ZERO-FREE POLYNOMIAL APPROXIMATION ON A CHAIN OF JORDAN DOMAINS

P. M. GAUTHIER AND GREG KNESE

ABSTRACT. Sur un compact du plan dont le complémentaire est connexe, est-il possible d'approcher uniformément une fonction continue, holomorphe et sans zéros à l'intérieur, par des polynômes n'ayant aucun zéros sur le compact tout entier? Dans cette note brève, nous rappelons le raport surprenant entre ce problème et l'hypothèse de Riemann et donnons une réponse affirmative pour une "chaine" de domaines de Jordan.

On a compact subset of the plane with connected complement, is it possible to uniformly approximate a continuous function, holomorphic and non-vanishing on the interior, with polynomials non-vanishing on the entire compact set? In this brief note, we recall the surprising connection between this question and the Riemann hypothesis and proceed to provide an affirmative answer for a "chain" of Jordan domains.

1. INTRODUCTION

For a compact set $K \subset \mathbb{C}$, we denote by $A(K)$ the family of continuous functions on K, which are holomorphic on the interior K^o of K. Mergelyan's theorem asserts that every $f \in A(K)$ is uniformly approximable by polynomials if and only if $\mathbb{C} \setminus K$ is connected.

Question 1.1. *Let K be compact subset of $\mathbb{C}$ with connected complement. Suppose $f \in A(K)$ has no zeros on K^o and $\epsilon > 0$. Is there a polynomial p_ϵ with no zeros on K such that $\max_{z \in K} |f(z) - p_\epsilon(z)| < \epsilon$?*

An affirmative answer has been given when K is strictly starlike [3], a closed Jordan domain [1], or a disjoint union of such compacta [4]. In this note, we investigate the case when K is a union of finitely many Jordan domains, not necessarily disjoint. Question 1.1 is related to the following question regarding approximation by vertical translates of the Riemann zeta-function.

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Question 1.2. *Let K be a compact subset of the strip $1/2 < \Re(z) < 1$ with connected complement. Suppose $f \in A(K)$ has no zeros on K^o and $\epsilon > 0$. Is the set of $t > 0$, such that $\max_{x \in K} |f(z) - \zeta(z+it)| < \epsilon$, of positive lower density?*

Recently, Johan Andersson has made the remarkable observation [1] that these two problems are equivalent. Under the stronger hypothesis that f has no zeros on K (rather than on K^o), the answer to Question 1.1 is positive, as an obvious consequence of Mergelyan's Theorem. Under this stronger hypothesis, Question 1.2 also has a positive answer, however this is far from obvious. It is a consequence of Voronin's spectacular universality theorem for the Riemann zeta-function, which has been refined by Bhaskar Bagchi [2] and Steven Mark Gonek [5].

For a measurable set E of positive numbers, we denote by $m(E)$ the measure of E and by $\underline{d}(E)$ and $\overline{d}(E)$ respectively the lower and upper densities of E

$$\underline{d}(E) = \liminf_{T \to \infty} \frac{m(E \cap [0,T])}{T} \qquad \overline{d}(E) = \limsup_{T \to \infty} \frac{m(E \cap [0,T])}{T}.$$

The following result of Bagchi suggests that these problems may be related to the Riemann Hypothesis and therefore might be difficult to solve in complete generality.

Theorem 1.3 (Bagchi). *The following assertions are equivalent.*
1) The Riemann hypothesis is true.
2) For each compact set K with connected complement lying in the strip $1/2 < Re(z) < 1$ and for each $\epsilon > 0$,

$$\overline{d}\left(\{t > 0 : \max_{z \in K} |\zeta(z+it) - \zeta(z)| < \epsilon\}\right) > 0.$$

3) For each compact set K with connected complement lying in the strip $1/2 < Re(z) < 1$ and for each $\epsilon > 0$,

$$\underline{d}\left(\{t > 0 : \max_{z \in K} |\zeta(z+it) - \zeta(z)| < \epsilon\}\right) > 0.$$

For a further discussion of this issue, we refer to [4].

2. Chain of Jordan domains

Our main theorem is the following.

Theorem 2.1. *Let $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$ be a chain of Jordan domains. That is, $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $|i - j| > 1$ and $\overline{\Omega}_i \cap \overline{\Omega}_j$ is a single point if $|i - j| = 1$. Suppose $f \in A(\overline{\Omega})$ and $f(z) \neq 0$, for $z \in \Omega$. Then,*

for each $\epsilon > 0$, there is a polynomial p_ϵ such that $|f - p_\epsilon| < \epsilon$ and $p_\epsilon(z) \neq 0$, for $z \in \overline{\Omega}$.

