets of formulae in Σ.

Definition 2.3 *A* basic entailment *is a binary relation* $\overset{1}{\vdash}$ *between formulae, that satisfies the following conditions:*[5] [6] [7]

1R 1-reflexivity: $\psi \overset{1}{\vdash} \psi$.

1C 1-cut: *if* $\psi \overset{1}{\vdash} \tau$ *and* $\tau \overset{1}{\vdash} \phi$ *then* $\psi \overset{1}{\vdash} \phi$.

Next we generalize the propositional connectives used in the original systems:

Definition 2.4 *Let* $\overset{1}{\vdash}$ *be some basic entailment.*

- *A connective* $\wedge$ *is called a* combining conjunction *(w.r.t.* $\overset{1}{\vdash}$*) if the following condition is satisfied:* $\tau \overset{1}{\vdash} \psi\wedge\phi$ *iff* $\tau \overset{1}{\vdash} \psi$ *and* $\tau \overset{1}{\vdash} \phi$.
- *A connective* $\vee$ *is called a* combining disjunction *(w.r.t.* $\overset{1}{\vdash}$*) if the following condition is satisfied:* $\psi\vee\phi \overset{1}{\vdash} \tau$ *iff* $\psi \overset{1}{\vdash} \tau$ *and* $\phi \overset{1}{\vdash} \tau$.

¿From now on, unless otherwise stated, we assume that $\overset{1}{\vdash}$ is a basic entailment, and $\wedge$ is a combining conjunction w.r.t. $\overset{1}{\vdash}$.

Definition 2.5

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[5] The "1" means that exactly one formula should appear on both sides of this relation.

[6] It could have been convenient to assume also that $\overset{1}{\vdash}$ is closed under substitutions of equivalents, but here we allow cases in which this is not the case.

[7] These conditions mean, actually, that basic entailment induces a category in which the objects are formulae.

- *A connective* $\vee$ *is called a* $\wedge$-combining disjunction *(w.r.t.* $\overset{1}{\vdash}$*) if it is a combining disjunction and:* $\sigma \wedge (\psi \vee \phi) \overset{1}{\vdash} \tau$ *iff* $\sigma \wedge \psi \overset{1}{\vdash} \tau$ *and* $\sigma \wedge \phi \overset{1}{\vdash} \tau$*.*
- *A connective* $\supset$ *is called a* $\wedge$-internal implication *(w.r.t.* $\overset{1}{\vdash}$*) if the following condition is satisfied:* $\tau \wedge \psi \overset{1}{\vdash} \phi$ *iff* $\tau \overset{1}{\vdash} \psi \supset \phi$*.*
- *A constant* t *is called a* $\wedge$-internal truth *(w.r.t.* $\overset{1}{\vdash}$*) if the following condition is satisfied:* $\psi \wedge t \overset{1}{\vdash} \phi$ *iff* $\psi \overset{1}{\vdash} \phi$*.*

Definition 2.6

a) *A formula* τ *is a* conjunct *of a formula* ψ *if* $\psi = \tau$*, or if* $\psi = \phi_1 \wedge \phi_2$ *and* τ *is a conjunct of either* ϕ_1 *or* ϕ_2*.*
b) *For every* $1 \leq i \leq n$ ψ_i *is called a* semiconjunction *of* $\psi_1, \ldots, \psi_n$*; If* ψ' *and* ψ'' *are semiconjunctions of* $\psi_1, \ldots, \psi_n$ *then so is* $\psi' \wedge \psi''$*.*
c) *A* conjunction *of* $\psi_1, \ldots, \psi_n$ *is a semiconjunction of* $\psi_1, \ldots, \psi_n$ *in which every* ψ_i *appears at least once as a conjunct.*

Lemma 2.7 *(Basic properties of* $\overset{1}{\vdash}$ *and* $\wedge$*)*

a) $\overset{1}{\vdash}$ *is monotonic: If* $\psi \overset{1}{\vdash} \tau$ *then* $\psi \wedge \phi \overset{1}{\vdash} \tau$ *and* $\phi \wedge \psi \overset{1}{\vdash} \tau$*.*
b) *If* τ *is a conjunct of* ψ *then* $\psi \overset{1}{\vdash} \tau$*.*
c) *If* ψ *is a conjunction of* $\psi_1, \ldots, \psi_n$ *and* ψ' *is a semiconjunction of* $\psi_1, \ldots, \psi_n$ *then* $\psi \overset{1}{\vdash} \psi'$*.*
d) *If* ψ *and* ψ' *are conjunctions of* $\psi_1, \ldots, \psi_n$ *then* ψ *and* ψ' *are equivalent:* $\psi \overset{1}{\vdash} \psi'$ *and* $\psi' \overset{1}{\vdash} \psi$*.*
e) *If* $\psi \overset{1}{\vdash} \phi$ *and* $\psi \wedge \phi \overset{1}{\vdash} \tau$ *then* $\psi \overset{1}{\vdash} \tau$*.*

PROOF. For part (a), suppose that $\psi \overset{1}{\vdash} \tau$. By 1-reflexivity, $\psi \wedge \phi \overset{1}{\vdash} \psi \wedge \phi$. Since $\wedge$ is a combining conjunction, $\psi \wedge \phi \overset{1}{\vdash} \psi$. A 1-cut with $\psi \overset{1}{\vdash} \tau$ yields $\psi \wedge \phi \overset{1}{\vdash} \tau$. The case of $\phi \wedge \psi$ is similar.
We leave the other parts to the reader. ■

Notation 2.8 *Let* $\Gamma = \{\psi_1, \ldots, \psi_n\}$*. Then* $\wedge\Gamma$ *and* $\psi_1 \wedge \ldots \wedge \psi_n$ *will both denote any conjunction of all the formulae in* Γ*.*

Note Because of Lemma 2.7 (especially part (d)), there will be no importance to the order according to which the conjunction of elements of Γ is taken in those cases below in which we use Notation 2.8.

Notation 2.9 $\psi \equiv \phi = (\psi \supset \phi) \wedge (\phi \supset \psi)$.

Lemma 2.10 *(Basic properties of* $\overset{1}{\vdash}$ *and* $\supset$*,* t*) Let* $\supset$ *be a* $\wedge$*-internal implication w.r.t.* $\overset{1}{\vdash}$ *and let* t *be a* $\wedge$*-internal truth w.r.t.* $\overset{1}{\vdash}$*. Then:*

a) *If* $t \overset{1}{\vdash} \tau$ *then* $\phi \overset{1}{\vdash} \tau$*.*
b) $\psi \overset{1}{\vdash} t$ *for every formula* ψ*.*
c) $\psi \wedge \phi \overset{1}{\vdash} \tau$ *iff* $\phi \overset{1}{\vdash} \psi \supset \tau$*.*

d) $\psi \overset{1}{\vdash} \phi$ *iff* $t \overset{1}{\vdash} \psi \supset \phi$*. Also,* $\psi \overset{1}{\vdash} \phi$ *and* $\phi \overset{1}{\vdash} \psi$ *iff* $t \overset{1}{\vdash} \psi \equiv \phi$*.*

e) If $\tau \overset{1}{\vdash} \psi \supset \phi$ *then* $t \overset{1}{\vdash} (\tau \wedge \psi) \supset (\tau \wedge \phi)$*; If* $\tau \overset{1}{\vdash} \psi \equiv \phi$ *then* $t \overset{1}{\vdash} (\tau \wedge \psi) \equiv (\tau \wedge \phi)$*.*

f) If ψ_1, ψ_2 *are conjunctions of the same set of formulae then* $t \overset{1}{\vdash} \psi_1 \equiv \psi_2$*.*

g) If $\psi \overset{1}{\vdash} \phi$ *and* $\psi \overset{1}{\vdash} \phi \supset \tau$ *then* $\psi \overset{1}{\vdash} \tau$*.*

PROOF. All the parts of the lemma are easily verified. We only give a proof of the first claim of part (e): If $\tau \overset{1}{\vdash} \psi \supset \phi$, then $\tau \wedge \psi \overset{1}{\vdash} \phi$. By Lemma 2.7(a), $\tau \wedge \psi \overset{1}{\vdash} \tau$. Thus $\tau \wedge \psi \overset{1}{\vdash} \tau \wedge \phi$ (combining conjunction), and so $t \overset{1}{\vdash} (\tau \wedge \psi) \supset (\tau \wedge \phi)$ by part (d). ■

Lemma 2.11 *Let* $\vee$ *be a combining disjunction w.r.t.* $\overset{1}{\vdash}$*.*

a) $\vee$ *is a* $\wedge$*-combining disjunction iff the following distributive law obtains:*

$$\phi \wedge (\psi_1 \vee \psi_2) \overset{1}{\vdash} (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2)$$

b) If $\overset{1}{\vdash}$ *has a* $\wedge$*-internal implication then* $\vee$ *is a* $\wedge$*-combining disjunction.*

PROOF. Part (a) is based on the facts that $\psi \overset{1}{\vdash} \psi \vee \phi$, $\phi \overset{1}{\vdash} \psi \vee \phi$, $\psi \wedge \phi \overset{1}{\vdash} \psi$, and $\psi \wedge \phi \overset{1}{\vdash} \phi$ (see the proof of Lemma 2.7(a)). We leave the details to the reader. Part (b) follows from (a), since it is easy to see that if $\overset{1}{\vdash}$ has a $\wedge$-internal implication then the above distributive law holds. ■

Note It is easy to see that the converse of the distributive law above, i.e. that

$$(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \overset{1}{\vdash} \phi \wedge (\psi_1 \vee \psi_2)$$

is true whenever $\wedge$ and $\vee$ are, respectively, a combining conjunction and a combining disjunction w.r.t. $\overset{1}{\vdash}$.

Definition 2.12 *Suppose that a language* Σ *of a basic entailment* $\overset{1}{\vdash}$ *contains a combining conjunction* $\wedge$*, a* $\wedge$*-internal implication* $\supset$*, and a* $\wedge$*-internal truth* t*. A binary relation* $\overset{1}{\mid\!\sim}$ *between formulae in* Σ *is called* $\{\wedge, \supset, t, \overset{1}{\vdash}\}$-cumulative *if it satisfies the following conditions:*

$\psi \overset{1}{\mid\!\sim} \psi$*.*
if $\psi \overset{1}{\mid\!\sim} \phi$ *and* $\psi \overset{1}{\mid\!\sim} \tau$*, then* $\psi \wedge \phi \overset{1}{\mid\!\sim} \tau$*.*
if $\psi \overset{1}{\mid\!\sim} \phi$ *and* $\psi \wedge \phi \overset{1}{\mid\!\sim} \tau$*, then* $\psi \overset{1}{\mid\!\sim} \tau$*.*
if $t \overset{1}{\vdash} \psi \equiv \phi$ *and* $\psi \overset{1}{\mid\!\sim} \tau$*, then* $\phi \overset{1}{\mid\!\sim} \tau$*.*
if $t \overset{1}{\vdash} \psi \supset \phi$ *and* $\tau \overset{1}{\mid\!\sim} \psi$*, then* $\tau \overset{1}{\mid\!\sim} \phi$*.*

Note In our notations, a KLM-cumulative relation (Definition 2.1) is $\{\wedge, \hookrightarrow, t, \vdash_{\mathrm{cl}}\}$-cumulative.

Lemma 2.10(d) allows us to further generalize the notion of a cumulative relation so that only the availability of a combining conjunction is assumed:

Definition 2.13 *A binary relation* $\overset{1}{\mid\!\sim}$ *between formulae is called* $\{\wedge, \overset{1}{\vdash}\}$-cumulative *if it satisfies the following conditions:*

1R	1-reflexivity:	$\psi \overset{1}{\mid\!\sim} \psi$.
1CM	1-cautious monotonicity:	*if* $\psi \overset{1}{\mid\!\sim} \phi$ *and* $\psi \overset{1}{\mid\!\sim} \tau$, *then* $\psi\wedge\phi \overset{1}{\mid\!\sim} \tau$.
1CC	1-cautious cut:	*if* $\psi \overset{1}{\mid\!\sim} \phi$ *and* $\psi\wedge\phi \overset{1}{\mid\!\sim} \tau$, *then* $\psi \overset{1}{\mid\!\sim} \tau$.
1LLE	1-left logical equivalence:	*if* $\psi \overset{1}{\vdash} \phi$ *and* $\phi \overset{1}{\vdash} \psi$ *and* $\psi \overset{1}{\mid\!\sim} \tau$, *then* $\phi \overset{1}{\mid\!\sim} \tau$.
1RW	1-right weakening:	*if* $\psi \overset{1}{\vdash} \phi$ *and* $\tau \overset{1}{\mid\!\sim} \psi$, *then* $\tau \overset{1}{\mid\!\sim} \phi$.

If, in addition, $\vee$ *is a* $\wedge$*-combining disjunction w.r.t.* $\overset{1}{\vdash}$, *and* $\overset{1}{\mid\!\sim}$ *satisfies the following rule:*

1Or	1-$\vee$ introduction:	*if* $\psi \overset{1}{\mid\!\sim} \tau$ *and* $\phi \overset{1}{\mid\!\sim} \tau$, *then* $\psi\vee\phi \overset{1}{\mid\!\sim} \tau$

then $\overset{1}{\mid\!\sim}$ *is called* $\{\vee, \wedge, \overset{1}{\vdash}\}$-preferential.

Proposition 2.14 *Let* $\supset$ *be a* $\wedge$*-internal implication w.r.t.* $\overset{1}{\vdash}$ *and let* t *be a* $\wedge$*-internal truth w.r.t.* $\overset{1}{\vdash}$. *Then a relation is* $\{\wedge, \supset, t, \overset{1}{\vdash}\}$*-cumulative iff it is* $\{\wedge, \overset{1}{\vdash}\}$*-cumulative.*

PROOF. Follows easily from Lemma 2.10. ■

Note From the note after Definition 2.12 and the last proposition it follows that in a language containing Σ_{cl}, $\overset{1}{\mid\!\sim}$ is a KLM-preferential relation (Definition 2.2) iff it is $\{\vee, \wedge, \hookrightarrow, t, \vdash_{\mathrm{cl}}\}$-preferential.

Proposition 2.15 *Every* $\{\wedge, \overset{1}{\vdash}\}$*-cumulative relation* $\overset{1}{\mid\!\sim}$ *is an extension of its corresponding basic entailment: If* $\psi \overset{1}{\vdash} \phi$ *then* $\psi \overset{1}{\mid\!\sim} \phi$.

