# Averaging Operators on the Unit Interval

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In working with negations and t-norms, it is not uncommon to call upon the arithmetic of the real numbers even though that is not part of the structure of the unit interval as a bounded lattice. To develop a self-contained system, we incorporate an averaging operator, which provides a (continuous) scaling of the unit interval that is not available from the lattice structure. The interest here is in the relations among averaging operators and t-norms, t-conorms, negations, and their generators. © 1999 John Wiley & Sons, Inc.

## 1. INTRODUCTION

An averaging operator is a binary operation $\dot{+}$ on the unit interval that is commutative, strictly increasing in each variable, convex (continuous), idempotent, and bisymmetric. We consider mean systems $(\mathbb{I}, \dot{+})$, where $\dot{+}$ is an averaging operator on the bounded lattice $\mathbb{I} = ([0,1], \leq, 0, 1)$ and note that these algebras have no nontrivial automorphisms.

All averaging operators are isomorphic to the arithmetic mean via an automorphism $\gamma$ of the unit interval (a generator for the averaging operator) that takes the given average of two elements $x$ and $y$ to the arithmetic mean of $\gamma(x)$ and $\gamma(y)$. This characterization and many other facts about averaging operators can be found in the references.[1–9] The averaging operators we consider are not "weighted" averages in the usual sense, although they share some of the basic properties. These averaging operators can be thought of as "skewed" averages. They provide a (continuous) scaling of the unit interval that is not provided by the lattice structure.

We show that each averaging operator on the unit interval naturally defines a negation $\eta$ by the property

$$x \dot{+} \eta(x) = 0 \dot{+} 1$$

and the averaging operator is "self-dual" with respect to this negation. We also relate the averaging operator to the nilpotent t-norms that determine the same

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negation and find a natural one-to-one correspondence between averaging operators and nilpotent t-norms, with corresponding averaging operators and nilpotent t-norms determining the same negation. This correspondence relates the Lukasiewicz t-norm to the arithmetic mean, both of which lead to the standard negation $1 - x$, for example. We consider what happens in the general case.

Each averaging operator on the unit interval induces a binary operation on the group of automorphisms (and also on the set of antiautomorphisms) of the unit interval. We use these induced operations to define special maps from the set of negations to the automorphism group of the unit interval and from the automorphism group of the unit interval onto the centralizer of the negation induced by the given averaging operator. In this new setting, we generalize theorems from an earlier paper[10] where we proved those theorems for the arithmetic means and its corresponding negation $1 - x$.

In the last section we consider de Morgan systems with averaging operators and generalize the families of Frank t-norms and nearly Frank t-norms in this setting.

## 2. AVERAGING OPERATORS ON THE UNIT INTERVAL

We denote by $\mathbb{I}$ the bounded lattice consisting of the unit interval $[0,1]$ with the standard partial order, that is, $\mathbb{I} = ([0,1], \leq, 0, 1)$. To develop systems $(\mathbb{I}, \eta, \dot{+}), (\mathbb{I}, \triangle, \dot{+}), (\mathbb{I}, \triangle, \eta, \dot{+})$ for negations $\eta$ and t-norms $\triangle$ that include the necessary arithmetic as part of the system, we use the following definition, which is a variant of those in the references.

Definition 1. *An averaging operator on* $\mathbb{I}$ *is a binary operation* $\dot{+}\colon \mathbb{I}^2 \to \mathbb{I}$ *satisfying for all* $x, y \in [0,1]$:

(1) $x \dot{+} y = y \dot{+} x$ *($\dot{+}$ is commutative).*
(2) $y < z$ *implies* $x \dot{+} y < x \dot{+} z$ *($\dot{+}$ is strictly increasing in each variable).*
(3) $x \dot{+} y \leq c \leq x \dot{+} z$ *implies there exists* $w \in [y, z]$ *with* $x \dot{+} w = c$ *($\dot{+}$ is convex, i.e., continuous).*
(4) $x \dot{+} x = x$ *($\dot{+}$ is idempotent).*
(5) $(x \dot{+} y) \dot{+} (z \dot{+} w) = (x \dot{+} z) \dot{+} (y \dot{+} w)$ *($\dot{+}$ is bisymmetric).*

The following properties of an averaging operator are well-known.

Proposition 2. *Let* $\dot{+}$ *be an averaging operator. Then for each* $x, y \in [0,1]$:

(1) $x \wedge y \leq x \dot{+} y \leq x \vee y$*—that is, the average of x and y lies between x and y, and*
(2) *the function* $A_x\colon \mathbb{I} \to [x \dot{+} 0, x \dot{+} 1]\colon y \mapsto x \dot{+} y$ *is an isomorphism—that is,* $A_x$ *is an increasing function that is both one-to-one and onto.*

*Proof.* If $x \le y$, then $x \wedge y = x = x \dotplus x \le x \dotplus y \le y \dotplus y = y = x \vee y$. Similarly, if $y \le x$, $x \wedge y \le x \dotplus y \le x \vee y$. Clearly the function $A_x$ is strictly increasing and, in particular, one-to-one. Suppose $x \dotplus 0 \le c \le x \dotplus 1$. Then by convexity, there is a number $w \in [0,1]$ with $x \dotplus w = c$. Thus $A_x$ is onto. ■

The standard averaging operator is the arithmetic mean:

$$\mathrm{av}(x, y) = \frac{x+y}{2}$$

Other examples include the power means and logarithmic means:

$$x \dotplus y = \left(\frac{x^a + y^a}{2}\right)^{1/a}$$

$$x \dotplus y = \log_a(a^x + a^y)$$

Indeed, for any automorphism or anti-automorphism $\gamma$ of $\mathbb{I}$,

$$x \dotplus y = \gamma^{-1}\left(\frac{\gamma(x) + \gamma(y)}{2}\right) = \gamma^{-1}(\mathrm{av}(\gamma(x), \gamma(y)))$$

is an averaging operator.