Let $D_1 = \{z : |z + 1/2| < 1/2\}, D_2 = \{z : |z - 1/2| < 1/2\}$.

We introduce three methods of approximation via three lemmas (whose proofs are trivial). We frequently use $f^{-1}(0)$ in place of $f^{-1}(\{0\})$.

Lemma 2.2. *Let D be a disk, $p \in \partial D, f \in A(\overline{D})$, and $f^{-1}(0) \subset \partial D$. Then, for each $\epsilon > 0$, there exists $f_\epsilon \in A(\overline{D})$ with $|f - f_\epsilon| < \epsilon, f^{-1}(0) \subset \{p\}, f_\epsilon(p) = f(p)$. We say that f_ϵ is an approximation via shrinking toward p.*

Proof. We may assume $D = D_2$ and $p = 0$. Set $f_\epsilon(z) = f(rz)$, for $0 < r < 1$, and r sufficiently near 1. $\qquad\square$

Lemma 2.3. *Let $f \in A(\overline{D}_2), f^{-1}(0) \subset \partial D_2$. Then, for each $\epsilon > 0$, there exists $f_\epsilon \in A(\overline{D}_2)$ such that $|f - f_\epsilon| < \epsilon, f^{-1}(0) \subset \{0, 1\}, f_\epsilon(0) = f(0), f_\epsilon(1) = f(1)$.*

Proof. For $0 < r < 1$ the mapping $L_r : D_2 \to D_2$

$$L_r(z) = \frac{\left(\frac{z}{1-z}\right)^r}{1 + \left(\frac{z}{1-z}\right)^r}$$

maps the disk D_2 onto a lens shaped region with corners at the points 0 and 1 of angle πr. Here the r-th root is chosen so that $1^r = 1$.

Set $f_\epsilon = f \circ L_r$ with r sufficiently close to 1. $\qquad\square$

Lemma 2.4. *Let $f \in A(\overline{D}_2)$ and $\epsilon > 0$. For $\delta = \delta(\epsilon) > 0$, set*

$$f_\epsilon(z) = f(w), \quad where \quad w = \left(1 - \frac{z-1}{z} - i\delta\right)^{-1}.$$

Then, $f_\epsilon(0) = f(0)$ and for sufficiently small δ, we have $|f_\epsilon - f| < \epsilon$. We call such an f_ϵ an approximation of f on $\overline{D}_2$ parabolic at 0.

Proof. Notice that for $\delta = 0$, we have $f_\epsilon = f$. It is then clear that $f_\epsilon \to f$ uniformly as $\delta \to 0$. $\qquad\square$

Removing a zero at a point of contact between two disks is the most technical part of our proof.

Lemma 2.5. *Let $D = D_1 \cup D_2$. Let $f \in A(\overline{D}), f^{-1}(0) \subset \{0, 1\}$. Then, there exists $f_\epsilon \in A(\overline{D})$ such that $|f - f_\epsilon| < \epsilon, f_\epsilon^{-1}(0) \subset \{1\}, f_\epsilon(\pm 1) = f(\pm 1)$.*

Proof. If $f(0) \neq 0$, there is nothing to prove. Set $f_\epsilon = f$.

Suppose $f(0) = 0$. Then, f is constant on neither D_1 nor D_2, for otherwise f would have interior zeros, contrary to the hypothesis. Set

$f_j = f \mid \overline{D}_j, j = 1, 2$. As in the proof of Lemma 2.2, let g_1 be an approximation of f_1 by shrinking $\overline{D}_1$ towards -1 and let g_2 be an approximation of f_2 by shrinking $\overline{D}_2$ towards $+1$. We choose the approximation so that $|f_j - g_j| < \epsilon/3$ on $\overline{D}_j$. We note that $g_j \in A(\overline{D}_j)$, $g_1^{-1}(0) = \emptyset$, $g_2^{-1}(0) \subset \{+1\}$, $g_1(-1) = f_1(-1)$, $g_2(+1) = f_2(+1)$. If $g_1(0) = g_2(0)$, we may set $f_\epsilon = g_j$ on $\overline{D}_j$, for $j = 1, 2$ and the proof is complete.