PROOF. By 1RW of $\psi \overset{1}{\vdash} \phi$ and $\psi \overset{1}{\mid\!\sim} \psi$. ■

Proposition 2.16 *Let* $\overset{1}{\mid\!\sim}$ *be a* $\{\wedge, \overset{1}{\vdash}\}$*-cumulative relation. Then:*

a) $\wedge$ *is a combining conjunction also w.r.t.* $\overset{1}{\mid\!\sim}$: $\tau \overset{1}{\mid\!\sim} \psi\wedge\phi$ *iff* $\tau \overset{1}{\mid\!\sim} \psi$ *and* $\tau \overset{1}{\mid\!\sim} \phi$.

b) If t *is a* $\wedge$*-internal truth w.r.t.* $\overset{1}{\vdash}$ *then it is also a* $\wedge$*-internal truth w.r.t.* $\overset{1}{\mid\!\sim}$: $\psi\wedge t \overset{1}{\mid\!\sim} \phi$ *iff* $\psi \overset{1}{\mid\!\sim} \phi$.

PROOF.
a) ($\Leftarrow$): Suppose that $\tau \overset{1}{\mid\!\sim} \psi$ and $\tau \overset{1}{\mid\!\sim} \phi$. Then by 1CM, [1]: $\tau\wedge\psi \overset{1}{\mid\!\sim} \phi$. On the other hand, by Lemma 2.7(c), $\tau\wedge\psi\wedge\phi \overset{1}{\vdash} \psi\wedge\phi$, and so by Proposition 2.15, [2]: $\tau\wedge\psi\wedge\phi \overset{1}{\mid\!\sim} \psi\wedge\phi$. A 1CC, of [1] and [2] yields $\tau\wedge\psi \overset{1}{\mid\!\sim} \psi\wedge\phi$. Another 1CC with $\tau \overset{1}{\mid\!\sim} \psi$ yields that $\tau \overset{1}{\mid\!\sim} \psi\wedge\phi$.
($\Rightarrow$): Suppose that $\tau \overset{1}{\mid\!\sim} \psi\wedge\phi$. By Lemma 2.7(c), $\tau\wedge(\psi\wedge\phi) \overset{1}{\vdash} \psi$. By Proposition 2.15 $\tau\wedge(\psi\wedge\phi) \overset{1}{\mid\!\sim} \psi$. A 1CC with $\tau \overset{1}{\mid\!\sim} \psi\wedge\phi$ yields that $\tau \overset{1}{\mid\!\sim} \psi$. Similarly, if $\tau \overset{1}{\mid\!\sim} \psi\wedge\phi$ then $\tau \overset{1}{\mid\!\sim} \phi$.
b) By Lemma 2.10(b) and Proposition 2.15, $\psi \overset{1}{\mid\!\sim} t$. Now, suppose that $\psi \overset{1}{\mid\!\sim} \phi$. A 1CM with

$\psi \overset{1}{\mid\!\sim} t$ yields $\psi \wedge t \overset{1}{\mid\!\sim} \phi$. For the converse, assume that $\psi \wedge t \overset{1}{\mid\!\sim} \phi$. A 1CC with $\psi \overset{1}{\mid\!\sim} t$ yields $\psi \overset{1}{\mid\!\sim} \phi$. ■

Note Unlike $\wedge$ and t, in general $\supset$ and $\vee$ do *not* always remain a $\wedge$-internal implication and a combining disjunction w.r.t $\overset{1}{\mid\!\sim}$. Counter-examples will be given in Section 3 (see Proposition 3.24 and the note that follows it).

It is possible to strengthen the conditions in Definition 2.13 as follows:

s-1R	*strong* 1R:	if ψ is a conjunct of γ then $\gamma \overset{1}{\mid\!\sim} \psi$.
s-1RW	*strong* 1RW:	if $\tau \wedge \psi \overset{1}{\vdash} \phi$ and $\tau \overset{1}{\mid\!\sim} \psi$, then $\tau \overset{1}{\mid\!\sim} \phi$.

Our next goal is to show that these stronger versions are really valid for any $\{\wedge, \overset{1}{\vdash}\}$-cumulative relation. Moreover, each property is in fact equivalent to the corresponding property under certain conditions, which are specified below.

Proposition 2.17

a) 1RW and s-1RW are equivalent in the presence of 1R and 1CC.

b) 1RW and s-1R are equivalent in the presence of 1R, 1CC, and 1LLE.

PROOF.

a) The fact that s-1RW implies 1RW follows from Lemma 2.7(a). For the converse assume that $\tau \wedge \psi \overset{1}{\vdash} \phi$. By Proposition 2.15 (the proof of which uses only 1R and 1RW), $\tau \wedge \psi \overset{1}{\mid\!\sim} \phi$. A 1CC with $\tau \overset{1}{\mid\!\sim} \psi$ yields $\tau \overset{1}{\mid\!\sim} \phi$.

b) Suppose that $\psi \overset{1}{\vdash} \phi$ and $\tau \overset{1}{\mid\!\sim} \psi$. ¿From Lemma 2.7 it easily follows that the first assumption entails that $\tau \wedge \psi \wedge \phi \overset{1}{\vdash} \tau \wedge \psi$ and $\tau \wedge \psi \overset{1}{\vdash} \tau \wedge \psi \wedge \phi$. By s-1R, $\tau \wedge \psi \wedge \phi \overset{1}{\mid\!\sim} \phi$. A 1LLE of the last three sequents yields $\tau \wedge \psi \overset{1}{\mid\!\sim} \phi$. Finally, by 1CC with $\tau \overset{1}{\mid\!\sim} \psi$ we get $\tau \overset{1}{\mid\!\sim} \phi$. In the other direction s-1R is obtained from 1RW as follows: Let ψ be a conjunct of γ. By Lemma 2.7(b) $\gamma \overset{1}{\vdash} \psi$. A 1RW with $\gamma \overset{1}{\mid\!\sim} \gamma$ yields that $\gamma \overset{1}{\mid\!\sim} \psi$. ■

Corollary 2.18

a) s-1R and s-1RW are equivalent in the presence of 1R, 1CC, and 1LLE.

b) A relation is $\{\wedge, \overset{1}{\vdash}\}$-cumulative if it satisfies s-1R, 1LLE, 1CM, and 1CC.

PROOF. Immediate from Proposition 2.17 and the fact that s-1R entails 1R. ■

2.4 *Tarskian consequence relations and Tarskian cautious consequence relations*

The next step in our generalizations is to allow several premises on the l.h.s. of the consequence relations.

Definition 2.19

a) *A (ordinary)* Tarskian *consequence relation [44] (*tcr*, for short) is a binary relation* $\vdash$ *between sets of formulae and formulae, that satisfies the following conditions:* [8]

s-TR	strong T-reflexivity:	$\Gamma \vdash \psi$ *for every* $\psi \in \Gamma$.
TM	T-monotonicity:	*if* $\Gamma \vdash \psi$ *and* $\Gamma \subseteq \Gamma'$ *then* $\Gamma' \vdash \psi$.
TC	T-cut:	*if* $\Gamma_1 \vdash \psi$ *and* $\Gamma_2, \psi \vdash \phi$ *then* $\Gamma_1, \Gamma_2 \vdash \phi$.

b) *A Tarskian* cautious *consequence relation (*tccr*, for short) is a binary relation* $\mid\!\sim$ *between sets of formulae and formulae in a language* Σ*, that satisfies the following conditions:*[9]

s-TR	strong T-reflexivity:	$\Gamma \mid\!\sim \psi$ *for every* $\psi \in \Gamma$.
TCM	T-cautious monotonicity:	*if* $\Gamma \mid\!\sim \psi$ *and* $\Gamma \mid\!\sim \phi$*, then* $\Gamma, \psi \mid\!\sim \phi$.
TCC	T-cautious cut:	*if* $\Gamma \mid\!\sim \psi$ *and* $\Gamma, \psi \mid\!\sim \phi$*, then* $\Gamma \mid\!\sim \phi$.

Proposition 2.20 *Any tccr* $\mid\!\sim$ *is closed under the following rules for every* n*:*

TCM$^{[n]}$	*if* $\Gamma \mid\!\sim \psi_i$ $(i=1,\ldots,n)$ *then* $\Gamma, \psi_1, \ldots, \psi_{n-1} \mid\!\sim \psi_n$.
TCC$^{[n]}$	*if* $\Gamma \mid\!\sim \psi_i$ $(i=1,\ldots,n)$ *and* $\Gamma, \psi_1, \ldots \psi_n \mid\!\sim \phi$*, then* $\Gamma \mid\!\sim \phi$.

PROOF. We show closure under TCM$^{[n]}$ by induction on n. The case $n = 1$ is trivial, and TCM$^{[2]}$ is simply TCM. Now, assume that TCC$^{[n]}$ is valid and $\Gamma \mid\!\sim \psi_i$ for $i = 1, \ldots, n+1$. By induction hypothesis $\Gamma, \psi_1, \ldots, \psi_{n-1} \mid\!\sim \psi_n$ and $\Gamma, \psi_1, \ldots, \psi_{n-1} \mid\!\sim \psi_{n+1}$. Hence $\Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \psi_{n+1}$ by TCM.
The proof of TCC$^{[n]}$ is also by induction on n. TCC$^{[1]}$ is just TCC. Assume now that $\Gamma \mid\!\sim \psi_i$ $(i=1, \ldots, n+1)$ and $\Gamma, \psi_1, \ldots \psi_n, \psi_{n+1} \mid\!\sim \phi$. By TCM$^{[n+1]}$ $\Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \psi_{n+1}$. A TCC of the last two sequents gives $\Gamma, \psi_1, \ldots \psi_n \mid\!\sim \phi$. Hence $\Gamma \mid\!\sim \phi$ by induction hypothesis. ■

The following definition is the multiple-assumptions analogue of Definition 2.4:

Definition 2.21 *Let* $\vdash$ *be a relation between a set of formulae and a formula in a language* Σ.

- *A connective* $\wedge$ *is called* combining conjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma \vdash \psi \wedge \phi$ *iff* $\Gamma \vdash \psi$ *and* $\Gamma \vdash \phi$.
- *A connective* $\wedge$ *is called* internal conjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma, \psi \wedge \phi \vdash \tau$ *iff* $\Gamma, \psi, \phi \vdash \tau$.
- *A connective* $\vee$ *is called* combining disjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma, \psi \vee \phi \vdash \tau$ *iff* $\Gamma, \psi \vdash \tau$ *and* $\Gamma, \phi \vdash \tau$.

In what follows we assume that $\vdash$ is a tcr and $\wedge$ is a combining conjunction with respect to $\vdash$.

Lemma 2.22 *(Basic properties of* $\vdash$ *and* $\wedge$*)*

a) *If* $\Gamma, \psi \vdash \tau$ *then* $\Gamma, \psi \wedge \phi \vdash \tau$.

b) *If* $\Gamma, \psi \vdash \tau$ *then* $\Gamma, \phi \wedge \psi \vdash \tau$.

c) *If* ψ *is a conjunction of* $\psi_1, \ldots, \psi_n$ *and* ψ' *is a semiconjunction of* $\psi_1, \ldots, \psi_n$ *then* $\psi \vdash \psi'$.

d) *If* ψ *and* ψ' *are conjunctions of* $\psi_1, \ldots, \psi_n$ *then* ψ *and* ψ' *are equivalent:* $\psi \vdash \psi'$ *and* $\psi' \vdash \psi$.

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[8] The prefix "T" denotes that these are Tarskian rules.

[9] A set of conditions which is similar to the one below was first proposed in [19], except that instead of cautious cut Gabbay uses cut.

e) If $\Gamma \neq \emptyset$ *then* $\Gamma \vdash \psi$ *iff* $\wedge\Gamma \vdash \psi$.

f) $\wedge$ *is an internal conjunction w.r.t.* $\vdash$.

PROOF. Similar to that of Lemma 2.7. ■

Our next goal is to generalize the notion of cumulative entailment relation (Definition 2.13). We shall first do it for consequence relations that have a combining conjunction.

Definition 2.23 *A tccr* $\mid\!\sim$ *is called* $\{\wedge, \vdash\}$-cumulative *if it satisfies the following conditions:*

w-TLLE	weak T-left logical equivalence:	*if* $\psi \vdash \phi$ *and* $\phi \vdash \psi$ *and* $\psi \mid\!\sim \tau$, *then* $\phi \mid\!\sim \tau$.
w-TRW	weak T-right weakening:	*if* $\psi \vdash \phi$ *and* $\tau \mid\!\sim \psi$, *then* $\tau \mid\!\sim \phi$.
TICR	T-internal conjunction reduction:	*for every* $\Gamma \neq \emptyset$, $\Gamma \mid\!\sim \psi$ *iff* $\wedge\Gamma \mid\!\sim \psi$.

If, in addition, $\vdash$ *has a combining disjunction* $\vee$, *and* $\mid\!\sim$ *satisfies*

TOr	T-$\vee$-introduction:	*if* $\Gamma, \psi \mid\!\sim \tau$ *and* $\Gamma, \phi \mid\!\sim \tau$, *then* $\Gamma, \psi \vee \phi \mid\!\sim \tau$

then $\mid\!\sim$ *is called* $\{\vee, \wedge, \vdash\}$-preferential.

Notes

1. Because of Proposition 2.22 and w-TLLE, it again does not matter what conjunction of Γ is used in TICR.
2. Condition TICR is obviously equivalent to the requirement that $\wedge$ is an internal conjunction w.r.t. $\mid\!\sim$ (see Definition 2.21).

Proposition 2.24 *In the definition of* $\{\wedge, \vdash\}$*-cumulative tccr one can replace condition s-TR with the following weaker condition:*

TR	T-reflexivity:	$\psi \mid\!\sim \psi$.