The preceding example is universal, that is, given an averaging operator $\dotplus$, there is an automorphism $\gamma$ of $\mathbb{I}$ that satisfies

$$\gamma(x \dotplus y) = \frac{\gamma(x) + \gamma(y)}{2}$$

for all $x, y \in [0,1]$. This automorphism can be defined inductively on the collection of elements of $[0,1]$ that are generated by $\dotplus$ from 0 and 1. Such elements can be written uniquely in one of the forms

$$x = 0, \quad x = 1, \quad x = 0 \dotplus 1, \quad \text{or}$$
$$x = \big(\big(\cdots((0 \dotplus 1) \dotplus a_1\big) \dotplus \cdots\big) \dotplus a_{n-1}\big) \dotplus a_n$$

for $a_1, \ldots, a_n \in \{0,1\}$, $n \ge 1$, and $\gamma$ is then defined inductively by

$$\gamma(0) = 0; \quad \gamma(1) = 1;$$
$$\gamma(x \dotplus a) = \frac{\gamma(x) + a}{2} \qquad \text{if } \gamma(x) \text{ is defined and } a \in \{0,1\}$$

The function $\gamma$ satisfies

$$\gamma\big(\big(\cdots((0 \dotplus x_1) \dotplus x_2) \dotplus \cdots\big) \dotplus x_n\big) = \sum_{k=1}^{n} \frac{1}{2^{n-k+1}} x_k$$

where $x_1, \ldots, x_n$ is any sequence of 0's and 1's. Now $\gamma$ is a strictly increasing function on a dense subset of $\mathbb{I}$, and thus $\gamma$ extends uniquely up to an automorphism of $\mathbb{I}$ (see, for example, Aczél,[5] p. 287). Moreover, there were no choices made in the definition of $\gamma$ on the dense subset. Thus, we have the following theorem.

THEOREM 3. *The automorphism $\gamma$ defined above satisfies*

$$\gamma(x \dotplus y) = \frac{\gamma(x) + \gamma(y)}{2}$$

*for all $x, y \in [0,1]$. Thus, every averaging operator on $[0,1]$ is isomorphic to the usual averaging operator on $[0,1]$, that is, the mean systems $(\mathbb{I}, \dotplus)$ and $(\mathbb{I}, \mathrm{av})$ are isomorphic as algebras. Moreover, $\gamma$ is the only isomorphism between $(\mathbb{I}, \dotplus)$ and $(\mathbb{I}, \mathrm{av})$.*

COROLLARY 4. *For any averaging operator $\dotplus$, the automorphism group of $(\mathbb{I}, \dotplus)$ has only one element.*

*Proof.* Suppose that $f$ is an automorphism of $(\mathbb{I}, \dotplus)$. Then $\gamma f$ is an isomorphism of $(\mathbb{I}, \dotplus)$ with $(\mathbb{I}, \mathrm{av})$, so by the previous theorem, $\gamma f = \gamma$. Thus $f = \gamma\gamma^{-1} = 1$. ∎

When an averaging operator is given by the formula

$$x \dotplus y = \gamma^{-1}\left(\frac{\gamma(x) + \gamma(y)}{2}\right)$$

for an automorphism $\gamma$ of $\mathbb{I}$, we will call $\gamma$ a *generator* of the operator $\dotplus$ and write $\dotplus = \dotplus_\gamma$. From the theorem above, the generator of an averaging operator is unique.

## 3. AVERAGING OPERATORS AND AUTOMORPHISMS

THEOREM 5. *If $f, g$ are automorphisms [antiautomorphisms] of $\mathbb{I}$, and $\dotplus$ is an averaging operator on $\mathbb{I}$, then $f \dotplus g$ defined by $(f \dotplus g)(x) = f(x) \dotplus g(x)$ is again an automorphism [antiautomorphism] of $\mathbb{I}$.*

*Proof.* Suppose $f$ and $g$ are automorphisms of $\mathbb{I}$. If $x < y$, then $f(x) < f(y)$ and $g(x) < g(y)$ imply that $f(x) \dotplus g(x) < f(y) \dotplus g(y)$ since $\dotplus$ is strictly increasing in each variable. Thus, the map $f \dotplus g$ is strictly increasing. Also, $(f \dotplus g)(0) = f(0) \dotplus g(0) = 0 \dotplus 0$ and $(f \dotplus g)(1) = f(1) \dotplus g(1) = 1 \dotplus 1 = 1$. It remains to show that $f$ maps $[0,1]$ onto $[0,1]$. Let $y \in [0,1]$. Then $f(0) \dotplus g(0) = 0 \leq y \leq 1 =$

$f(1) \dot{+} g(1)$. Let

$$u = \bigvee\{x \in [0,1]: f(x) \dot{+} g(x) \le y\}$$
$$v = \bigwedge\{x \in [0,1]: f(x) \dot{+} g(x) \ge y\}$$

If $u < w < v$, then $f(w) \dot{+} g(w) > y$ and $f(w) \dot{+} g(w) < y$, an impossibility. Thus, $u = v$ and $f(u) \dot{+} g(u) = y$. This completes the proof for automorphisms. Similar remarks hold if $f$ and $g$ are antiautomorphisms of $\mathbb{I}$. ■

## 4. AVERAGING OPERATORS AND NEGATIONS

In this section we show that each averaging operator naturally determines a negation, with respect to which the averaging operator is self-dual.

THEOREM 6. *For each averaging operator* $\dot{+}$ *on* $\mathbb{I}$, *the equation*

$$x \dot{+} \eta(x) = 0 \dot{+} 1$$

*defines a negation* $\eta = \eta_{\dot{+}}$ *on* $\mathbb{I}$ *with fixed point* $0 \dot{+} 1$.