Suppose $g_1(0) \neq g_2(0)$. Let h_ϵ be an approximation of g_1 on $\overline{D}_1$ parabolic at -1 in the sense of Lemma 2.4. Note that $h_\epsilon(\overline{D}_1) = g_1(\overline{D}_1)$, so h_ϵ omits zero on $\overline{D}_1$. We claim that there are arbitrarily close such approximations such that $h_\epsilon(0)$ is not on the line determined by 0 and $g_2(0)$. If not, it follows from the construction of h_ϵ that $g_1(z)$ is on this line, for all $z \in \partial D_1$ near 0. Therefore, $g_1(\partial D_1)$ is in this line. This can be proved via conformal mapping using the fact that a function analytic on a neighborhood of the closed upper half plane and real valued on an interval of the real line must be real valued on the entire real line. Consequently, $g_1(\overline{D}_1)$ is also in this line. But g_1 is non-constant and hence open on D_1, which is a contradiction. Thus, we may choose h_ϵ such that $|h_\epsilon - g_1| < \epsilon/3$ on $\overline{D}_1$ and $h_\epsilon(0)$ is not on the line determined by 0 and $g_2(0)$.

Note that $|g_2(0) - h_\epsilon(0)| \leq |g_2(0) - f(0)| + |f(0) - g_1(0)| + |g_1(0) - h_\epsilon(0)| < \epsilon$. Let us write $g_2(0) - h_\epsilon(0)$ in polar coordinates:

$$g_2(0) - h_\epsilon(0) = re^{i\alpha},$$

where $r < \epsilon$. We note that 0 is not on the line segment

$$h_\epsilon(0) + te^{i\alpha}, \quad 0 \leq t \leq r$$

by choice of h_ϵ. Consider the pie piece:

$$P = \{te^{i(\alpha+\varphi)} : 0 \leq t \leq r, |\varphi| \leq \delta_1\}.$$

Choose $\delta_1 > 0$ so small that 0 is not on the translated closed pie piece given by

$$h_\epsilon(0) + P.$$

By the continuity of h_ϵ, there is a $\delta_2 > 0$ such that 0 is not in the set

$$(2.1) \qquad h_\epsilon(z) + P, \quad |z| \leq \delta_2, \quad z \in \overline{D}_1.$$

Let us define a mapping $w = \eta(z)$ on $\overline{D}_1$ via a series of transformations

$$z \mapsto z_1 = -\frac{z+1}{z}, \qquad (D_1 \to RHP := \text{right half plane})$$

$$z_1 \mapsto z_2 = z_1^{2\delta_1/\pi}, \quad z_2(1) = 1, \quad (RHP \to \text{ sector with angle } 2\delta_1)$$

$$z_2 \mapsto z_3 = \delta_3 z_2, \quad \delta_3 > 0, \qquad \text{(contraction of the sector)}$$

$$z_3 \mapsto z_4 = r\frac{z_3}{z_3 + 1}, \qquad (\text{sector} \to \text{lens})$$

$$z_4 \mapsto w = e^{i\alpha} z_4. \qquad (\text{rotation of the lens})$$

Thus, $\eta \in A(\bar{D}_1)$ maps $\overline{D}_1$ to a "lens" of angular opening $2\delta_1$, whose end points are $\eta(-1) = 0$ and $\eta(0) = re^{i\alpha}$. The parameter δ_3 will be chosen momentarily.

Define $f_\epsilon(z) = h_\epsilon(z) + \eta(z)$, for $z \in \overline{D}_1$ and $f_\epsilon(z) = g_2(z)$, for $z \in \overline{D}_2$. Then, $f_\epsilon(z) \in A(\overline{D})$ since $\eta(0) = g_2(0) - h_\epsilon(0)$. Also, $|f - f_\epsilon| < 2\epsilon$. On D_2 this is because $f_\epsilon = g_2$. On D_1, this follows since $|f - g_1|, |g_1 - h_\epsilon| < \epsilon/3$ and $|\eta| < \epsilon$. Since ϵ is an arbitrary positive number, there remains only to show that $f_\epsilon^{-1}(0) \subset \{+1\}$ and since we already know that $g_2^{-1}(0) \subset \{+1\}$, it is sufficient to show that $f_\epsilon(z) \neq 0$ for $z \in \overline{D}_1$. We break into cases $|z| \geq \delta_2$ and $|z| \leq \delta_2$.