PROOF. Let $\psi \in \Gamma$. A w-T-RW of $\wedge\Gamma \vdash \psi$ and $\wedge\Gamma \mid\!\sim \wedge\Gamma$ yields $\wedge\Gamma \mid\!\sim \psi$. By TICR, $\Gamma \mid\!\sim \psi$. ■

We now show that the concept of a $\{\wedge, \vdash\}$-cumulative tccr is equivalent to the notion of $\{\wedge, \overset{1}{\vdash}\}$-cumulative relation:

Definition 2.25 *Let* $\overset{1}{\vdash}$ *be a basic entailment with a combining conjunction* $\wedge$. *Let* $\overset{1}{\mid\!\sim}$ *be a* $\{\wedge, \overset{1}{\vdash}\}$*-cumulative relation. Define two binary relations* $(\overset{1}{\vdash})'$ *and* $(\overset{1}{\mid\!\sim})'$ *between sets of formulae and formulae in a language* Σ *as follows:*

a) $\Gamma(\overset{1}{\vdash})'\phi$ *iff either* $\Gamma \neq \emptyset$ *and* $\wedge\Gamma \overset{1}{\vdash} \phi$, *or* $\Gamma = \emptyset$ *and* $\psi \overset{1}{\vdash} \phi$ *for every* ψ.

b) $\Gamma(\overset{1}{\mid\!\sim})'\phi$ *iff* $\Gamma \neq \emptyset$ *and* $\wedge\Gamma \overset{1}{\mid\!\sim} \phi$. [10]

Definition 2.26 *Let* $\vdash$ *be a tcr with a combining conjunction* $\wedge$. *Suppose that* $\mid\!\sim$ *is a* $\{\wedge, \vdash\}$*-cumulative tccr. Define two binary relations* $(\vdash)^*$ *and* $(\mid\!\sim)^*$ *between formulae in* Σ *as follows:*

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[10] Since $\overset{1}{\mid\!\sim}$ is $\{\wedge, \overset{1}{\vdash}\}$-cumulative, it satisfies, in particular, 1LLE. Hence, the order in which the conjunction of Γ is taken has no importance (see Lemma 2.7d). Thus $(\overset{1}{\mid\!\sim})'$ is well-defined.

a) $\psi\ (\vdash)^*\ \phi$ *iff* $\{\psi\} \vdash \phi$.
b) $\psi\ (\mid\!\sim)^*\ \phi$ *iff* $\{\psi\} \mid\!\sim \phi$.

Proposition 2.27 *Let* $\overset{1}{\vdash}$, $\overset{1}{\mid\!\sim}$, $\vdash$, *and* $\mid\!\sim$ *be as in the last two definitions. Then:*

a) $(\overset{1}{\vdash})'$ *is a tcr for which* $\wedge$ *is a combining conjunction.*
b) $(\overset{1}{\mid\!\sim})'$ *is a* $\{\wedge, (\overset{1}{\vdash})'\}$*-cumulative tccr.*
c) $(\vdash)^*$ *is a basic entailment for which* $\wedge$ *is a combining conjunction.*
d) $(\mid\!\sim)^*$ *is a* $\{\wedge, (\vdash)^*\}$*-cumulative entailment.*
e) $((\overset{1}{\vdash})')^* = \overset{1}{\vdash}$.
f) $((\overset{1}{\mid\!\sim})')^* = \overset{1}{\mid\!\sim}$.
g) If $\vdash$ *is a normal tcr (i.e., if* $\forall\psi\ \psi \vdash \phi$ *then* $\vdash \phi$*), then* $((\vdash)^*)' = \vdash$.
h) If $\Gamma \neq \emptyset$ *then* $\Gamma\ ((\mid\!\sim)^*)'\ \psi$ *iff* $\Gamma \mid\!\sim \psi$.
i) If $\vee$ *is a* $\wedge$*-combining disjunction w.r.t.* $\overset{1}{\vdash}$ *and* $\overset{1}{\mid\!\sim}$ *satisfies* 1*-Or, then* $(\overset{1}{\mid\!\sim})'$ *is* $\{\vee, \wedge, \vdash\}$*-preferential.*
j) If $\vee$ *is a combining disjunction w.r.t.* $\vdash$ *and* $\mid\!\sim$ *satisfies* T*-Or, then* $(\mid\!\sim)^*$ *is* $\{\vee, \wedge, \overset{1}{\vdash}\}$*-preferential.*

PROOF. All the parts of the claim are easily verified. We show parts (h) and (i) as examples: (h): Suppose that $\Gamma \neq \emptyset$. Then $\Gamma\ ((\mid\!\sim)^*)'\ \psi$ iff $\wedge\Gamma\ (\mid\!\sim)^*\ \phi$ iff $\wedge\Gamma \mid\!\sim \phi$, iff (by TICR) $\Gamma \mid\!\sim \phi$.
(i): By (b) we only need to show that $(\overset{1}{\mid\!\sim})'$ satisfies TOr. So assume that $\gamma_1, \gamma_2, \ldots, \gamma_n, \psi\ (\overset{1}{\mid\!\sim})'\ \tau$ and $\gamma_1, \gamma_2, \ldots, \gamma_n, \phi\ (\overset{1}{\mid\!\sim})'\ \tau$. Then $(\bigwedge_{i=1}^n \gamma_i) \wedge \psi \overset{1}{\mid\!\sim} \tau$ and $(\bigwedge_{i=1}^n \gamma_i) \wedge \phi \overset{1}{\mid\!\sim} \tau$. By 1-Or, $((\bigwedge_{i=1}^n \gamma_i) \wedge \psi) \vee ((\bigwedge_{i=1}^n \gamma_i) \wedge \phi) \overset{1}{\mid\!\sim} \tau$. By Lemma 2.11, the note that follows it, and 1-LLE, $((\bigwedge_{i=1}^n \gamma_i) \wedge (\psi \vee \phi) \overset{1}{\mid\!\sim} \tau$. Thus, $\gamma_1, \gamma_2, \ldots, \gamma_n, \psi \vee \phi\ (\overset{1}{\mid\!\sim})'\ \tau$. ∎

Corollary 2.28 *Suppose that* $\mid\!\sim$ *is* $\{\wedge, \vdash_{\rm cl}\}$*-cumulative [*$\{\vee, \wedge, \vdash_{\rm cl}\}$*-preferential]. Define* $\psi \overset{1}{\mid\!\sim} \phi$ *iff* $\psi \mid\!\sim \phi$*. Then w.r.t.* $\Sigma_{\rm cl}$, $\overset{1}{\mid\!\sim}$ *is cumulative [preferential] in the sense of [24] (Definitions 2.1 and 2.2).*

We next generalize the definition of a cumulative tccr to make it independent of the existence of *any* specific connective in the language. In particular, we do not want to assume anymore that a combining conjunction is available.

Proposition 2.29 *Let* $\vdash$ *be a tcr, and let* $\mid\!\sim$ *be a tccr in the same language. The following connections between* $\vdash$ *and* $\mid\!\sim$ *are equivalent:*

TCum	T-cumulativity:	*for every* $\Gamma \neq \emptyset$*, if* $\Gamma \vdash \psi$ *then* $\Gamma \mid\!\sim \psi$.
TLLE	T-left logical equivalence:	*if* $\Gamma, \psi \vdash \phi$ *and* $\Gamma, \phi \vdash \psi$ *and* $\Gamma, \psi \mid\!\sim \tau$*, then* $\Gamma, \phi \mid\!\sim \tau$.
TRW	T-right weakening:	*if* $\Gamma, \psi \vdash \phi$ *and* $\Gamma \mid\!\sim \psi$*, then* $\Gamma \mid\!\sim \phi$.
TMiC	T-mixed cut:	*for every* $\Gamma \neq \emptyset$*, if* $\Gamma \vdash \psi$ *and* $\Gamma, \psi \mid\!\sim \phi$*, then* $\Gamma \mid\!\sim \phi$.

PROOF. We show that each property is equivalent to TCum:

TCum $\Rightarrow$ TLLE: Suppose that $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$. By TCum we have that $\Gamma, \psi \mid\!\sim \phi$ and $\Gamma, \phi \mid\!\sim \psi$. A T-cautious monotonicity of the first sequent with $\Gamma, \psi \mid\!\sim \tau$ yields $\Gamma, \psi, \phi \mid\!\sim \tau$, and by T-cautious cut with $\Gamma, \phi \mid\!\sim \psi$ we are done.

TLLE $\Rightarrow$ TCum: Let $\gamma \in \Gamma$, and suppose that $\Gamma \vdash \psi$. This entails that $\Gamma, \gamma \vdash \psi$. Also, by s-R, $\Gamma, \psi \vdash \gamma$. Since $\Gamma, \psi \mid\!\sim \psi$ then by TLLE we have that $\Gamma, \gamma \mid\!\sim \psi$. But $\gamma \in \Gamma$, so $\Gamma \mid\!\sim \psi$.

TCum $\Rightarrow$ TRW: Suppose that $\Gamma, \psi \vdash \phi$. By TCum $\Gamma, \psi \mid\!\sim \phi$. TCC with $\Gamma \mid\!\sim \psi$ yields $\Gamma \mid\!\sim \phi$.

TRW $\Rightarrow$ TCum: Suppose that $\Gamma \neq \emptyset$ and $\Gamma \vdash \psi$. Then there exists some $\gamma \in \Gamma$, and so $\Gamma, \gamma \vdash \psi$. By s-TR, $\Gamma \mid\!\sim \gamma$, and by TRW $\Gamma \mid\!\sim \psi$.

TCum $\Rightarrow$ TMiC: If Γ is a nonempty set of assertions s.t. $\Gamma \vdash \psi$, then by TCum, $\Gamma \mid\!\sim \psi$. A T-cautious cut of this sequent and $\Gamma, \psi \mid\!\sim \phi$ gives $\Gamma \mid\!\sim \phi$.

TMiC $\Rightarrow$ TCum: Suppose that Γ is a nonempty set of assertions and $\Gamma \vdash \psi$. By T-reflexivity, $\Gamma, \psi \mid\!\sim \psi$, and by TMiC, $\Gamma \mid\!\sim \psi$. ■

Notes

1. If there is a formula ψ s.t. $\mid\!\sim \psi$, then one can remove the requirement $\Gamma \neq \emptyset$ from the definition of TCum. Indeed, suppose that $\mid\!\sim \psi$. If $\vdash \phi$ then $\psi \vdash \phi$. Since the l.h.s. of the last entailment is nonempty, then by the original version of Cum, $\psi \mid\!\sim \phi$, and by TCC with $\mid\!\sim \psi$ we have $\mid\!\sim \phi$. The other direction is, however, not true: Let, for instance, $\vdash$ be some tcr for which there exists ψ_0 s.t. $\vdash \psi_0$. Define $\Gamma \mid\!\sim \phi$ if $\Gamma \vdash \phi$ and $\Gamma \neq \emptyset$. It is easy to verify that all the conditions of Definition 2.19 as well as TCum are valid for this $\mid\!\sim$, but $\not\mid\!\sim \psi_0$.
2. Being the "complement" of TMiC, one might consider TRW as another kind of "mixed cut".

Definition 2.30 *Let $\vdash$ be a tcr. A tccr $\mid\!\sim$ in the same language is called* $\vdash$-cumulative *if it satisfies* any *of the conditions of Proposition 2.29. If, in addition, $\vdash$ has a combining disjunction $\vee$, and $\mid\!\sim$ satisfies TOr, then $\mid\!\sim$ is called* $\{\vee, \vdash\}$-preferential.

Note Since $\Gamma \vdash \psi$ for every $\psi \in \Gamma$, TCum implies s-TR, and so a binary relation that satisfies TCum, TCM, and TCC is a $\vdash$-cumulative tccr.

Proposition 2.31 *Suppose that $\vdash$ is a tcr with a combining conjunction $\wedge$. A tccr $\mid\!\sim$ is a $\{\wedge, \vdash\}$-cumulative iff it is $\vdash$-cumulative. If $\vdash$ has also a combining disjunction $\vee$, then $\mid\!\sim$ is $\{\vee, \wedge, \vdash\}$-preferential iff it is $\{\vee, \vdash\}$-preferential.*

For proving Proposition 2.31 we first show the following lemmas:

Lemma 2.32 *Suppose that $\vdash$ is a tcr with a combining conjunction $\wedge$, and let $\mid\!\sim$ be a $\vdash$-cumulative tccr. Then $\bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$ iff $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \phi$.*

PROOF. For the proof we need two simple claims:

Claim 2.32-A: $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \bigwedge_{i=1}^{n} \psi_i$.
Proof: Clearly, $\psi_1, \psi_2, \ldots, \psi_{n-1}, \psi_n \vdash \bigwedge_{i=1}^{n} \psi_i$ and $\psi_1, \psi_2, \ldots, \psi_{n-1} \bigwedge_{i=1}^{n} \psi_i \vdash \psi_n$. Now, since $\psi_1, \psi_2, \ldots, \psi_{n-1}, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \bigwedge_{i=1}^{n} \psi_i$, then by TLLE, $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \bigwedge_{i=1}^{n} \psi_i$.

Claim 2.32-B: Let $1 \leq j \leq n$. Then $\Gamma, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$ iff $\Gamma, \psi_j, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$.
Proof: ($\Rightarrow$) Follows by applying TLLE on $\Gamma, \bigwedge_{i=1}^{n} \psi_i, \psi_j \vdash \bigwedge_{i=1}^{n} \psi_i$, and $\Gamma, \bigwedge_{i=1}^{n} \psi_i, \bigwedge_{i=1}^{n} \psi_i \vdash \psi_j$, and $\Gamma, \bigwedge_{i=1}^{n} \psi_i, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$.
($\Leftarrow$) By applying TLLE on $\Gamma, \bigwedge_{i=1}^{n} \psi_i, \psi_j \vdash \bigwedge_{i=1}^{n} \psi_i$ and $\Gamma, \bigwedge_{i=1}^{n} \psi_i, \bigwedge_{i=1}^{n} \psi_i, \vdash \psi_j$, and $\Gamma, \psi_j, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$, we get that $\Gamma, \bigwedge_{i=1}^{n} \psi_i, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$. Thus $\Gamma, \bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$.

Lemma 2.32 now easily follows from the above claims: If $\bigwedge_{i=1}^{n} \psi_i \mathrel{|\!\sim} \phi$ then by repeated applications of Claim 2.32-B, $\bigwedge_{i=1}^{n} \psi_i, \psi_1, \psi_2, \ldots, \psi_n \mathrel{|\!\sim} \phi$. A T-cautious cut with the property of Claim 2.32-A yields $\psi_1, \psi_2, \ldots, \psi_n \mathrel{|\!\sim} \phi$. For the converse suppose that $\psi_1, \psi_2, \ldots, \psi_n \mathrel{|\!\sim} \phi$. By T-cautious monotonicity with the property of Claim 2.32-A, $\bigwedge_{i=1}^{n} \psi_i, \psi_1, \psi_2, \ldots, \psi_n \mathrel{|\!\sim} \phi$, and by Claim 2.32-B (applied n times), $\bigwedge_{i=1}^{n} \psi_i \mathrel{|\!\sim} \phi$. ■

Lemma 2.33 *Let* $\mathrel{|\!\sim}$ *be a* $\{\wedge, \vdash\}$*-cumulative relation. Then* $\mathrel{|\!\sim}$ *satisfies* TRW.