*Proof.* Since $x \dot{+} 0 = 0 \dot{+} x \le 0 \dot{+} 1 \le x \dot{+} 1$, by Condition (3) of Definition 1, for each $x \in [0,1]$ there is a number $y \in [0,1]$ such that $x \dot{+} y = 0 \dot{+} 1$, and since $A_x$ is strictly increasing, by Proposition 2, there is only one such $y$ for each $x$. Thus, the equation defines a function $y = \eta(x)$. Clearly $\eta(0) = 1$ and $\eta(1) = 0$. Suppose $0 \le x < y \le 1$. We know $x \dot{+} \eta(x) = y \dot{+} \eta(y) = 0 \dot{+} 1$. If $\eta(x) \le \eta(y)$, then $x \dot{+} \eta(x) < y \dot{+} \eta(x) \le y \dot{+} \eta(y)$, which is not the case. Thus, $\eta(x) > \eta(y)$ and $\eta$ is a strictly decreasing function. Now $\eta(\eta(x))$ is defined by $\eta(x) \dot{+} \eta(\eta(x)) = 0 \dot{+} 1$. But also, $\eta(x) \dot{+} x = x \dot{+} \eta(x) = 0 \dot{+} 1$. Thus, applying Proposition 2 part (2) to $\eta(x)$, we see that $\eta(\eta(x)) = x$. It follows that $\eta$ is a negation. If $x$ is the fixed point of $\eta$, then $x = x \dot{+} x = x \dot{+} \eta(x) = 0 \dot{+} 1$. ■

THEOREM 7. *Every homomorphism between mean systems respects the natural negation—that is, is a homomorphism of mean systems with natural negation.*

*Proof.* Suppose $f: (\mathbb{I}, \dot{+}_1) \to (\mathbb{I}, \dot{+}_2)$ is a homomorphism. Then

$$f(x) \dot{+}_2 f\big(\eta_{\dot{+}_1}(x)\big) = f\big(x \dot{+}_1 \eta_{\dot{+}_1}(x)\big) = f(0 \dot{+}_1 1)$$
$$= f(0) \dot{+}_2 f(1) = 0 \dot{+}_2 1$$

Thus, $f(\eta_{\dot{+}_1}(x)) = \eta_{\dot{+}_2}(f(x))$. ■

For this reason, *mean systems with natural negation* $(\mathbb{I}, \dot{+}, \eta_{\dot{+}})$ will be often be referred to simply as *mean systems*.

COROLLARY 8. *If* $\gamma$ *is the generator of* $\dotplus$, *then* $\eta_{\dotplus} = \gamma^{-1}\alpha\gamma$.

*Proof.* From Theorem 7, we see that $\gamma\eta_{\dotplus} = \alpha\gamma$. ■

A negation of the form $\gamma^{-1}\alpha\gamma$ is said to be *generated* by $\gamma$ and will be written as $\alpha_\gamma = \gamma^{-1}\alpha\gamma$.

*Example 9.*

$$\text{For } x \dotplus y = \frac{x+y}{2}, \quad \eta_{\dotplus}(x) = 1 - x$$

$$\text{For } x \dotplus y = \left(\frac{x^a + y^a}{2}\right)^{1/a}, \quad \eta_{\dotplus}(x) = (1 - x^a)^{1/a}$$

$$\text{For } x \dotplus y = \log_a(x^a + y^a), \quad \eta_{\dotplus_a}(x) = \log_a(1 + a - a^x) \quad \text{and}$$

$$\lim_{a \to 1} \eta_{\dotplus_a}(x) = 1 - x$$

The following theorem shows that $\dotplus$ is self-dual with respect to its natural negation—that is,

$$x \dotplus y = \eta_{\dotplus}(\eta_{\dotplus}(y) \dotplus \eta_{\dotplus}(x))$$

THEOREM 10. *Let* $\dotplus$ *be an averaging operator on* $\mathbb{I}$. *Then* $\eta_{\dotplus}$ *is an antiautomorphism of the system* $(\mathbb{I}, \dotplus)$. *Moreover, it is the only antiautomorphism of* $(\mathbb{I}, \dotplus)$.

*Proof.* Let $\eta = \eta_{\dotplus}$. Since $\eta$ is an antiautomorphism of $\mathbb{I}$, we need only show that $\eta(x \dotplus y) = \eta(y) \dotplus \eta(x)$ for all $x, y \in [0,1]$. Now $\eta(x \dotplus y)$ is the unique value satisfying the equation $(x \dotplus y) \dotplus \eta(x \dotplus y) = 0 \dotplus 1$. But by bisymmetry,

$$\begin{aligned}(x \dotplus y) \dotplus (\eta(y) \dotplus \eta(x)) &= (x \dotplus \eta(x)) \dotplus (y \dotplus \eta(y)) \\ &= (0 \dotplus 1) \dotplus (0 \dotplus 1) = 0 \dotplus 1\end{aligned}$$

It follows that $\eta(x \dotplus y) = \eta(y) \dotplus \eta(x)$. The last statement follows from Corollary 4. ■

Let Map($\mathbb{I}$) denote the group of all automorphisms and antiautomorphisms of $\mathbb{I}$. For any subset $S$ of Map($\mathbb{I}$) the set $Z(S) = \{f \in \text{Map}(\mathbb{I})\text{: } fs = sf \text{ for all } s \in S\}$ is the *centralizer* of $S$ in Map($\mathbb{I}$) and is a subgroup of Map($\mathbb{I}$). We are only concerned with cases where $S = \{\eta\}$ is a single antiautomorphism, and we are only interested in those $f$ which are in Aut($\mathbb{I}$), that is, in the centralizer of $\eta$ in Aut($\mathbb{I}$), which is the group

$$Z(\{\eta\}) \cap \text{Aut}(\mathbb{I}) = \{f \in \text{Aut}(\mathbb{I})\text{: } f\eta = \eta f\}$$

For ease of notation, we are going to denote this group by $Z(\eta)$ and refer to it as the centralizer of $\eta$.

Let Neg($\mathbb{I}$) denote the set of all (strong) negations—antiautomorphisms $\beta$ of $\mathbb{I}$ satisfying $\beta(\beta(x)) = x$ for all $x \in [0,1]$. The following three theorems generalize Theorems 2, 22, and 23 of our paper on de Morgan systems,[10] where these theorems are proved for the arithmetic mean and the corresponding negation $\alpha(x) = 1 - x$.