Let $m = \min\{|h_\epsilon(z)| : z \in \overline{D}_1\}$. We now choose δ_3 so small that η maps the region $\{z \in \overline{D}_1 : |z| \geq \delta_2\}$ into the set $\{w : |w| < m\}$. Then, for $z \in \overline{D}_1, |z| \geq \delta_2$,

$$|f_\epsilon(z)| \geq |h_\epsilon(z)| - |\eta(z)| > m - m = 0.$$

Now, suppose $z \in \overline{D}_1, |z| \leq \delta_2$. Then

$$f_\epsilon(z) = h_\epsilon(z) + \eta(z) = h_\epsilon(z) + te^{i(\alpha+\varphi)},$$

with $|\varphi| \leq \delta_1$ and $t \leq r$, because the lens $\eta(\overline{D}_1)$ lies in the pie piece $|w| \leq r, |\arg w - \alpha| \leq \delta_1$. Thus, by (2.1), $f_\epsilon(z) \neq 0$.

We have shown that $f_\epsilon^{-1}(0) \subset \{+1\}$. Since $\eta(-1) = 0$, we also have that $f_\epsilon(-1) = f(-1)$ and moreover $f_\epsilon(+1) = g_2(+1) = f(+1)$. This concludes the proof. $\square$

Lemma 2.6. *Let $D = D_1 \cup D_2 \cup \cdots \cup D_n$, where the D_j are discs of radius $1/2$ whose respective centers are the points $1/2, 3/2, \cdots, (2n-1)/2$ and whose points of tangency are $1, 2, \cdots, n-1$. Suppose $f \in A(\overline{D})$ and $f(z) \neq 0$, for $z \in D$. Then, for each $\epsilon > 0$, there is an $f_\epsilon \in A(\overline{D})$ such that $|f - f_\epsilon| < \epsilon$ and $f_\epsilon(z) \neq 0$, for $z \in \overline{D}$.*

Proof. Set $f_j = f \mid \overline{D}_j$. By Lemma 2.2, we may assume that $f_1^{-1}(0) \subset \{1\}$ (by "shrinking toward 1") and $f_n^{-1}(0) \subset \{n-1\}$ (by "shrinking toward $n-1$").

By Lemma 2.3, we may assume that, for $j = 2, 3, \cdots, n-1$, we have $f_j^{-1}(0) \subset \{j-1, j\}$.

Now, we proceed by finite induction to eliminate the only possible remaining zeros $1, 2, \cdots, n-1$. Applying Lemma 2.5 to $D_1 \cup D_2$, we may get rid of the the possible zero 1. Then, applying Lemma 2.5 to $D_2 \cup D_3$, we get rid of the possible zero 2. After $n-1$ steps, we have eliminated all possible zeros. This concludes the proof of the lemma. $\square$

Proof of Theorem. It is sufficient to approximate f uniformly by a function $f_\epsilon \in A(\overline{\Omega})$ such that $f_\epsilon(z) \neq 0$, for $z \in \overline{\Omega}$, since such an f_ϵ can in turn be uniformly approximated by polynomials which are zero-free on $\overline{\Omega}$ by Mergelyan's theorem.

For each $j = 1, 2, \cdots, n$, let $\phi_j(w) = z$ be a conformal mapping of the disc D_j from the previous lemma onto the Jordan domain Ω_j. By the Osgood-Carathéodory Theorem, ϕ_j extends to a homeomorphism of $\overline{D}_j$ onto $\overline{\Omega}_j$ and we may assume that ϕ_j maps the points of tangency of D_j with neighboring discs to the points of tangency of Ω_j with neighboring Jordan domains. Let ϕ be the map from $\overline{D}$ to $\overline{\Omega}$, defined by setting $\phi = \phi_j$ on $\overline{D}_j$. Setting $g = f \circ \phi$, we have $g \in A(\overline{D})$ and $g(w) \neq 0$ for $w \in D$. By Lemma 2.6, there is a $g_\epsilon \in A(\overline{D})$ such that $|g - g_\epsilon| < \epsilon$ and $g_\epsilon(w) \neq 0$, for $w \in \overline{D}$. We may set $f_\epsilon(z) = g(\phi^{-1}(z)) = g(w)$. $\square$

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Département de mathématiques et de statistique, Université de Montréal, CP-6128 Centreville, Montréal, H3C3J7, CANADA
  E-mail address: gauthier@dms.umontreal.ca

Department of Mathematics, University of Alabama, Box 870350 Tuscaloosa, AL 35487-0350
  E-mail address: geknese@bama.ua.edu