PROOF. Suppose that $\Gamma, \psi \vdash \phi$. By Lemma 2.22(e) $(\wedge\Gamma) \wedge \psi \vdash \phi$. Since $\wedge\Gamma \wedge \psi \mathrel{|\!\sim} \wedge\Gamma \wedge \psi$ (s-R), then by w-TRW we have that $(\wedge\Gamma) \wedge \psi \mathrel{|\!\sim} \phi$. By TICR, $\Gamma, \psi \mathrel{|\!\sim} \phi$, and a TCC with $\Gamma \mathrel{|\!\sim} \psi$ yields that $\Gamma \mathrel{|\!\sim} \phi$. ■

Note In fact, we have proved a stronger claim, since in the course of the proof we haven't used CM and w-TLLE.

Now we can show Proposition 2.31:

PROOF. **of Proposition 2.31** $(\Leftarrow)$ Suppose that $\mathrel{|\!\sim}$ is a $\vdash$-cumulative tccr. It obviously satisfies w-TLLE and w-TRW (take $\Gamma = \emptyset$ and $\Gamma = \{\tau\}$, respectively). Lemma 2.32 shows that $\mathrel{|\!\sim}$ also satisfies TICR. Thus $\mathrel{|\!\sim}$ is a $\{\wedge, \vdash\}$-cumulative tccr.
$(\Rightarrow)$ Suppose that $\mathrel{|\!\sim}$ is a $\{\wedge, \vdash\}$-cumulative tccr. By Lemma 2.33 it satisfies TRW, and so it is $\vdash$-cumulative.
We leave the second part concerning $\vee$ to the reader. ■

Corollary 2.34 *Let* $\mathrel{|\!\sim}$ *be a* $\vdash$*-cumulative relation, and let* $\wedge$ *be a combining conjunction w.r.t.* $\vdash$*. Then* $\wedge$ *is a combining conjunction w.r.t.* $\mathrel{|\!\sim}$ *as well.*

PROOF. For a $\{\wedge, \vdash\}$-cumulative relation the proof is similar to that of Proposition 2.16(a). Hence the claim follows from Proposition 2.31. ■

Another characterization of $\vdash$-cumulative tccr which resembles more that of a cumulative entailment (Definition 2.13) is given in the following proposition:

Proposition 2.35 *A relation* $\mathrel{|\!\sim}$ *is a* $\vdash$*-cumulative tccr iff it satisfies* TR*,* TCM*,* TCC*,* TLLE *and* TRW*.*

PROOF. If $\mathrel{|\!\sim}$ is a $\vdash$-cumulative tccr then by Proposition 2.29 and the fact that s-TR implies TR, it obviously has all the above properties. The converse follows from the fact that TRW and s-TR are equivalent in the presence of TR, TCC, and TLLE. The proof of this fact is similar to that of Proposition 2.17. ■

2.5 Scott consequence relations and Scott cautious consequence relations

The last generalization that we consider in this section concerns with consequence relations in which *both* the premises and the conclusions may contain more than one formula.

Definition 2.36

a) A Scott *consequence relation [37, 38] (*scr*, for short) is a binary relation* $\vdash$ *between sets of formulae, that satisfies the following conditions:*

s-R	strong reflexivity:	*if* $\Gamma \cap \Delta \neq \emptyset$ *then* $\Gamma \vdash \Delta$.
M	monotonicity:	*if* $\Gamma \vdash \Delta$ *and* $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ *then* $\Gamma' \vdash \Delta'$.
C	cut:	*if* $\Gamma_1 \vdash \psi, \Delta_1$ *and* $\Gamma_2, \psi \vdash \Delta_2$ *then* $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

b) A Scott cautious *consequence relation (*sccr*, for short) is a binary relation* $\mid\!\sim$ *between nonempty*[11] *sets of formulae, that satisfies the following conditions:*

s-R	strong reflexivity:	*if* $\Gamma \cap \Delta \neq \emptyset$ *then* $\Gamma \mid\!\sim \Delta$.
CM	cautious monotonicity:	*if* $\Gamma \mid\!\sim \psi$ *and* $\Gamma \mid\!\sim \Delta$ *then* $\Gamma, \psi \mid\!\sim \Delta$.
CC$^{[1]}$	cautious 1-cut:	*if* $\Gamma \mid\!\sim \psi$ *and* $\Gamma, \psi \mid\!\sim \Delta$ *then* $\Gamma \mid\!\sim \Delta$.

The following definition is a natural analogue for the multiple-conclusion case of Definition 2.21:[12]

Definition 2.37 *Let* $\vdash$ *be a relation between sets of formulae.*

- *A connective* $\wedge$ *is called* combining conjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma \vdash \psi \wedge \phi, \Delta$ *iff* $\Gamma \vdash \psi, \Delta$ *and* $\Gamma \vdash \phi, \Delta$.
- *A connective* $\wedge$ *is called* internal conjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma, \psi \wedge \phi \vdash \Delta$ *iff* $\Gamma, \psi, \phi \vdash \Delta$.
- *A connective* $\vee$ *is called* combining disjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma, \psi \vee \phi \vdash \Delta$ *iff* $\Gamma, \psi \vdash \Delta$ *and* $\Gamma, \phi \vdash \Delta$.
- *A connective* $\vee$ *is called* internal disjunction *(w.r.t.* $\vdash$*) if the following condition is satisfied:* $\Gamma \vdash \psi \vee \phi, \Delta$ *iff* $\Gamma \vdash \psi, \phi, \Delta$.

Note Again, it can be easily seen that if $\vdash$ is an scr then $\wedge$ is an internal conjunction iff it is a combining conjunction, and similarly for $\vee$. This, however, is not true in general.

A natural requirement from a Scott cumulative consequence relation is that its single-conclusion counterpart will be a Tarskian cumulative consequence relation. Such a relation should also use disjunction on the r.h.s. like it uses conjunction on the l.h.s. The following definition formalizes these requirements.

Definition 2.38 *Let* $\vdash$ *be an scr with a combining disjunction* $\vee$*. A relation* $\mid\!\sim$ *between nonempty finite sets of formulae is called* $\{\vee, \vdash\}$-cumulative sccr *if it is an sccr that satisfies the following two conditions:*

a) Let $\vdash_{\mathrm{T}}$ *and* $\mid\!\sim_{\mathrm{T}}$ *be, respectively, the single-conclusion counterparts of* $\vdash$ *and* $\mid\!\sim$ *(i.e.,* $\Gamma \vdash_{\mathrm{T}} \psi$ *iff* $\Gamma \vdash \{\psi\}$ *and* $\Gamma \mid\!\sim_{\mathrm{T}} \psi$ *iff* $\Gamma \mid\!\sim \{\psi\}$*). Then* $\vdash_{\mathrm{T}}$ *is a tcr and* $\mid\!\sim_{\mathrm{T}}$ *is a* $\vdash_{\mathrm{T}}$*-cumulative tccr.*

b) For $\Delta = \{\psi_1, \ldots, \psi_n\}$*, denote by* $\vee\Delta$ *(or by* $\psi_1 \vee \ldots \vee \psi_n$*) any disjunction of all the formulae in* Δ*.*[13] *Then for every* $\Delta \neq \emptyset$*,* $\mid\!\sim$ *satisfies the following property:*[14]

IDR	internal disjunction reduction:	$\Gamma \mid\!\sim \Delta$ *iff* $\Gamma \mid\!\sim \vee\Delta$.

Following the line of what we have done in the previous section, we next specify conditions that are equivalent to those of Definition 2.38, but are independent of the existence of *any* specific connective in the language. In particular, we do not want to assume anymore that a combining disjunction is available:

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[11] The condition of non-emptiness is just technically convenient here. It is possible to remove it with the expense of complicating somewhat the definitions and propositions. It is preferable instead to employ (whenever necessary) the propositional constants t and f to represent the empty l.h.s. and the empty r.h.s., respectively.

[12] This definition is taken from [7]. Definitions 2.4 and 2.21 are obvious adaption of it.

[13] It easily follows from (a) above and from the properties of $\vee$ in $\vdash$ that the order according to which $\vee\Delta$ is taken has no importance here.

[14] This property is dual to the property of internal conjunction reduction (TICR, see Definition 2.23) of a $\vdash$-cumulative tccr.

Definition 2.39 *Let* $\vdash$ *be an scr. An sccr* $\mathrel{|\!\sim}$ *in the same language is called* weakly $\vdash$-cumulative *if it satisfies the following conditions:*

Cum	cumulativity:	*if* $\Gamma, \Delta \neq \emptyset$ *and* $\Gamma \vdash \Delta$, *then* $\Gamma \mathrel{|\!\sim} \Delta$.
RW$^{[1]}$	right weakening:	*if* $\Gamma, \psi \vdash \phi$ *and* $\Gamma \mathrel{|\!\sim} \psi, \Delta$ *then* $\Gamma \mathrel{|\!\sim} \phi, \Delta$.
RM	right monotonicity:	*if* $\Gamma \mathrel{|\!\sim} \Delta$ *then* $\Gamma \mathrel{|\!\sim} \psi, \Delta$.

Notes

1. Since $\Gamma, \psi \vdash \psi, \Delta$, Cum implies s-R, and so a binary relation that satisfies Cum, CM, CC$^{[1]}$, RW$^{[1]}$, and RM, is a weakly $\vdash$-cumulative sccr.
2. Any weakly $\vdash$-cumulative relation satisfies the following condition:

LLE *left logical equiv.:* if $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \mathrel{|\!\sim} \Delta$ then $\Gamma, \phi \mathrel{|\!\sim} \Delta$

Indeed, by Cum on $\Gamma, \psi \vdash \phi$ we have that $\Gamma, \psi \mathrel{|\!\sim} \phi$, and CM with $\Gamma, \psi \mathrel{|\!\sim} \Delta$ yields $\Gamma, \psi, \phi \mathrel{|\!\sim} \Delta$. Also, since $\Gamma, \phi \vdash \psi$ then by Cum $\Gamma, \phi \mathrel{|\!\sim} \psi$. A CC$^{[1]}$ with $\Gamma, \psi, \phi \mathrel{|\!\sim} \Delta$ yields $\Gamma, \phi \mathrel{|\!\sim} \Delta$.

Proposition 2.40 *Let* $\vdash$ *and* $\vee$ *be as in Definition 2.38. A relation* $\mathrel{|\!\sim}$ *is a* $\{\vee, \vdash\}$*-cumulative sccr iff it is a weakly* $\vdash$*-cumulative sccr.*

Proof. ($\Leftarrow$) Since $\vdash$ is an scr, $\vdash_{\mathrm{T}}$ is obviously a tcr. Also, since $\mathrel{|\!\sim}$ is a weakly $\vdash$-cumulative sccr, it satisfies s-R, CM, CC$^{[1]}$, and Cum, thus $\mathrel{|\!\sim}_{\mathrm{T}}$ obviously satisfies s-tR, tCM, tCC and tCum, therefore $\mathrel{|\!\sim}_{\mathrm{T}}$ is a $\vdash_{\mathrm{T}}$-cumulative tccr. It remains to show that $\mathrel{|\!\sim}$ satisfies IDR: Suppose first that $\Gamma \mathrel{|\!\sim} \vee\Delta$ for $\Delta \neq \emptyset$. Since $\Gamma, \vee\Delta \vdash \Delta$, then by Cum, $\Gamma, \vee\Delta \mathrel{|\!\sim} \Delta$. A CC$^{[1]}$ with $\Gamma \mathrel{|\!\sim} \vee\Delta$ yields $\Gamma \mathrel{|\!\sim} \Delta$. For the converse, we first show that if $\Gamma \mathrel{|\!\sim} \psi, \phi, \Delta$ then $\Gamma \mathrel{|\!\sim} \psi \vee \phi, \Delta$. Indeed, RW$^{[1]}$ of $\Gamma \mathrel{|\!\sim} \psi, \phi, \Delta$ and $\Gamma, \psi \vdash \psi \vee \phi$ yields $\Gamma \mathrel{|\!\sim} \psi \vee \phi, \phi, \Delta$. Another RW$^{[1]}$ with $\Gamma, \phi \vdash \psi \vee \phi$ yields $\Gamma \mathrel{|\!\sim} \psi \vee \phi, \psi \vee \phi, \Delta$. Thus, $\Gamma \mathrel{|\!\sim} \psi \vee \phi, \Delta$. Now, by an induction on the number of formulae in Δ it follows that if $\Delta \neq \emptyset$ and $\Gamma \mathrel{|\!\sim} \Delta$, then $\Gamma \mathrel{|\!\sim} \vee\Delta$.
($\Rightarrow$) Let $\mathrel{|\!\sim}$ be a $\{\vee, \vdash\}$-cumulative sccr. Suppose that $\Gamma, \Delta \neq \emptyset$ and $\Gamma \vdash \Delta$. Then $\Gamma \vdash \vee\Delta$. Hence $\Gamma \vdash_{\mathrm{T}} \vee\Delta$, and since $\mathrel{|\!\sim}_{\mathrm{T}}$ is a $\vdash_{\mathrm{T}}$-cumulative tccr, $\Gamma \mathrel{|\!\sim}_{\mathrm{T}} \vee\Delta$. Thus $\Gamma \mathrel{|\!\sim} \vee\Delta$, and by IDR, $\Gamma \mathrel{|\!\sim} \Delta$. This shows that $\mathrel{|\!\sim}$ satisfies Cum. For RW$^{[1]}$, assume that $\Gamma, \psi \vdash \phi$ and $\Gamma \mathrel{|\!\sim} \psi, \Delta$. Since $\vdash$ is an scr and $\vee$ is a combining disjunction for it, the first assumption implies that $\Gamma, \psi \vee (\vee\Delta) \vdash \phi \vee (\vee\Delta)$. By IDR the second assumption implies that $\Gamma \mathrel{|\!\sim} \psi \vee (\vee\Delta)$. Hence $\Gamma, \psi \vee (\vee\Delta) \vdash_{\mathrm{T}} \phi \vee (\vee\Delta)$ and $\Gamma \mathrel{|\!\sim}_{\mathrm{T}} \psi \vee (\vee\Delta)$. By tRW (see Proposition 2.29) applied to $\mathrel{|\!\sim}_{\mathrm{T}}$ we get $\Gamma \mathrel{|\!\sim}_{\mathrm{T}} \phi \vee (\vee\Delta)$. Hence $\Gamma \mathrel{|\!\sim} \phi \vee (\vee\Delta)$. By IDR again, $\Gamma \mathrel{|\!\sim} \phi, \Delta$. It remains to show that $\mathrel{|\!\sim}$ satisfies RM. Suppose then that $\Gamma \mathrel{|\!\sim} \Delta$ and let $\delta \in \Delta$. Then $\Gamma \mathrel{|\!\sim} \Delta, \delta$, and RW$^{[1]}$ with $\Gamma, \delta \vdash \psi \vee \delta$ yields $\Gamma \mathrel{|\!\sim} \psi \vee \delta, \Delta$. Using IDR it easily follows that $\Gamma \mathrel{|\!\sim} \psi, \delta, \Delta$, and since $\delta \in \Delta$ we have that $\Gamma \mathrel{|\!\sim} \psi, \Delta$. ■

Note A careful inspection of the proof of Proposition 2.40 shows that if a combining disjunction is available for $\vdash$, then RM follows from the other conditions for a weakly $\vdash$-cumulative sccr. It follows that in this case Cum, CM, CC$^{[1]}$, and RW$^{[1]}$ suffice for defining a weakly $\vdash$-cumulative sccr.