THEOREM 11. *Let* $\dot{+}$ *be an averaging operator on* $\mathbb{I}$ *and let* $\eta$ *be the negation determined by the equation* $x \dot{+} \eta(x) = 0 \dot{+} 1$. *Then the centralizer* $Z(\eta)$ *of* $\eta$ *is the set of elements of the form*

$$\eta f \eta \dot{+} f$$

*for automorphisms* $f$ *of* $\mathbb{I}$. *Moreover, if* $f \in Z(\eta)$, *then* $\eta f \eta \dot{+} f = f$.

*Proof.* To show $\eta f \eta \dot{+} f$ is in the centralizer of $\eta$, we need to show that $(\eta f \eta \dot{+} f)(\eta(x)) = \eta((\eta f \eta \dot{+} f)(x))$. We prove this by showing that $(\eta f \eta \dot{+} f)\eta$ satisfies the defining property for $\eta$, that is, that $(\eta f \eta \dot{+} f)(x) \dot{+} (\eta f \eta \dot{+} f)\eta(x) = 0 \dot{+} 1$. Now

$$(\eta f \eta \dot{+} f)(\eta(x)) = \eta f \eta \eta(x) \dot{+} f\eta(x) = \eta f(x) \dot{+} f\eta(x)$$

and

$$\eta\big((\eta f \eta \dot{+} f)(x)\big) = \eta(\eta f \eta(x) \dot{+} f(x))$$

By bisymmetry,

$$\begin{aligned}\big[\eta f \eta(x) \dot{+} f(x)\big] \dot{+} \big[\eta f(x) \dot{+} f\eta(x)\big] &= \big[\eta f \eta(x) \dot{+} f\eta(x)\big] \dot{+} \big[\eta f(x) \dot{+} f(x)\big] \\ &= [0 \dot{+} 1] \dot{+} [0 \dot{+} 1] = [0 \dot{+} 1]\end{aligned}$$

Thus, the expression $(\eta f \eta \dot{+} f)(\eta(x)) = \eta f(x) \dot{+} f\eta(x)$ satisfies the defining equality for $\eta(\eta f \eta(x) \dot{+} f(x))$, and we conclude that

$$(\eta f \eta \dot{+} f)\eta = \eta(\eta f \eta \dot{+} f)$$

Clearly, if $f \in Z(\eta)$, then $\eta f \eta \dot{+} f = f$. It follows that every element of $Z(\eta)$ is of the form $\eta f \eta \dot{+} f$ for some automorphism $f$ of $\mathbb{I}$. ■

THEOREM 12. *For any negation* $\eta$, *all negations are conjugates of* $\eta$ *by automorphisms of* $\mathbb{I}$. *More specifically, if* $\beta$ *is a negation, then*

$$\beta = f^{-1} \eta f$$

*for* $f$ *the automorphism of* $\mathbb{I}$ *defined by*

$$f(x) = \eta\beta(x) \dot{+} x$$

*where* $\dot{+}$ *is any averaging operator such that* $\eta = \eta_{\dot{+}}$. *Moreover,* $\beta = g^{-1}\eta g$ *if and only if* $gf^{-1} \in Z(\eta)$.

*Proof.* The map $\eta\beta \dotplus \mathrm{id}$ is an automorphism of $\mathbb{I}$ since the composition of two negations is an automorphism and the average of two automorphisms is an automorphism (Theorem 5). To show that $\beta = (\eta\beta \dotplus \mathrm{id})^{-1}\eta(\eta\beta \dotplus \mathrm{id})$, we show that $(\eta\beta \dotplus \mathrm{id})\beta = \eta(\eta\beta \dotplus \mathrm{id})$. For any $x \in [0, 1]$,

$$(\eta\beta \dotplus \mathrm{id})\beta(x) = (\eta\beta\beta(x) \dotplus \beta(x)) = \eta(x) \dotplus \beta(x)$$

and

$$\eta(\eta\beta \dotplus \mathrm{id})(x) = \eta(\eta\beta(x) \dotplus x)$$

Now, by bisymmetry

$$\begin{aligned}[\eta\beta(x) \dotplus x] \dotplus [\eta(x) \dotplus \beta(x)] &= [\eta\beta(x) \dotplus \beta(x)] \dotplus [\eta(x) \dotplus x] \\ &= [0 \dotplus 1] \dotplus [0 \dotplus 1] = [0 \dotplus 1]\end{aligned}$$

Thus, using the defining property of $\eta$, $\eta(x) \dotplus \beta(x) = \eta(\eta\beta(x) \dotplus x)$, or

$$(\eta\beta \dotplus \mathrm{Id})\beta = \eta(\eta\beta \dotplus \mathrm{id})$$

as claimed. ■

The next theorem follows easily.

THEOREM 13. *Let $\dotplus$ be an averaging operator on $\mathbb{I}$, and let $\eta$ be the negation determined by the equation $x \dotplus \eta(x) = 0 \dotplus 1$. The map*

$$\mathrm{Neg}(\mathbb{I}) \to \mathrm{Aut}(\mathbb{I})/Z(\eta) \colon \beta \mapsto Z(\eta)(\eta\beta \dotplus \mathrm{Id})$$

*is a one-to-one correspondence between the negations of $\mathbb{I}$ and the set of right cosets of the centralizer $Z(\eta)$ of $\eta$.*

## 5. AVERAGING OPERATORS AND NILPOTENT T-NORMS

A commutative, associative binary operation $\vartriangle$ on $\mathbb{I}$ is a *convex*, *Archimedean t-norm* if the following conditions hold:

(1) $1 \vartriangle x = x$ for all $x \in [0, 1]$.
(2) The operation $\vartriangle$ is increasing in each variable, that is, if $x, y, x_1, y_1 \in [0, 1]$ with $x \le x_1$ and $y \le y_1$, then $x \vartriangle y \le x_1 \vartriangle y_1$.
(3) The operation $\vartriangle$ is Archimedean, that is, $x \vartriangle x < x$ for all $x \in (0, 1)$.
(4) The operation $\vartriangle$ is convex, that is, if $x \vartriangle y \le c \le x_1 \vartriangle y_1$, there is an $r$ between $x$ and $x_1$ and an $s$ between $y$ and $y_1$ such that $c = r \vartriangle s$.