The last proposition and its proof show, in particular, the following claim:

Corollary 2.41 *Let* $\vdash$ *be an scr with a combining disjunction* $\vee$*, and let* $\mathrel{|\!\sim}$ *be a weakly* $\vdash$*-cumulative sccr. Then* $\vee$ *is an internal disjunction w.r.t.* $\mathrel{|\!\sim}$*.*

Part (a) of the following proposition shows that a similar claim about conjunction also holds:

Proposition 2.42 *Let* $\vdash$ *be an scr with a combining conjunction* $\wedge$*, and let* $\mid\!\sim$ *be a weakly* $\vdash$*-cumulative sccr. Then:*

a) $\wedge$ *is an internal conjunction w.r.t.* $\mid\!\sim$*. I.e.,* $\mid\!\sim$ *satisfies the following property:*

ICR internal conjunction reduction: *for every* $\Gamma \neq \emptyset$*,* $\Gamma \mid\!\sim \Delta$ *iff* $\wedge\Gamma \mid\!\sim \Delta$

b) $\wedge$ *is a "half" combining conjunction w.r.t.* $\mid\!\sim$*. I.e, the following rules are valid for* $\mid\!\sim$*:*[15]

$$[\mid\!\sim \wedge]_{\mathrm{E}} \quad \frac{\Gamma \mid\!\sim \psi \wedge \phi, \Delta}{\Gamma \mid\!\sim \psi, \Delta} \qquad \frac{\Gamma \mid\!\sim \psi \wedge \phi, \Delta}{\Gamma \mid\!\sim \phi, \Delta}$$

PROOF.
a) The proof is similar to that of in the Tarskian case (see Lemma 2.32 and Note 2 after Definition 2.39), using Δ instead of ϕ.
b) $\Gamma \mid\!\sim \psi, \Delta$ is obtained by applying $\mathrm{RW}^{[1]}$ to $\Gamma \mid\!\sim \psi \wedge \phi, \Delta$ and $\Gamma, \psi \wedge \phi \vdash \psi$. Similarly for $\Gamma \mid\!\sim \phi, \Delta$. ■

Note Clearly, the condition ICR in part (a) of Proposition 2.42 is equivalent to the following conditions:

$$[\wedge \mid\!\sim]_{\mathrm{I}} \quad \frac{\Gamma, \psi, \phi \mid\!\sim \Delta}{\Gamma, \psi \wedge \phi \mid\!\sim \Delta} \qquad [\wedge \mid\!\sim]_{\mathrm{E}} \quad \frac{\Gamma, \psi \wedge \phi \mid\!\sim \Delta}{\Gamma, \psi, \phi \mid\!\sim \Delta}$$

Definition 2.43 *Suppose that an scr* $\vdash$ *has a combining conjunction* $\wedge$*. A weakly* $\vdash$*-cumulative sccr* $\mid\!\sim$ *is called* $\{\wedge, \vdash\}$-cumulative *if it satisfies the following condition:*

$$[\mid\!\sim \wedge]_{\mathrm{I}} \quad \frac{\Gamma \mid\!\sim \psi, \Delta \qquad \Gamma \mid\!\sim \phi, \Delta}{\Gamma \mid\!\sim \psi \wedge \phi, \Delta}$$

Corollary 2.44 *If* $\vdash$ *is an scr with a combining conjunction* $\wedge$ *and* $\mid\!\sim$ *is a* $\{\wedge, \vdash\}$*-cumulative sccr, then* $\wedge$ *is a combining conjunction w.r.t.* $\mid\!\sim$ *as well.*

PROOF. Follows from Proposition 2.42(b). ■

As usual, we provide an equivalent notion in which one does not have to assume that a combining conjunction is available:

Definition 2.45 *A weakly* $\vdash$*-cumulative sccr* $\mid\!\sim$ *is called* $\vdash$-cumulative *if for every finite* n *the following condition is satisfied:*

$\mathbf{RW}^{[n]}$ *if* $\Gamma \mid\!\sim \psi_i, \Delta$ $(i = 1, \ldots, n)$ *and* $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$ *then* $\Gamma \mid\!\sim \phi, \Delta$.

Proposition 2.46 *Let* $\wedge$ *be a combining conjunction for* $\vdash$*. An sccr* $\mid\!\sim$ *is* $\{\wedge, \vdash\}$*-cumulative iff it is* $\vdash$*-cumulative.*

PROOF. We have to show that if $\wedge$ is a combining conjunction w.r.t. $\vdash$, then $\mathrm{RW}^{[n]}$ is equivalent to $[\mid\!\sim \wedge]_{\mathrm{I}}$. Suppose first that $\mid\!\sim$ satisfies $[\mid\!\sim \wedge]_{\mathrm{I}}$. From $\Gamma \mid\!\sim \psi_i, \Delta$ $(i = 1, \ldots, n)$ it follows, by $[\mid\!\sim \wedge]_{\mathrm{I}}$, that $\Gamma \mid\!\sim \psi_1 \wedge, \ldots, \wedge \psi_n, \Delta$. From $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$ it follows that $\Gamma, \psi_1 \wedge, \ldots, \wedge \psi_n \vdash \phi$. By a $\mathrm{RW}^{[1]}$ on these two sequents, $\Gamma \mid\!\sim \phi, \Delta$. For the converse, assume that $\Gamma \mid\!\sim \psi, \Delta$ and $\Gamma \mid\!\sim \phi, \Delta$. Since $\Gamma, \psi, \phi \vdash \psi \wedge \phi$, $\mathrm{RW}^{[2]}$ yields that $\Gamma \mid\!\sim \psi \wedge \phi, \Delta$. ■

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[15] The subscripts "I" and "E" in the following rules stand for "Introduction" and "Elimination", respectively.

Corollary 2.47 *If $\vdash$ is an scr with a combining conjunction $\wedge$ and $\mid\!\sim$ is a $\vdash$-cumulative sccr, then $\wedge$ is a combining conjunction and an internal conjunction w.r.t. $\mid\!\sim$.*

PROOF. By Proposition 2.42(a), Corollary 2.44, and Proposition 2.46. ■

Next we consider the dual property, i.e.: conditions for assuring that a combining disjunction $\vee$ w.r.t. an scr $\vdash$ will remain a combining disjunction w.r.t. a weakly $\vdash$-cumulative sccr $\mid\!\sim$. Our first observation is that one direction of the combining disjunction property for $\mid\!\sim$ of $\vee$ yields monotonicity of $\mid\!\sim$:

Lemma 2.48 *Suppose that $\vee$ is a combining disjunction for $\vdash$ and $\mid\!\sim$ is a weakly $\vdash$-cumulative sccr. Suppose also that $\mid\!\sim$ satisfies the following condition:*

$$[\vee\mid\!\sim]_{\mathrm{E}} \qquad \frac{\Gamma, \psi\vee\phi \mid\!\sim \Delta}{\Gamma, \psi \mid\!\sim \Delta} \qquad \frac{\Gamma, \psi\vee\phi \mid\!\sim \Delta}{\Gamma, \phi \mid\!\sim \Delta}$$

Then $\mid\!\sim$ is (left) monotonic.

PROOF. Suppose that $\Gamma \mid\!\sim \Delta$, and let $\gamma \in \Gamma$. Then $\Gamma, \gamma \mid\!\sim \Delta$. Since $\Gamma, \gamma \vdash \gamma\vee\psi$ we have also $\Gamma, \gamma \mid\!\sim \gamma\vee\psi$. Hence, by CM, $\Gamma, \gamma, \gamma\vee\psi \mid\!\sim \Delta$. By $[\vee\mid\!\sim]_{\mathrm{E}}$ this implies that $\Gamma, \gamma, \psi \mid\!\sim \Delta$ and so $\Gamma, \psi \mid\!\sim \Delta$. ■

It follows that requiring $[\vee\mid\!\sim]_{\mathrm{E}}$ from a weakly $\vdash$-cumulative sccr is too strong. It is reasonable, however, to require the other direction of the combining disjunction property:

Definition 2.49 *A weakly $\vdash$-cumulative sccr $\mid\!\sim$ is called* weakly $\{\vee, \vdash\}$-preferential *if it satisfies the following condition, (also denoted by $[\vee\mid\!\sim]_{\mathrm{I}}$):*

Or left $\vee$-introduction: *if* $\Gamma, \psi \mid\!\sim \Delta$ *and* $\Gamma, \phi \mid\!\sim \Delta$, *then* $\Gamma, \psi\vee\phi \mid\!\sim \Delta$.

Unlike in the Tarskian case, this time we are able to provide an equivalent condition in which one does not have to assume that a combining disjunction is available:

Definition 2.50 *Let $\vdash$ be an scr. A weakly $\vdash$-cumulative sccr is called* weakly $\vdash$-preferential *if it satisfies the following rule:*

CC cautious cut*:* *if* $\Gamma \mid\!\sim \psi, \Delta$ *and* $\Gamma, \psi \mid\!\sim \Delta$ *then* $\Gamma \mid\!\sim \Delta$.

Proposition 2.51 *Let $\vdash$ be an scr and let $\mid\!\sim$ be a weakly $\vdash$-cumulative sccr. Then $\mid\!\sim$ is a weakly $\vdash$-preferential sccr iff for every finite n it satisfies* cautious n-cut*:*

CC$^{[n]}$ *if* $\Gamma, \psi_i \mid\!\sim \Delta$ $(i=1, \ldots, n)$ *and* $\Gamma \mid\!\sim \psi_1, \ldots, \psi_n$ *then* $\Gamma \mid\!\sim \Delta$.

PROOF. ($\Leftarrow$) We have to show that $\mid\!\sim$ satisfies CC. Suppose that $\Delta = \{\delta_1, \ldots, \delta_k\}$ for some $k \geq 1$. Since for every $1 \leq i \leq k$ we have that $\Gamma, \delta_i \mid\!\sim \Delta$ and since by assumption $\Gamma, \psi \mid\!\sim \Delta$, a cautious $(k+1)$-cut of these $k+1$ sequents with $\Gamma \mid\!\sim \psi, \Delta$ yields that $\Gamma \mid\!\sim \Delta$.
($\Rightarrow$) Suppose that $\mid\!\sim$ satisfies CC. We show the following stronger condition by induction on n:

If $\Gamma \mid\!\sim \psi_1, \ldots, \psi_n, \Delta_0$ and $\Gamma, \psi_i \mid\!\sim \Delta_i$ $(i=1, \ldots, n)$ then $\Gamma \mid\!\sim \Delta_0, \Delta_1, \ldots, \Delta_n$.

- For the case $n = 1$, assume that $\Gamma \mathrel{|\!\sim} \psi_1, \Delta_0$ and $\Gamma, \psi_1 \mathrel{|\!\sim} \Delta_1$. By RM on each sequent we have that $\Gamma \mathrel{|\!\sim} \psi_1, \Delta_0, \Delta_1$ and $\Gamma, \psi_1 \mathrel{|\!\sim} \Delta_0, \Delta_1$. A CC gives the desired result.
- Assume the claim for n; We prove it for $n{+}1$: Suppose that $\Gamma, \psi_i \mathrel{|\!\sim} \Delta_i$ for $i = 1,\ldots, n{+}1$ and $\Gamma \mathrel{|\!\sim} \psi_1,\ldots, \psi_{n+1}, \Delta_0$. By induction hypothesis applied to the last sequent and $\Gamma, \psi_i \mathrel{|\!\sim} \Delta_i$, for $i = 1,\ldots, n$, we get $\Gamma \mathrel{|\!\sim} \Delta_0, \Delta_1,\ldots, \Delta_n, \psi_{n+1}$. From this and $\Gamma, \psi_{n+1} \mathrel{|\!\sim} \Delta_{n+1}$ we get that $\Gamma \mathrel{|\!\sim} \Delta_0, \Delta_1, \ldots, \Delta_{n+1}$ like in the case of $n = 1$. $\blacksquare$

Note By Proposition 2.20, the single conclusion counterpart of CC$^{[n]}$ is valid for *any* sccr (not only the cumulative or preferential ones).

Proposition 2.52 *Let $\vdash$ be an scr with a combining disjunction $\vee$. A weakly $\vdash$-cumulative sccr $\mathrel{|\!\sim}$ satisfies Or iff it is closed under CC$^{[n]}$ for every finite n.*

Proof. Suppose first that $\mathrel{|\!\sim}$ satisfies Or. Then from $\Gamma, \psi_i \mathrel{|\!\sim} \Delta$ $(i = 1, \ldots, n)$ it easily follows that $\Gamma, \psi_1 \vee \ldots \vee \psi_n \mathrel{|\!\sim} \Delta$. On the other hand, $\Gamma \mathrel{|\!\sim} \psi_1 \vee \ldots \vee \psi_n$ follows from $\Gamma \mathrel{|\!\sim} \psi_1, \ldots, \psi_n$ by IDR and Proposition 2.40. Thus, $\Gamma \mathrel{|\!\sim} \Delta$ by CC$^{[1]}$. For the converse, suppose that $\mathrel{|\!\sim}$ is a weakly $\vdash$-cumulative sccr that satisfies CC$^{[n]}$ for every finite n, and suppose that $\Gamma, \psi \mathrel{|\!\sim} \Delta$ and $\Gamma, \phi \mathrel{|\!\sim} \Delta$. Now, since $\Gamma, \psi \vdash \psi \vee \phi$ then by Cum $\Gamma, \psi \mathrel{|\!\sim} \psi \vee \phi$, and CM with $\Gamma, \psi \mathrel{|\!\sim} \Delta$ yields [1]: $\Gamma, \psi, \psi \vee \phi \mathrel{|\!\sim} \Delta$. Similarly, since $\Gamma, \phi \vdash \psi \vee \phi$ then by Cum and CM with $\Gamma, \phi \mathrel{|\!\sim} \Delta$ we have [2]: $\Gamma, \phi, \psi \vee \phi \mathrel{|\!\sim} \Delta$. Also, since $\Gamma, \psi \vee \phi \vdash \psi, \phi$ then by Cum, [3]: $\Gamma, \psi \vee \phi \mathrel{|\!\sim} \psi, \phi$. A CC$^{[2]}$ of [1], [2], and [3] yields $\Gamma, \psi \vee \phi \mathrel{|\!\sim} \Delta$. $\blacksquare$

Corollary 2.53 *et $\vdash$ be an scr with a combining disjunction $\vee$. An sccr $\mathrel{|\!\sim}$ is weakly $\{\vee, \vdash\}$-preferential iff it is weakly $\vdash$-preferential.*

Proof. By Propositions 2.51 and 2.52. $\blacksquare$

Proposition 2.54 *Let $\vdash$ be an scr. Then $\mathrel{|\!\sim}$ is weakly $\vdash$-preferential iff it satisfies Cum, CM, CC, and RM.*

Proof. One direction is obvious. For the other direction, we have to show that if $\mathrel{|\!\sim}$ satisfies the above conditions then it also satisfies RW$^{[1]}$ and CC$^{[1]}$. For RW$^{[1]}$, assume that $\Gamma, \psi \vdash \phi$ and $\Gamma \mathrel{|\!\sim} \psi, \Delta$. By Cum and RM on the first assumption, $\Gamma, \psi \mathrel{|\!\sim} \phi, \Delta$. By RM on the second assumption, $\Gamma \mathrel{|\!\sim} \psi, \phi, \Delta$. A CC on the last two sequents yields $\Gamma \mathrel{|\!\sim} \phi, \Delta$. We leave the proof of CC$^{[1]}$ to the reader. $\blacksquare$

Corollary 2.55 *Let $\vdash$ be an scr. A relation $\mathrel{|\!\sim}$ is a weakly $\vdash$-preferential iff it satisfies Cum, CM, and the following rule:*

s-AC strong additive cut*:* *if* $\Gamma \mathrel{|\!\sim} \psi, \Delta_1$ *and* $\Gamma, \psi \mathrel{|\!\sim} \Delta_2$ *then* $\Gamma \mathrel{|\!\sim} \Delta_1, \Delta_2$

Proof. Suppose first that $\mathrel{|\!\sim}$ satisfies Cum, CM, and s-AC. By Proposition 2.54 we have to show that $\mathrel{|\!\sim}$ satisfies CC and RM. CC is obtained by taking $\Delta_1 = \Delta_2$ in s-AC. For RM, Suppose that $\Gamma \mathrel{|\!\sim} \Delta$ and let $\delta \in \Delta$. Then $\Gamma \mathrel{|\!\sim} \delta, \Delta$. On the other hand, since $\Gamma, \delta \vdash \delta, \psi$, then by Cum, $\Gamma, \delta \mathrel{|\!\sim} \delta, \psi$. s-AC with $\Gamma \mathrel{|\!\sim} \delta, \Delta$ yields $\Gamma \mathrel{|\!\sim} \psi, \Delta$. For the converse, suppose that $\mathrel{|\!\sim}$ is a weakly $\vdash$-preferential sccr for which $\Gamma \mathrel{|\!\sim} \psi, \Delta_1$ and $\Gamma, \psi \mathrel{|\!\sim} \Delta_2$. By RM, $\Gamma \mathrel{|\!\sim} \psi, \Delta_1, \Delta_2$ and $\Gamma, \psi \mathrel{|\!\sim} \Delta_1, \Delta_2$. Thus, $\Gamma \mathrel{|\!\sim} \Delta_1, \Delta_2$, by CC. $\blacksquare$

We are now ready to introduce our strongest notions of nonmonotonic Scott consequence relation:

Definition 2.56 *Let $\vdash$ be an scr. An sccr $\mathrel{|\!\sim}$ is called* $\vdash$-preferential *iff it satisfies Cum, CM, CC, RM, and RW$^{[n]}$ for every n.*

Proposition 2.57 *Let $\vdash$ be an scr. The following conditions are equivalent:*

a) $\mid\!\sim$ is $\vdash$-preferential,

b) $\mid\!\sim$ is a $\vdash$-cumulative sccr that satisfies CC,

c) $\mid\!\sim$ is a weakly $\vdash$-preferential sccr that satisfies RW$^{[n]}$ for every n.

The proof is left to the reader.

Proposition 2.58 *Let $\vdash$ be an scr and let $\mid\!\sim$ be a $\vdash$-preferential sccr.*

a) A combining conjunction $\wedge$ w.r.t. $\vdash$ is also an internal conjunction and a combining conjunction w.r.t. $\mid\!\sim$.

b) A combining disjunction $\vee$ w.r.t. $\vdash$ is also an internal disjunction and "half" combining disjunction w.r.t. $\mid\!\sim$.[16]

PROOF. Part (a) follows from Corollary 2.47. Part (b) follows from Corollary 2.41 and Corollary 2.53. ■

CC$^{[n]}$ $(n \geq 1)$ is a natural generalization of cautious cut. A dual generalization, which seems equally natural, is given in the following rule from [25]:

$$\text{LCC}^{[n]} \quad \frac{\Gamma \mid\!\sim \psi_1, \Delta \;\;\ldots\;\; \Gamma \mid\!\sim \psi_n, \Delta, \quad \Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta}$$

Definition 2.59 *[25] A binary relation $\mid\!\sim$ is a* plausibility logic *if it satisfies Inclusion ($\Gamma, \psi \mid\!\sim \psi$), CM, RM, and LCC$^{[n]}$ ($n \geq 1$).*

Definition 2.60 *Let $\vdash$ be an scr. A relation $\mid\!\sim$ is called* $\vdash$-plausible *if it is a $\vdash$-preferential sccr and a plausibility logic.*

A more concise characterization of a $\vdash$-plausible relation is given in the following proposition:

Proposition 2.61 *Let $\vdash$ be an scr. A relation $\mid\!\sim$ is $\vdash$-plausible iff it satisfies Cum, CM, RM, and LCC$^{[n]}$ for every n.*

PROOF. Since CC is just LCC$^{[1]}$, we only need to show the derivability for all n of RW$^{[n]}$. So assume that $\Gamma \mid\!\sim \psi_i, \Delta$ $(i=1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$. By Cum and RM this implies that $\Gamma \mid\!\sim \psi_i, \phi, \Delta$ $(i = 1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \phi, \Delta$. Hence $\Gamma \mid\!\sim \phi, \Delta$ follows by LCC$^{[n]}$. ■

Proposition 2.62 *Let $\vdash$ be an scr with a combining conjunction $\wedge$. A relation $\mid\!\sim$ is $\vdash$-preferential iff it is $\vdash$-plausible.*

PROOF. One direction is obvious. By the last proposition, for showing the converse we have to prove that if $\mid\!\sim$ is $\vdash$-preferential and $\vdash$ has a combining conjunction $\wedge$, then $\mid\!\sim$ satisfies LCC$^{[n]}$ for every finite n. This follows from Corollary 2.47 and the following lemma:

Lemma 2.62-A: Let $\mid\!\sim$ be a $\vdash$-preferential sccr, where $\vdash$ is an scr with a combining conjunction $\wedge$. Then $[\mid\!\sim \wedge]_{\text{I}}$ is equivalent to LCC$^{[n]}$.
Proof: $(\Rightarrow)$ If $\Gamma \mid\!\sim \psi_1, \Delta \;\ldots\; \Gamma \mid\!\sim \psi_n, \Delta$ then by $[\mid\!\sim \wedge]_{\text{I}}$, $\Gamma \mid\!\sim \psi_1 \wedge \ldots \wedge \psi_n, \Delta$. Also, if

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[16] I.e., $\mid\!\sim$ satisfies left $\vee$-introduction (but *not* necessarily left $\vee$-elimination).

$\Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \Delta$ then by ICR (see Proposition 2.42(a)), $\Gamma, \psi_1 \wedge \ldots \wedge \psi_n \mid\!\sim \Delta$. By CC, then, $\Gamma \mid\!\sim \Delta$.
($\Leftarrow$) Suppose that $\Gamma \mid\!\sim \psi, \Delta$ and $\Gamma \mid\!\sim \phi, \Delta$. By RM, $\Gamma \mid\!\sim \psi, \psi \wedge \phi, \Delta$ and $\Gamma \mid\!\sim \phi, \psi \wedge \psi, \Delta$. Also, by Cum on $\Gamma, \psi, \phi \vdash \psi \wedge \phi, \Delta$ we have that $\Gamma, \psi, \phi \mid\!\sim \psi \wedge \phi, \Delta$. By LCC$^{[2]}$ on these three sequents, $\Gamma \mid\!\sim \psi \wedge \phi, \Delta$. ∎

Table 1 and Figure 1 summarize the various types of Scott relations considered in this section and their relative strengths. $\vdash$ is assumed there to be an scr, and $\vee$, $\wedge$ are combining disjunction and conjunction (respectively) w.r.t. $\vdash$, whenever they are mentioned.

TABLE 1. Scott relations

consequence relation	general conditions
	valid conditions with $\wedge$ and $\vee$
sccr	**s-R, CM, CC**$^{[1]}$
weakly $\vdash$-cumulative sccr	**Cum, CM, CC**$^{[1]}$**, RW**$^{[1]}$**, RM**
	$[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$
$\vdash$-cumulative sccr	**Cum, CM, CC**$^{[1]}$**, RW**$^{[n]}$**, RM**
	$[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$
weakly $\vdash$-preferential sccr	**Cum, CM, CC, RM**
	$[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$,
$\vdash$-preferential sccr	**Cum, CM, CC, RW**$^{[n]}$**, RM**
	$[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$
$\vdash$-plausible sccr	**Cum, CM, LCC**$^{[n]}$**, RM**
	$[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$
scr extending $\vdash$	**Cum, M, C**
	$[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\vee\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$

3 A semantical point of view

In this section we present a general method of constructing nonmonotonic consequence relations of the strongest type considered in the previous section, i.e.: preferential and plausible sccrs. Our approach is based on a multiple-valued semantics. This will allow us to define in a natural way consequence relations that are not only nonmonotonic, but also paraconsistent (i.e.: capable of reasoning with inconsistency in a nontrivial way).

A basic idea behind our method is that of using a set of *preferential models* for making inferences. Preferential models were introduced by McCarthy [30] and later by Shoham [40, 41] as a generalization of the notion of circumscription. The essential idea is that only a subset of models should be relevant for making inferences from a given theory. These models are the most preferred ones according to some conditions that can be specified syntactically by a set of (usually second-order) propositions, the satisfaction of which yields the exact kind of preference one wants to work with.

Here we choose the preferred models according to preference criteria, specified by preorders on the set of models of a given theory. The resulting consequence relations are shown to be plausible Scott relations.

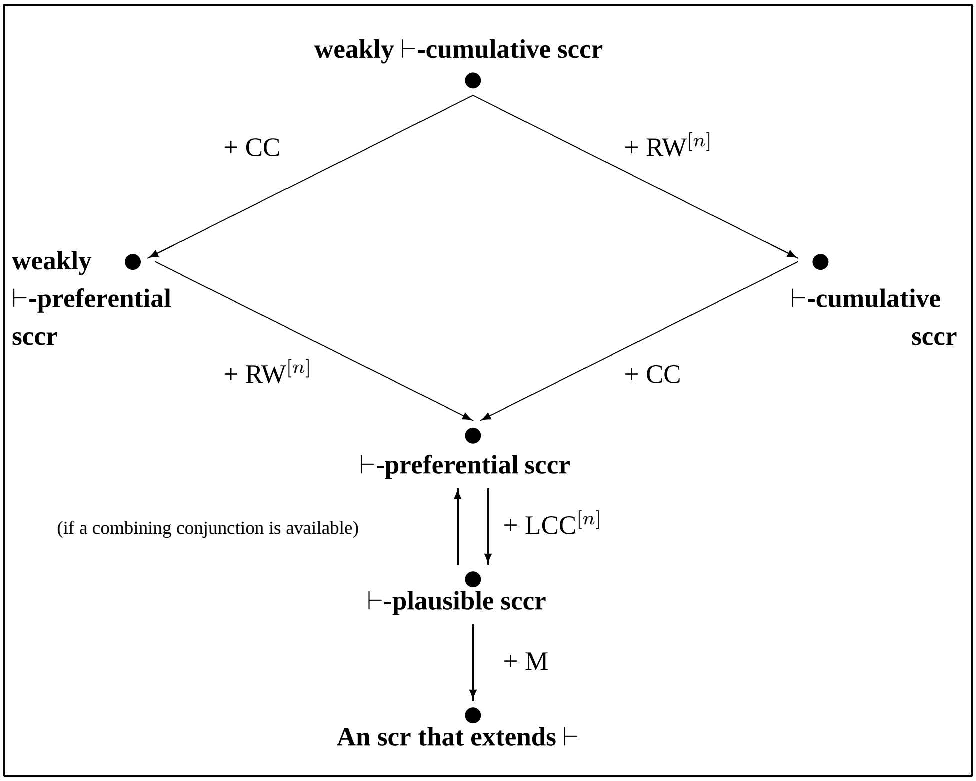

FIG. 1. Relative strength of the Scott relations

3.1 *Multiple-valued models and Scott consequence relations*

Definition 3.1 *Let Σ be an arbitrary propositional language. A* multiple-valued structure *for Σ is a triple $(\mathcal{L}, \mathcal{F}, \mathcal{S})$, where $\mathcal{L}$ is set of elements ("truth values"), $\mathcal{F}$ is a nonempty proper subset of $\mathcal{L}$, and $\mathcal{S}$ is a set of operations on $\mathcal{L}$ that correspond to the connectives in Σ.*

The set $\mathcal{F}$ consists of the *designated* values of $\mathcal{L}$, i.e.: those that represent true assertions. In what follows we shall assume that $\mathcal{L}$ contains at least the classical values t, f, and that $t \in \mathcal{F}$, $f \notin \mathcal{F}$.