The condition of convexity for an operation $\mathbb{I}^2 \to \mathbb{I}$ is equivalent to continuity of that binary operation in the usual topology on the unit interval. All of the t-norms and t-conorms we consider are convex and Archimedean.

A t-norm is *nilpotent* if for each $x \in [0, 1)$ there is a positive integer $n$ for which

$$\overbrace{x \triangle x \cdots \triangle x}^{n \text{ times}} = 0$$

or, equivalently, if there exists an element $y \in (0, 1)$ with $x \triangle y = 0$. In a paper on negations and nilpotent t-norms,[11] we showed that a negation is naturally associated with a nilpotent t-norm by the condition

$$\eta_{\triangle}(x) = \bigvee \{ y \colon x \triangle y = 0 \}$$

that is, $x \triangle y = 0$ if and only if $y \leq \eta_{\triangle}(x)$.

NOTATION 14. *We will use the symbol* $\blacktriangle$ *for the Lukasiewicz t-norm*

$$x \blacktriangle y = (x + y - 1) \vee 0$$

*and the symbol* $\alpha$ *for the common negation*

$$\alpha(x) = 1 - x$$

*Recall that for an automorphism* $\gamma$ *of* $\mathbb{I}$, *the nilpotent t-norm* $\blacktriangle_\gamma$ *generated by* $\gamma$ *is defined by*

$$x \blacktriangle_\gamma y = \gamma^{-1}((\gamma(x) + \gamma(y) - 1) \vee 0)$$

*The averaging operator* $\dot{+}_\gamma$ *generated by* $\gamma$ *is defined by*

$$x \dot{+}_\gamma y = \gamma^{-1}\left( \frac{\gamma(x) + \gamma(y)}{2} \right)$$

*and the negation* $\alpha_\gamma$ *generated by* $\gamma$ *is defined by*

$$\alpha_\gamma(x) = \gamma^{-1}\alpha\gamma(x) = \gamma^{-1}(1 - \gamma(x))$$

*Also recall that the negation* $\eta_{\dot{+}}$ *determined by* $\dot{+}$ *is defined by*

$$x \dot{+} \eta_{\dot{+}}(x) = 0 \dot{+} 1$$

It was observed in Theorem 6 that the negation generated by $\gamma$ is the same as the negation associated with the averaging operator $\dot{+}_\gamma$—that is, $\alpha_\gamma = \eta_{\dot{+}_\gamma}$. A similar relationship holds for the nilpotent t-norm $\blacktriangle_\gamma$.

PROPOSITION 15. *For an automorphism* $\gamma$ *of* $\mathbb{I}$, *the negations* $\alpha_\gamma$, $\eta_{\blacktriangle_\gamma}$, *and* $\eta_{\dot{+}_\gamma}$, *coincide*, *that is*,

$$x \blacktriangle_\gamma y = 0 \qquad \text{if and only if } y \leq \alpha_\gamma(x)$$

*and*

$$x \dotplus_\gamma \eta_{\blacktriangle_\gamma}(x) = x \dotplus_\gamma \alpha_\gamma(x) = 0 \dotplus_\gamma 1$$

*Proof.* Since $x \blacktriangle_\gamma y = \gamma^{-1}((\gamma(x) + \gamma(y) - 1) \vee 0)$, we have $x \vartriangle_\gamma y = 0$ if and only if $\gamma(x) + \gamma(y) - 1 \leq 0$ if and only if $\gamma(y) \leq 1 - \gamma(x)$ if and only if $y \leq \gamma^{-1}(1 - \gamma(x)) = \alpha_\gamma(x)$. The last equation follows. ■

We remark that this same negation is often represented in the form

$$\eta(x) = f^{-1}\left(\frac{f(0)}{f(x)}\right)$$

for a *multiplicative generator* $f$ of the nilpotent t-norm. See our paper on negations and nilpotent t-norms,[11] for example.

There are a number of different averaging operators that give the same negation, namely one for each automorphism in the centralizer of that negation. The same can be said for nilpotent t-norms. However, there is a closer connection between averaging operators and nilpotent t-norms than a common negation. Given an averaging operator one can determine the particular nilpotent t-norm that has the same generator, and conversely, as shown in the following theorem. This correspondence is a natural one, that is, it does not depend on the generator. Recall that for a nilpotent t-norm, the function defined by $\eta_\vartriangle(x) = \bigvee\{y\colon x \vartriangle y = 0\}$ is a negation.[11]

THEOREM 16. *The condition*

$$x \vartriangle y \leq z \qquad \text{if and only if } x \dotplus y \leq z \dotplus 1$$

*determines a one-to-one correspondence between nilpotent t-norms and averaging operators, namely, given an averaging operator* $\dotplus$, *define* $\vartriangle_{\dotplus}$ *by*

$$x \vartriangle_{\dotplus} y = \bigwedge\left\{z\colon x \dotplus y \leq z \dotplus 1\right\}$$

*This correspondence preserves generators.*

*Proof.* By Theorem 3, we may assume that $\dotplus = \dotplus_\gamma$ for any automorphism $\gamma$ of $\mathbb{I}$. Then

$$\begin{aligned}
x \vartriangle_{\dotplus} y &= \bigwedge\left\{z\colon x \dotplus_\gamma y \leq z \dotplus_\gamma 1\right\} \\
&= \bigwedge\left\{z\colon \gamma^{-1}\left(\frac{\gamma(x) + \gamma(y)}{2}\right) \leq \gamma^{-1}\left(\frac{\gamma(z) + \gamma(1)}{2}\right)\right\} \\
&= \bigwedge\{z\colon \gamma(x) + \gamma(y) \leq \gamma(z) + 1\} \\
&= \bigwedge\{z\colon \gamma(x) + \gamma(y) - 1 \leq \gamma(z)\} \\
&= \bigwedge\{z\colon (\gamma(x) + \gamma(y) - 1) \vee 0 \leq \gamma(z)\} \\
&= \bigwedge\left\{z\colon \gamma^{-1}((\gamma(x) + \gamma(y) - 1) \vee 0) \leq z\right\} \\
&= \gamma^{-1}((\gamma(x) + \gamma(y) - 1) \vee 0)
\end{aligned}$$