Definition 3.2 *Let $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ be a multiple-valued structure, and let Γ be a set of formulae in a language Σ.*

a) A (multiple-valued) valuation *ν is a function that assigns an element of $\mathcal{L}$ to each atomic formula. A valuation is extended to complex formulae in the standard way. The set of all the valuations into $\mathcal{L}$ is denoted by $\mathcal{V}$.*

b) A valuation ν satisfies *a formula ψ (notation: $\nu \models^{\mathcal{L},\mathcal{F}} \psi$) if $\nu(\psi) \in \mathcal{F}$. The relation $\models^{\mathcal{L},\mathcal{F}} \in \mathcal{V} \times \Sigma$ is called a* satisfaction relation.

c) A valuation ν is a model *of Γ (notation: $\nu \models^{\mathcal{L},\mathcal{F}} \Gamma$) if it satisfies every formula in Γ. The set of the models of Γ is denoted by $mod(\Gamma)$.*

Definition 3.3 *Let* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ *be a multiple-valued structure. Denote* $\Gamma \vdash^{\mathcal{L},\mathcal{F}} \Delta$ *if every model of* Γ *is a model of some formula in* Δ*.*

Example 3.4 *Many well-known formalisms correspond to Definition 3.3, especially when a lattice structure is defined on the elements of* $\mathcal{L}$*, and the elements of* $\mathcal{F}$ *form a filter in this lattice. Classical logic, for instance, is obtained by taking the two-valued lattice* $(\{t, f\}, f <_L t)$ *with* $\mathcal{F} = \{t\}$*. For Kleene three-valued logic [22] take* $\mathcal{L} = \{t, f, \bot\}$ *with* $\mathcal{F} = \{t\}$*. The connectives in* $\mathcal{S}$ *correspond to the lattice operations of a lattice in which* $f <_L \bot <_L t$ *together with a negation operation defined by:* $\neg f = t, \neg t = f, \neg\bot = \bot$*. Belnap four-valued logics [9, 10] is obtained from* $\mathcal{L} = \{t, f, \top, \bot\}$*,* $\mathcal{F} = \{t, \top\}$*, and* $\mathcal{S}$ *that contains the lattice operations of the the four-valued lattice in which* $f <_L (\bot, \top) <_L t$*, and a negation operation defined by:* $\neg f = t, \neg t = f, \neg\bot = \bot, \neg\top = \top$*.*

Proposition 3.5 $\vdash^{\mathcal{L},\mathcal{F}}$ *is an scr.*

PROOF. Reflexivity and Monotonicity immediately follow from the definition of $\vdash^{\mathcal{L},\mathcal{F}}$. For cut, assume that $M \in mod(\Gamma_1 \cup \Gamma_2)$. In particular, $M \in mod(\Gamma_1)$, and since $\Gamma_1 \vdash^{\mathcal{L},\mathcal{F}} \psi, \Delta_1$, either $M \models^{\mathcal{L},\mathcal{F}} \delta$ for some $\delta \in \Delta_1$, or $M \models^{\mathcal{L},\mathcal{F}} \psi$. In the former case we are done. In the latter case $M \in mod(\Gamma_2 \cup \{\psi\})$ and since $\Gamma_2, \psi \vdash^{\mathcal{L},\mathcal{F}} \Delta_2$, we have that $M \vdash^{\mathcal{L},\mathcal{F}} \delta$ for some $\delta \in \Delta_2$. ∎

Definition 3.6 *Let* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ *be a multiple-valued structure.*

a) A binary operation $\triangle \in \mathcal{S}$ *is* conjunctive *if for all* $x, y \in \mathcal{L}$*,* $x \triangle y \in \mathcal{F}$ *iff* $x \in \mathcal{F}$ *and* $y \in \mathcal{F}$*.*

b) A binary operation $\nabla \in \mathcal{S}$ *is* disjunctive *if for all* $x, y \in \mathcal{L}$*,* $x \nabla y \in \mathcal{F}$ *iff* $x \in \mathcal{F}$ *or* $y \in \mathcal{F}$*.*

The following result is immediate from the definitions:

Proposition 3.7 *Let* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ *be a multiple-valued structure for a language* Σ*.*

a) If $\wedge$ *is a connective of* Σ *s.t. the corresponding operation of* S *is conjunctive, then* $\wedge$ *is a combining conjunction and an internal conjunction w.r.t.* $\vdash^{\mathcal{L},\mathcal{F}}$*.*

b) If $\vee$ *is a connective of* Σ *s.t. the corresponding operation of* S *is disjunctive, then* $\vee$ *is a combining disjunction and an internal disjunction w.r.t.* $\vdash^{\mathcal{L},\mathcal{F}}$*.*

3.2 Preferential models and Scott cautious consequence relations

3.2.1 The relation $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$

Definition 3.8 *A* preferential system *in a structure* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ *is a triple* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$*, where* $\mathcal{V}$ *is the set of all the valuations on* $\mathcal{L}$*,* $\models^{\mathcal{L},\mathcal{F}} \in \mathcal{V} \times \Sigma$ *is the satisfaction relation defined in 3.2, and* $\preceq$ *is a preorder on* $\mathcal{V}$*.*

Definition 3.9 *Let* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ *be a preferential system in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$*. A valuation* $M \in mod(\Gamma)$ *is a* $\mathcal{P}$-preferential model *of* Γ *if there is no other valuation* $M' \in mod(\Gamma)$ *s.t.* $M' \prec M$*. The set of all the preferential models of* Γ *in* $\mathcal{P}$ *is denoted by* $!(\Gamma, \mathcal{P})$*.*

Definition 3.10 *[29] A preferential system* $\mathcal{P}$ *is called* stoppered[17] *if for every set of formulae* Γ *and every* $M \in mod(\Gamma)$ *there is an* $M' \in !(\Gamma, \mathcal{P})$ *s.t.* $M' \preceq M$*.*

---

[17] In [24] the same property is called *smoothness*.

Note that if $\mathcal{V}$ is well-founded under $\preceq$ (i.e., $\mathcal{V}$ does not have an infinitely descending chain under $\prec$), then $\mathcal{P}$ is stoppered.

Definition 3.11 *Let* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ *be a preferential system in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$*. A set of formulae* Γ $\mathcal{P}$-preferentially entails *a set of formulae* Δ *(notation:* $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \Delta$*) if for every* $M \in\, !(\Gamma, \mathcal{P})$ *there is a* $\delta \in \Delta$ *s.t.* $M \models^{\mathcal{L},\mathcal{F}} \delta$*.*[18] *We say that* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ *is the consequence relation*[19] induced *by* $\mathcal{P}$*.*

Proposition 3.12 *If* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ *is a stoppered preferential system in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$*, then* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ *is a* $\vdash^{\mathcal{L},\mathcal{F}}$*-plausible sccr.*

For proving Proposition 3.12 we first show the following lemma:

Lemma 3.13 *Let* $\mathcal{P}$ *be a preferential system and let* Γ_1, Γ_2 *be two sets of formulae s.t.* $mod(\Gamma_1) \subseteq mod(\Gamma_2)$*. Then* $!(\Gamma_2, \mathcal{P}) \cap mod(\Gamma_1) \subseteq\, !(\Gamma_1, \mathcal{P})$*.*

PROOF. Suppose that $M \in\, !(\Gamma_2, \mathcal{P}) \cap mod(\Gamma_1)$, but $M \notin\, !(\Gamma_1, \mathcal{P})$. Then there is an $N \in mod(\Gamma_1)$ s.t. $N \prec M$. But $mod(\Gamma_1) \subseteq mod(\Gamma_2)$ so $N \in mod(\Gamma_2)$, therefore $M \notin\, !(\Gamma_2, \mathcal{P})$. ∎

PROOF. [of Proposition 3.12] The validity of Cum immediately follows from the definition of $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$. This is also the case with RM. By Proposition 2.61 it remains to show CM, and LCC$^{[n]}$:

- $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ satisfies cautious monotonicity:

Suppose that $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \psi$, and $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \Delta$. Let $M \in\, !(\Gamma \cup \{\psi\}, \mathcal{P})$. In particular, M is a model of Γ. Moreover, $M \in\, !(\Gamma, \mathcal{P})$, since otherwise by the fact that $\mathcal{P}$ is stoppered, there would have been a model $N \in\, !(\Gamma, \mathcal{P})$ that is strictly $\preceq$-smaller than M. Since $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \psi$, this N would have been a model of $\Gamma \cup \{\psi\}$, which is $\preceq$-smaller than M – a contradiction. Thus $M \in\, !(\Gamma, \mathcal{P})$. Now, since $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \Delta$, M is a model of some $\delta \in \Delta$. Hence $\Gamma, \psi \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \Delta$.

- $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ satisfies LCC$^{[n]}$ for every n:

Let $M \in\, !(\Gamma, \mathcal{P})$. If M is a model of some $\delta \in \Delta$ we are done. Otherwise, since $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \psi_i, \Delta$ for $i = 1, \ldots n$, M is a model of $\psi_1, \ldots, \psi_n$. By Lemma 3.13, $M \in\, !(\Gamma \cup \{\psi_1, \ldots, \psi_n\}, \mathcal{P})$. Since $\Gamma, \psi_1, \ldots, \psi_n \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \Delta$, there exists $\delta \in \Delta$ s.t. $M \in mod(\delta)$ in this case as well. ∎

Corollary 3.14 *Let* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ *be a stoppered preferential system in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$*.*

a) If $\wedge$ *is a connective s.t. the corresponding operation of* $\mathcal{S}$ *is conjunctive, then* $\wedge$ *is an internal conjunction and a combining conjunction w.r.t.* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$*.*

b) If $\vee$ *is a connective s.t. the corresponding operation of* $\mathcal{S}$ *is disjunctive, then* $\vee$ *is an internal disjunction w.r.t.* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$*, which satisfies left* $\vee$*-introduction.*

PROOF. By Propositions 3.12 $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ is $\vdash^{\mathcal{L},\mathcal{F}}$-plausible, and so it is obviously a $\vdash^{\mathcal{L},\mathcal{F}}$-preferential sccr. The claim now follows from Proposition 2.58. ∎

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[18] Note that we do *not* require that $M \in\, !(\{\delta\}, \mathcal{P})$, or that $M \in\, !(\Gamma \cup \{\delta\}, \mathcal{P})$.

[19] Here and in what follows we use the notion "consequence relation" in a wider sense than that of Tarski and Scott. In particular, we don't assume monotonicity.

3.2.2 Pointwise preferential systems

Let $\mathcal{P}$ be a preferential system in $(\mathcal{L}, \mathcal{F}, \mathcal{S})$. In Proposition 3.12 we have shown that a sufficient condition for assuring that the consequence relation induced by $\mathcal{P}$ would be a $\vdash^{\mathcal{L},\mathcal{F}}$-plausible sccr is that $\mathcal{P}$ is stoppered. However, as noted in [24] and in [29], it is not easy to check whether this property holds. In what follows we consider another property, which is more easily verified:

Definition 3.15 *A preferential system* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ *in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ *is called* pointwise, *if there is a well-founded partial order* $\leq$ *on* $\mathcal{L}$ *s.t.* $\forall \nu_1, \nu_2 \in \mathcal{V}$ $\nu_1 \preceq \nu_2$ *iff for every atomic formula* p, $\nu_1(p) \leq \nu_2(p)$.

Note If $\mathcal{L}$ is finite, then a preferential system $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ in $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ is pointwise iff there is a partial order $\leq$ on $\mathcal{L}$ s.t. $\forall \nu_1, \nu_2 \in \mathcal{V}$ $\nu_1 \preceq \nu_2$ iff for every atomic formula p, $\nu_1(p) \leq \nu_2(p)$.

Proposition 3.16 *Let* $\mathcal{P}$ *be a pointwise preferential system in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$. *Then* $\mathcal{P}$ *is stoppered.*

PROOF. Suppose that M is some model of Γ. We have to show that there is a model $N \in\ !(\Gamma, \mathcal{P})$ s.t. $N \preceq M$. So let $S_M = \{M_i \mid M_i$ is a model of $\Gamma,\ M_i \preceq M\}$ and let $C \subseteq S_M$ be a chain w.r.t. $\preceq$. We shall show that C is bounded below in S_M, so by Zorn's lemma S_M has a minimal element, which is the required $\preceq$-minimal model. Indeed, define a valuation N as follows: For each atom q let $N(q) = \min_{\leq}\{M_i(q) \mid M_i \in C\}$ ($N(q)$ exists since C is a chain and $\leq$ is well-founded). Obviously N bounds C. It remains to show that $N \in S_M$. Indeed, assume that $\psi \in \Gamma$ and let $\mathcal{A}(\psi) = \{p_1, \ldots, p_n\}$ be the set of the atomic formulae in ψ. For each $1 \leq j \leq n$ let $M_{p_j} \in \{M_i \in C \mid M_i(p_j) = N(p_j)\}$. Then: $N(p_1) = M_{p_1}(p_1), \ldots, N(p_n) = M_{p_n}(p_n)$. Since C is a chain we may assume, without a loss of generality, that $M_{p_1} \succeq \ldots \succeq M_{p_n}$, and so N is the same as M_{p_n} on every atom in $\mathcal{A}(\psi)$. Since M_{p_n} is a model of ψ, so is N. This is true for every $\psi \in \Gamma$ and so $N \in S_M$ as required. ■

Theorem 3.17 *Let* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ *be a pointwise preferential system in* $(\mathcal{L}, \mathcal{F}, \mathcal{S})$. *Then* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ *is* $\vdash^{\mathcal{L},\mathcal{F}}$*-plausible. Moreover:*

a) *If* $\wedge$ *is a connective with a corresponding conjunctive operation in* $\mathcal{S}$, *then* $\wedge$ *is an internal conjunction and a combining conjunction w.r.t.* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$.

b) *If* $\vee$ *is a connective with a corresponding disjunctive operation in of* S *is disjunctive, then* $\vee$ *is an internal disjunction, which satisfies left* $\vee$*-introduction.*

PROOF. By Propositions 3.12, 3.16 and Corollary 3.14. ■

3.3 *Examples*

Many well-known formalisms can be viewed as particular instances of the relation defined in 3.11. In this section we consider some of these formalisms.

In what follows we assume $\mathcal{L}$ to be a lattice and not only an arbitrary set of truth values. We further assume that the set $\mathcal{F}$ of the designated values is a filter on $\mathcal{L}$, and that $\mathcal{S}$ contains the basic lattice operations. The pair $(\mathcal{L}, \mathcal{F})$ is sometimes called a *logical lattice*.[20]

Note that in all the examples below the preferential systems under consideration are pointwise. Thus, by Theorem 3.17, the induced consequence relation is $\vdash^{\mathcal{L},\mathcal{F}}$-plausible.

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[20] To simplify notations we shall omit explicit references to $\mathcal{S}$ in what follows.

Example 3.18 *When taking the two-valued lattice and a degenerated preference order* $\leq$*, then* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ *is the same as the consequence relation of classical logic. Similarly, all the other formalisms of Example 3.4 are obtained from* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$ *by taking the appropriate multi-valued structure and a degenerated preferential order.*

Example 3.19 – Closed Word Assumption
Consider the two-valued lattice $f < t$ *with* t *as the designated element. Define a preferential relation* $\preceq$ *by* $\nu_1 \preceq \nu_2$ *if* $\nu_1(p) \leq \nu_2(p)$ *for every* p*. The preferential models of a theory are here its minimal models, and the induced consequence relation of the system corresponds to Reiter's closed-world assumption [36].*[21]

Example 3.20 – The logic LPm of Priest
Denote by $\vdash^3_{\mathrm{LP}}$ *the consequence relation of the logic LP.*[22] *It is well known that LP invalidates the Disjunctive Syllogism (*$\psi, \neg\psi \vee \phi \not\vdash^3_{\mathrm{LP}} \phi$*). In [34, 35] Priest argues that this is a drawback: a consistent theory should preserve classical conclusions. He suggests to resolve this drawback by considering as the relevant models of a set* Γ *only those that are* minimally inconsistent. *Such models assign the inconsistency value* $\top$ *only to some minimal set of atomic formulae. The consequence relation that is obtained is in our notations* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq}$*, where* $\mathcal{L}$ *is the three-valued lattice* $\{f, t, \top\}$*, in which* $f <_t \top <_t t$*,* $\mathcal{F} = \{t, \top\}$*, and* $\forall \nu_1, \nu_2 \in \mathcal{V}$*,* $\nu_1 \preceq \nu_2$ *iff for every atom* p $\nu_1(p) \leq_k \nu_2(p)$*, where the partial order* $\leq_k$ *is defined by* $f <_k \top$ *and* $t <_k \top$*.*[23]

Example 3.21 – The logic $\vdash^{\mathcal{L},\mathcal{F}}_{k}$
The following family of multiple-valued preferential systems is considered in [3, 5]. The algebraic structures that provide their semantics are sometimes called logical bilattices*. Bilattices were introduced by Ginsberg in [20, 21] as a general framework for a diversity of applications in AI (see also [1, 2, 8, 13, 14]). In these structures there are two partial orders according to which the truth values are represented, and each one of them induces a complete lattice on their common underlying structure. One order is usually denoted by* $\leq_t$*. It intuitively measures differences in the amount of* truth *that the elements represent. The other one is usually denoted by* $\leq_k$*. It is intuitively understood as representing differences in the amount of* knowledge *that each element exhibits. According to Ginsberg ([20, 21]), the two partial orders of a bilattice are related by a negation operation* $\neg$*, which is an involution w.r.t.* $\leq_t$ *(like in many logical lattices) and an order preserving w.r.t.* $\leq_k$*. Logical bilattices is a family of bilattices, proposed in [1, 2], which is particularly useful for constructing bilattice-based* logics*. A logical bilattice is a pair* $(\mathcal{L}, \mathcal{F})$*, where* $\mathcal{L}$ *is a bilattice, and* $\mathcal{F}$ *is a set of designated elements that form a prime* bifilter *in* $\mathcal{L}$ *i.e.: a prime filter w.r.t. both partial orders of* $\mathcal{L}$*.*

Assume now that $\leq_k$ *is well-founded, and let* $\nu_1 \preceq_k \nu_2$ *iff for every atom* p*,* $\nu_1(p) \leq_k \nu_2(p)$*. In the pointwise preferential system* $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq_k)$ *that is obtained,* $!(\Gamma, \preceq_k)$ *is the set of the* $\preceq_k$*-minimal models of* Γ*. In the induced consequence relation* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$ *one draws conclusions according to models that assume minimal knowledge concerning the premises. The intuition behind this approach is that one should not assume anything that is not* really *known.*

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[21] This can be extended to the first-order case in the usual way, in which case the preferential models of a theory are its minimal Herbrand models.

[22] Kleene three-valued logic with middle element designated [22], also known as basic J_3 – see, e.g., [12, Chap.IX] as well as [6, 31, 32, 33]. In the present notations, $\Gamma \vdash^3_{\mathrm{LP}} \Delta$ if $\Gamma \vdash^{\mathcal{L},\mathcal{F}} \Delta$, where $\mathcal{L}$ is a three-valued lattice defined by $f <_t \top <_t t$ and $\mathcal{F} = \{t, \top\}$.

[23] Note that the interpretation of $\vee$ and $\wedge$ are determined by $\leq_t$, while $\preceq$ is defined using $\leq_k$.

Here are some basic properties of $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$:

Proposition 3.22 *[3, 5] let $(\mathcal{L},\mathcal{F})$ be any logical bilattice.*

a) $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$ *is paraconsistent.*

b) $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$ *is nonmonotonic.*

c) If $\inf_k \mathcal{F} \in \mathcal{F}$[24]*, and if the formulae in* Γ, Δ *are in* Σ_{cl}*, then* $\Gamma \vdash^{\mathcal{L},\mathcal{F}} \Delta$ *iff* $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k} \Delta$.

Note In Theorem 3.17 no connection was assumed between the lattice order that defines the semantics of $\vee$ and $\wedge$, and the partial order that underlies $\preceq$. However, in bilattices there are strong connections between the two partial orders. As a result, the condition of the well-foundedness of $\leq_k$ can, in fact, be removed from the definition of a pointwise preferential system in case $(\mathcal{B},\mathcal{F})$ is a logical bilattice, provided that $\inf_k \mathcal{F} \in \mathcal{F}$. See [3] for more details.

Part (c) of the last proposition implies that in Σ_{cl}, in order to check whether $\Gamma \vdash^{\mathcal{L},\mathcal{F}} \Delta$ it is sufficient to consider only the $\preceq_k$-minimal models of Γ. However, as Proposition 3.22(b) shows, in the general case $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$ is *not* equivalent to $\vdash^{\mathcal{L},\mathcal{F}}$. The next proposition (3.24) is another evidence for that. Its proof easily provides an example for the note after Proposition 2.16:

Definition 3.23 *[7, 2] Let $(\mathcal{L},\mathcal{F})$ be a logical [bi-]lattice. Define: $a \supset b = b$ if $a \in \mathcal{F}$, and $a \supset b = t$ otherwise.*[25]

Note It is well known that in multiple-valued semantics it is usually no longer true that every classical tautology remains valid. For instance, in Kleene three-valued logic logic [22], as well as in Belnap 4-valued logic [9, 10], excluded middle is not valid. This implies that when switching to multiple-valued semantics the material implication $\psi \hookrightarrow \phi = \neg\psi \vee \phi$ does not act like an implication connective anymore. As the following proposition implies, $\supset$ *does* function like an implication in logical [bi-]lattices. Note also that on $\{t, f\}$ the material implication $\hookrightarrow$ and the implication connective $\supset$ are identical, and both of them are generalizations of the classical implication.

Proposition 3.24 *Let $(\mathcal{L},\mathcal{F})$ be a logical [bi-]lattice, and let $\supset$ be the connective defined in 3.23. Then:*

a) $\supset$ is an internal implication w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}$: $\Gamma, \psi \vdash^{\mathcal{L},\mathcal{F}} \phi, \Delta$ *iff* $\Gamma \vdash^{\mathcal{L},\mathcal{F}} \psi \supset \phi, \Delta$.

b) $\supset$ is not *an internal implication w.r.t.* $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$.

PROOF. Part (a) immediately follows from the definition of $\supset$. For part (b), consider Belnap four valued bilattice where $f <_t (\top, \bot) <_t t$ and $\bot <_k (t, f) <_k \top$ and $\mathcal{F} = \{t, \top\}$ (see [9, 10] and Example 3.4). For atoms p, q we have that $p \vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k} \neg p \supset q$ (the only $\preceq_k$-minimal model here assigns t to p and $\bot$ to q), while $p, \neg p \not\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k} q$ (a counter-model assigns $\top$ to p and $\bot$ to q). $\blacksquare$

Note Since the $\leq_t$-meet operation is obviously conjunctive in $\mathcal{L}$, then by Corollary 3.17, the corresponding connective $\wedge$ is an internal conjunction and a combining conjunction w.r.t.

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[24] This is true, in particular, whenever $\mathcal{L}$ is finite.

[25] Although we are using the same symbol ($\supset$) for denoting general implication connectives and the specific implication operation defined above, this should not cause any conflicts in the sequel.

$\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$. Similarly, it is possible to define a $\leq_k$-meet operation on $\mathcal{L}$ and by Corollary 3.17, the corresponding connective, $\otimes$, is also an internal conjunction and a combining conjunction w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$. By the same corollary, the connectives $\vee$ and $\oplus$, which respectively correspond to the the $\leq_t$-join and to the $\leq_k$-join on $\mathcal{L}$, are internal disjunctions w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$. Note, however, that like in the case of $\supset$, the connectives $\vee$ and $\oplus$ do not remain a combining disjunction w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$. This follows from Lemma 2.48, since it is shown there that one direction of the combining disjunction property yields monotonicity, whereas $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$ is nonmonotonic. For a specific example that shows that $[\vee\,{\mid\!\sim}]_{\mathrm{E}}$ is not valid, consider again the four-valued bilattice mentioned in the proof of Proposition 3.24(b). Then $(p \wedge \neg p) \vee p \vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k} \neg p \supset f$, while $(p \wedge \neg p) \nvdash^{\mathcal{L},\mathcal{F}}_{\preceq_k} \neg p \supset f$.

Example 3.25 – The logic $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$
Another useful preferential system that is based on logical bilattices is considered in [2, 3]: Let $(\mathcal{L},\mathcal{F})$ be a logical bilattice where $\mathcal{L} = (L, \leq_t, \leq_k)$. A subset $\mathcal{I}$ of L is called an inconsistency set, *if for every $b \in L$, $b \in \mathcal{I}$ iff $\neg b \in \mathcal{I}$, and $b \in \mathcal{F} \cap \mathcal{I}$ iff $b, \neg b \in \mathcal{F}$. Intuitively, $\mathcal{I}$ contains the elements of L that are understood as representing inconsistent knowledge or belief. Define a partial order $\leq_{\mathcal{I}}$ on $\mathcal{L}$ by $a <_{\mathcal{I}} b$ if $a \in \mathcal{L} \setminus \mathcal{I}$ and $b \in \mathcal{I}$. $\leq_{\mathcal{I}}$ is trivially well-founded. In the pointwise preferential system $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq_{\mathcal{I}})$ that is obtained, $!(\Gamma, \preceq_{\mathcal{I}})$ are the models that assume minimal inconsistency (w.r.t. $\mathcal{I}$) of the premises.These models are called the* $\mathcal{I}$-most consistent models *($\mathcal{I}$-mcms, for short) of Γ. The intuition this time is that contradictory data corresponds to inadequate information about the real world, and therefore should be minimized.*[26]

$\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$ might be viewed as a generalization of the three-valued logic LPm of Priest (see Example 3.20).[27] *In our terms, Priest considers the inconsistency set $\mathcal{I} = \{b \in L \mid b \in \mathcal{F}, \neg b \in \mathcal{F}\}$. In the 3-valued case this is the only inconsistency set, and it consists only of $\top$. In the general (multiple-valued) case, however, there are many other inconsistency sets. For a more detailed comparison between the logic of $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$ and LPm, see [3].*

Kifer and Lozinskii [23] also propose a similar relation (denoted there $|\!\approx_{\Delta}$, where Δ denotes the values that are considered as representing inconsistent knowledge). This relation is considered in the framework of annotated logics [42, 43]. See [2] for a discussion on the similarities and the differences between $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$ and $|\!\approx_{\Delta}$.

Proposition 3.26 *[2, 3] For any logical bilattice $(\mathcal{L},\mathcal{F})$ and an inconsistency set $\mathcal{I}$,*

a) $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$ is paraconsistent and nonmonotonic.

b) If Γ and Δ are in the language of $\{\neg, \wedge, \vee, \supset, f, t\}$ and $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}} \Delta$, then the disjunction of the sentences in Δ classically follows from Γ.

c) Let Γ be a classically consistent set in the language of $\{\neg, \wedge, \vee, f, t\}$, and let ψ be a formula in CNF that none of its conjuncts contains an atomic formula and its negation. If ψ classically follows from Γ, then $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}} \psi$.

Again, like in the case of $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_k}$, the connectives $\wedge$, $\otimes$ are internal conjunctions and combining conjunctions w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$, and the connectives $\vee$, $\oplus$ are internal disjunctions w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\preceq_{\mathcal{I}}}$.

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[26] In [2, 3] this preferential system is defined in a somewhat different way. We omit the details here.

[27] Note, however, that the three-valued structure is not a bilattice, but what is sometime called pseudo lower-bilattice [17].

4 Conclusion and further work

In this work we have studied logical approaches to nonmonotonic reasoning, based on the notion of a nonmonotonic consequence relation. We considered a sequence of generalizations of the works of Gabbay [18, 19], Makinson [28], and Kraus, Lehmann, Magidor [24]. These generalizations allow the use of monotonic nonclassical logics as the underlying logic upon which nonmonotonic reasoning may be based. We have found that multiple conclusion consequence relations are the best framework for defining plausible nonmonotonic systems. Our study yields intuitive justifications for the rules of the nonmonotonic systems mentioned above. It also clarifies the connections among some of these systems. For instance, it relates the work in [24] to that of [25].

We have also presented a general method for constructing plausible nonmonotonic relations. This method is based on a multiple-valued semantics, and on Shoham's idea of preferential models. It allows us to define in a uniform way consequence relations that are not only nonmonotonic, but also paraconsistent.

The question whether this semantical approach also characterizes nonmonotonic plausible consequence relations is still open. Formally, is it true that for every scr $\vdash$ and a $\vdash$-plausible sccr $\mid\!\sim$ there is a multiple-valued structure $(\mathcal{L}, \mathcal{F}, \mathcal{S})$ and a (pointwise?) preferential system $\mathcal{P} = (\mathcal{V}, \models^{\mathcal{L},\mathcal{F}}, \preceq)$ such that for every sets of formulae Γ, Δ in a language Σ we have that $\Gamma \mid\!\sim \Delta$ iff $\Gamma \vdash^{\mathcal{L},\mathcal{F}}_{\preceq} \Delta$. This is a matter for a further research.

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