Thus, in particular, $x \vartriangle_{\dot{+}} y$ is a nilpotent t-norm. Moreover, $\vartriangle_{\dot{+}}$ has the same generator as $\dot{+}$. Thus, the one-to-one correspondence $\dot{+}_{\gamma} \leftrightarrow \blacktriangle_{\gamma}$ is the natural one defined in the statement of the theorem. ■

To describe the inverse correspondence directly, that is, without reference to a generating function, given a nilpotent t-norm $\vartriangle$, define a binary operation $*_{\vartriangle}$ by

$$x *_{\vartriangle} y = \bigwedge \{z : z \vartriangle z \leq x \vartriangle y\}$$

and define $\dot{+}_{\vartriangle}$ by

$$x \dot{+}_{\vartriangle} y = (x *_{\vartriangle} y) \wedge \left(\eta_{\vartriangle}(\eta_{\vartriangle}(x) *_{\vartriangle} \eta_{\vartriangle}(y))\right)$$

This definition relies on the fact that for an averaging operator $\dot{+}$, $\eta_{\dot{+}}$ is an antiautomorphism of the system $(\mathbb{I}, \dot{+})$ (Theorem 10) and, in particular, $\dot{+}$ is self-dual relative to $\eta_{\dot{+}}$:

$$x \dot{+} y = \eta_{\dot{+}}\left(\eta_{\dot{+}}(x) \dot{+} \eta_{\dot{+}}(y)\right)$$

The situation with strict t-norms is somewhat more complicated. We explore that in the next section.

## 6. DE MORGAN SYSTEMS WITH AVERAGING OPERATORS

Given a convex, Archimedean t-norm $\vartriangle$ and a negation $\eta$, the operation $\triangledown$ on $\mathbb{I}$ defined by

$$x \triangledown y = \eta(\eta(x) \vartriangle \eta(y))$$

is the *convex*, *Archimedean t-conorm dual to* $\vartriangle$ *relative to* $\eta$. The dual *t*-conorm is *nilpotent* if the t-norm is nilpotent, and strict if the t-norm is strict. An algebra of the form $\mathbb{A} = (\mathbb{I}, \vartriangle, \eta, \triangledown)$, where $\vartriangle$ is a convex, Archimedean t-norm, $\eta$ is a negation (involution), and $\triangledown$ is the t-conorm dual to $\vartriangle$ relative to $\eta$, is called a *de Morgan system*. Since the conorm is determined algebraically by $\vartriangle$ and $\eta$, we will often refer to an algebra $(\mathbb{I}, \vartriangle, \eta)$ as a de Morgan system.

The family of t-norms $\vartriangle$ that satisfy the equation

$$(x \vartriangle y) + (x \triangledown y) = x + y$$

for $x \triangledown y = \alpha(\alpha(x) \vartriangle \alpha(x))$, the t-conorm dual to $\vartriangle$ relative to $\alpha(x) = 1 - x$, are called *Frank t-norms*.[12] Frank showed that this is the one-parameter family of t-norms of the form

$$x \vartriangle_{F_a} y = \log_a \left[1 + \frac{(a^x - 1)(a^y - 1)}{a - 1}\right], \qquad a > 0, a \neq 1$$

with limiting values

$$x \triangle_{F_0} y = x \wedge y$$

$$x \triangle_{F_1} y = xy$$

$$x \triangle_{F_\infty} y = (x + y - 1) \wedge 0$$

NOTATION 17. *Given an automorphism* $\gamma$ *of* $\mathbb{I}$, *the strict t-norm* $\triangle_\gamma$ *generated by* $\gamma$ *is defined by*

$$x \triangle_\gamma y = \gamma^{-1}(\gamma(x)\gamma(y))$$

*If* $\circ$ *is an arbitrary strict t-norm, and* $\gamma$ *is an automorphism of* $\mathbb{I}$, *we will use the notation* $\circ_\gamma$ *for the t-norm defined by*

$$x \circ_\gamma y = \gamma^{-1}(\gamma(x) \circ \gamma(y))$$

Note that all the Frank t-norms for $0 < a < \infty$ are strict. The strict Frank t-norms are generated by functions of the form

$$F_a(x) = \frac{a^x - 1}{a - 1}, \qquad a > 0, a \neq 1$$

$$F_1(x) = x$$

A t-norm $\triangle$ is called *nearly Frank*[13] if there is an isomorphism $h$: $(\mathbb{I}, \triangle, \alpha) \to (\mathbb{I}, \triangle_F, \alpha)$ of de Morgan systems for some Frank t-norm $\triangle_F$, that is, for all $x \in [0, 1]$,

$$h(x \triangle y) = h(x) \triangle_F h(y)$$

$$h\alpha(x) = \alpha h(x)$$

We generalize the notion of Frank t-norm to a de Morgan system with an arbitrary averaging operator $\dot{+}$.

DEFINITION 18. *A system* $(\mathbb{I}, \triangle, \eta, \triangledown, \dot{+})$ *is a Frank system if* $\triangle$ *is a t-norm (nilpotent or strict),* $\eta$ *is a negation,* $\triangledown$ *is a t-conorm,* $\dot{+}$ *is an averaging operator, and the identities*

(1) $x \triangledown y = \eta(\eta(x) \triangle \eta(y))$ [$(\mathbb{I}, \triangle, \eta, \triangledown)$ *is a de Morgan system.*]
(2) $x \dot{+} \eta(x) = 0 \dot{+} 1$ [$(\mathbb{I}, \eta, \dot{+})$ *is a mean system with* $\eta = \eta_{\dot{+}}$.]
(3) $(x \triangle y) \dot{+} (x \triangledown y) = x \dot{+} y$ [*The Frank equation is satisfied.*]

*hold for all* $x, y \in [0, 1]$. *A Frank system will be called a standard Frank system if* $\dot{+} = \mathrm{av} = \dot{+}_{\mathrm{id}}$.

Note that in a standard Frank system $(\mathbb{I}, \vartriangle, \eta, \triangledown, \dot{+})$, $\vartriangle$ is a Frank t-norm (nilpotent or strict) and $\eta = \alpha$. Also note that if $\dot{+}$ is generated by $h \in \mathrm{Aut}(\mathbb{I})$, the Frank equation is

$$h^{-1}\left(\frac{h(x \vartriangle y) + h(x \triangledown y)}{2}\right) = h^{-1}\left(\frac{h(x) + h(y)}{2}\right)$$

which is equivalent to

$$h(x \vartriangle y) + h(x \triangledown y) = h(x) + h(y)$$

If $(\mathbb{I}, \vartriangle, \eta, \triangledown, \dot{+})$ is a Frank system, we will say the reduct $(\mathbb{I}, \vartriangle, \dot{+})$ determines a Frank system, since $\eta$ is determined algebraically by $\dot{+}$, and $\triangledown$ by $\eta$ and $\vartriangle$.

THEOREM 19. *The system $(\mathbb{I}, \circ, \eta, \triangledown, \dot{+})$ is a Frank system if and only if it is isomorphic to a standard Frank system.*

*Proof.* Suppose $(\mathbb{I}, *, \dot{+})$ determines a Frank system. There is an automorphism $g$ of $\mathbb{I}$ such that $\dot{+} = \dot{+}_g$, and $g$ is also an automorphism of Frank systems

$$g\colon \left(\mathbb{I}, \circ, \dot{+}_g\right) \approx \left(\mathbb{I}, \circ_{g^{-1}}, \dot{+}_{\mathrm{id}}\right)$$

where $\dot{+}_{\mathrm{id}} = \mathrm{av}$. Thus $\circ_{g^{-1}}$ is a Frank t-norm. The converse is clear. ■

The Frank systems induce relations

$$R_n\colon \mathrm{Av}(\mathbb{I}) \to \mathrm{Nilp}(\mathbb{I}) \text{ and } R_s\colon \mathrm{Av}(\mathbb{I}) \to \mathrm{Strict}(\mathbb{I})$$

where $\mathrm{Av}(\mathbb{I})$ denotes the set of averaging operators, $\mathrm{Nilp}(\mathbb{I})$ the set of nilpotent t-norms, and $\mathrm{Strict}(\mathbb{I})$ the set of strict t-norms.

COROLLARY 20. *Let $R_n \subset \mathrm{Av}(\mathbb{I}) \times \mathrm{Nilp}(\mathbb{I})$ be the relation defined by $(\dot{+}, \circ) \in R_n$ if and only if $(\mathbb{I}, \circ, \dot{+})$ determines a nilpotent Frank system. Then $(\dot{+}, \circ) \in R_n$ if and only if $x \circ y = \bigwedge\{z\colon x \dot{+} y \le z \dot{+} 1\}$. Moreover, $R_n$ determines a one-to-one correspondence between $\mathrm{Av}(\mathbb{I})$ and $\mathrm{Nilp}(\mathbb{I})$.*

*Proof.* Suppose $(\mathbb{I}, \circ, \dot{+})$ determines a Frank system. By Theorem 19, there is an automorphism $g$ of $\mathbb{I}$ such that $\dot{+} = \dot{+}_g$, and $g$ is also an isomorphism of Frank systems

$$g\colon \left(\mathbb{I}, \circ, \dot{+}_g\right) \approx \left(\mathbb{I}, \circ_{g^{-1}}, \dot{+}_{\mathrm{id}}\right)$$

where $\dot{+}_{\mathrm{id}} = \mathrm{av}$. Thus $\circ_{g^{-1}}$ is a Frank t-norm, whence $\circ_{g^{-1}} = \blacktriangle$ is the Lukasiewicz t-norm and $\circ = \blacktriangle_g$ and $(\mathbb{I}, \circ, \dot{+}) = (\mathbb{I}, \blacktriangle_g, \dot{+}_g)$. Then from Theorem 16, we know that $R_n$ is the natural one-to-one correspondence given by the identity $x \circ y = \bigwedge\{z\colon x \dot{+} y \le z \dot{+}\}$. ■

COROLLARY 21. *Let* $R_s \subset \mathrm{Av}(\mathbb{I}) \times \mathrm{Strict}(\mathbb{I})$ *be the relation defined by* $(\dot{+}, \circ) \in R_s$ *if and only if* $(\mathbb{I}, \circ, \dot{+})$ *determines a strict Frank system. The following are equivalent*:

(1) $(\dot{+}, \circ) \in R_s$.
(2) $\circ = \triangle_{F_a g}$ *and* $\dot{+} = \dot{+}_g$ *for some automorphism* $g$ *of* $\mathbb{I}$ *and* $a \in (0, \infty)$.
(3) $\circ = \triangle_f$ *and* $\dot{+} = \dot{+}_{F_a^{-1} rf}$ *for some automorphism* $f$ *of* $\mathbb{I}$, $a, r \in (0, \infty)$.

*Proof.* If $\circ$ is strict, $\circ = \triangle_f$ for some automorphism $f$ of $\mathbb{I}$. Then, in the notation of the proof of Theorem 19, $\dot{+} = \dot{+}_g$ and $\circ_{g^{-1}} = \triangle_{fg^{-1}} = \triangle_{F_a}$ for some $a \in \mathbb{R}^+$, so $\circ = \triangle_f = \triangle_{F_a g}$. This means $rfg^{-1} = F_a$ for some $r, a \in \mathbb{R}^+$, where $r$ is interpreted as the automorphism $r(x) = x^r$. Thus, $(\mathbb{I}, \circ, \dot{+}) = (\mathbb{I}, \triangle_{F_a g}, \dot{+}_g) = (\mathbb{I}, \triangle_f, \dot{+}_{F_a^{-1} rf})$. ■

Thus, $(\mathbb{I}, \triangle_f, \dot{+}_g)$ determines a strict Frank system if and only if $f$ and $g$ are related by $g \in F_a^{-1} \mathbb{R}^+ f$ for some $a$. Note that for every strict Archimedean, convex to-norm $\circ = \triangle_f$ there is a two-parameter family of Frank systems

$$F_a^{-1} rf \colon \left(\mathbb{I}, \triangle_f, \dot{+}_{F_a^{-1} rf}\right) \approx \left(\mathbb{I}, \triangle_{F_a}, \mathrm{av}\right)$$

and for every averaging operator $\dot{+} = \dot{+}_g$ there is a one-parameter family of strict Frank systems

$$g \colon \left(\mathbb{I}, \triangle_{F_a g}, \dot{+}_g\right) \approx \left(\mathbb{I}, \triangle_{F_a}, \mathrm{av}\right)$$

Also, for every nilpotent Archimedean, convex t-norm $\circ = \blacktriangle_\gamma$ there is a unique Frank system

$$\gamma \colon \left(\mathbb{I}, \blacktriangle_\gamma, \dot{+}_\gamma\right) \approx \left(\mathbb{I}, \blacktriangle, \mathrm{av}\right)$$

Thus, every system of the form $(\mathbb{I}, \triangle)$ or $(\mathbb{I}, \dot{+})$ is a reduct of one or more Frank systems. However, not every de Morgan system can be extended to a Frank system. The following theorems identify those that can. A nilpotent de Morgan system is called a *Boolean system*[11] if the negation is the one naturally determined by the t-norm.

THEOREM 22. *A de Morgan system with nilpotent t-norm can be extended to a Frank system if and only if the system is Boolean. A de Morgan system* $(\mathbb{I}, \circ, \eta)$ *with strict t-norm* $\circ$ *can be extended to a Frank system if and only if there exists* $a \in \mathbb{R}^+$ *such that for* $f, g \in \mathrm{Aut}(\mathbb{I})$ *with* $\circ = \triangle_f$ *and* $\eta = \alpha_g$:

$$F_a^{-1} \mathbb{R}^+ f \cap Z(\alpha) g \neq \varnothing$$

*In this case*,

$$F_a^{-1} \mathbb{R}^+ f \cap Z(\alpha) g = \{h\}$$

*and the Frank system is*

$$\left(\mathbb{I}, \triangle_f, \alpha_g, \dot{+}_h\right) = \left(\mathbb{I}, \triangle_{F_a h}, \alpha_h, \dot{+}_h\right)$$

*Moreover, there is at most one such a. The t-norm in the Frank system is nearly Frank if and only if g is in the centralizer of α.*

*Proof.* If the t-norm in a Frank system is nilpotent, it is generated by the same automorphism as the averaging operator. Thus, the negation is also generated by the same automorphism as the t-norm.

Consider the de Morgan system $(\mathbb{I}, \triangle_f, \alpha_g)$ with strict t-norm. Assume

$$F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha)g \neq \varnothing$$

Then by Theorems 28 and 29 in the de Morgan systems paper[10]

$$F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha)g = \{h\}.$$

Thus, for some $r \in \mathbb{R}^+$ and $k \in Z(\alpha)$

$$F_a^{-1} rf = kg = h$$

Thus, $rf = F_a h$, implying $\triangle_f = \triangle_{F_a h}$. Thus, $(\mathbb{I}, \triangle_f, \alpha_g, \dot{+}_h) = (\mathbb{I}, \triangle_{F_a h}, \alpha_h, \dot{+}_h)$ is a Frank system. If $F_b^{-1}\mathbb{R}^+ f \cap Z(\alpha)g = \{k\}$ for some $b \in \mathbb{R}^+$, then $\triangle_{F_b k} = \triangle_f = \triangle_{F_a h}$, implying that $\triangle_{F_b k h^{-1}} = \triangle_{F_a}$. But no t-norm is both Frank and nearly Frank[13] from which it follows that $kh^{-1} = 1$ and, from that, that $a = b$.[12]

Now suppose $(\mathbb{I}, \triangle_f, \alpha_g)$ can be extended to the Frank system $(\mathbb{I}, \triangle_f, \alpha_g, \dot{+}_h)$. Then $\alpha_g$ is the negation for $\dot{+}_h$ so that $\alpha_g = \alpha_h$ and we have $hg^{-1} \in Z(\alpha)$, or $h = kg$ with $k \in Z(\alpha)$. Then by Corollary 21, $(\mathbb{I}, \triangle_f, \alpha_g, \dot{+}_h) = (\mathbb{I}, \triangle_{F_a h}, \alpha_h, \dot{+}_h)$, so $rf = F_a h$ for some $r, a \in \mathbb{R}^+$, or $F_a^{-1} rf = h = kg \in F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha)g$. ■

The intersection $F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha)g$ may be empty for all $a > 0$. That is the case when the equation $F_a^{-1} rf = hg$ has no solution for $r, a > 0$, and $h \in Z(\alpha)$. For a particular example of this, take $f = \mathrm{id}$, $g(x) = x$ for $0 \le x \le \frac{1}{2}$. Then $F_a^{-1} r = hg$ implies that $h(x) = F_a^{-1} r(x)$ for $0 \le x \le \frac{1}{2}$ and since $h \in Z(\alpha)$, $h(x) = 1 - F_a^{-1} r(1 - x) = \alpha(F_a^{-1} r(\alpha(x)))$ for $\frac{1}{2} \le x \le 1$. But then

$$g(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ \alpha r^{-1} F_a \alpha F_a^{-1} r(x) & \frac{1}{2} \le x \le 1 \end{cases}$$

Now simply choose $g$ such that $g(x_0)$ is not differentiable for some $x_0 \in (\frac{1}{2}, 1)$, and such an equality cannot hold for any choice of $a$ and $r$. So there are de Morgan systems $(\mathbb{I}, \circ_f, \alpha_g)$ that are not reducts of Frank systems.

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