DISQUISITIONES GENERALES CIRCA SUPERFICIES CURVAS

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1.

Disquisitiones, in quibus de directionibus variarum rectarum in spazio agitur, plerumque ad maius perspicuitatis et simplicitatis fastigium evehuntur, in auxilium vocando superficiem sphaericam radio $= 1$ circa centrum arbitrarium descriptam, cuius singula puncta repraesentare censebuntur directiones rectarum radiis ad illa terminatis parallelarum. Dum situs omnium punctorum in spatio per tres coordinatas determinatur, puta per distantias a tribus planis fixis inter se normalibus, ante omnia consideranda veniunt directiones axium his plains normalium: puncta superficiei sphaericae, quae has directiones axium rapraesentant, per (1), (2), (3) denotabimus; mutua igitur horum distantia erit quadrans. Ceterum axium directiones versus eas partes acceptas supponemus, versus quas coordinatae respondentes crescunt.

2.

Haud inutile erit, quasdam propositiones, quae in huiusmodi quaestionibus usum frequentem offerunt, hic in conspectum producere.

I. Angulus inter duas rectas se secantes mensuratur per arcum inter puncta, quae in superficie sphaerica illarum directionibus respondent.

II. Situs cuiuslibet plani repraesentari potest per circulum maximum in superficie sphaerica, cuius planum illi est parallelum.

III. Angulus inter duo plana aequalis est angulo sphaerico inter circulos maximos illa rapraesentantes, et proin etiam per arcum inter horum circulorum maximorum polos interceptum mensurantur. Et perinde inclinatio rectae ad planum mensuratur per arcum, a puncto, quod respondet directioni rectae, ad circulum maximum, qui plani situm rapraesentat, normaliter ductum.

IV. Denotantibus x, y, z; x', y', z' coordinatas duorum punctorum, r eorundem distantiam, atque L punctum, quod in superficie sphaerica rapraesentat directionem rectae a puncto priore ad posterius ductae, erit

$$x' = x + \cos(1)L, \quad y' = y + \cos(2)L, \quad z' = z + \cos(3)L$$

V. Hinc facile sequitur, haberi generaliter

$$\cos(1)L^2 + \cos(2)L^2 + \cos(3)L^2 = 1$$

nec non, denotante L' quodcunque aliud punctum superficiei sphaericae, esse

$$\cos(1)L.\cos(1)L' + \cos(2)L.\cos(2)L' + \cos(3)L.\cos(3)L' = \cos LL'$$

VI. THEOREMA. *Denotantibus L, L', L'', L''' quatuor puncta in superficie sphaerae, atque A angulum, quem arcus LL', $L''L'''$ in puncto concursus sui formant, erit*

$$\cos LL''.\cos L'L''' - \cos LL'''.\cos L'L''' = \sin LL'.\sin L''L'''.\cos A$$

Dem. Denotet litera A insuper punctum concursus ipsum, statuaturque

$$AL = t, \qquad AL' = t', \qquad AL'' = t'', \qquad AL''' = t'''$$

Habimus itaque:

$$\begin{aligned}
\cos LL'' &= \cos t \cos t'' + \sin t \sin t'' \cos A \\
\cos L'L''' &= \cos t' \cos t''' + \sin t' \sin t''' \cos A \\
\cos LL''' &= \cos t \cos t''' + \sin t \sin t''' \cos A \\
\cos L'L'' &= \cos t' \cos t'' + \sin t' \sin t'' \cos A
\end{aligned}$$

et proin

$$\begin{aligned}
&\cos LL''.\cos L'L''' - \cos LL'''.\cos L'L'' \\
&\quad = \cos A(\cos t \cos t'' \sin t' \sin t''' + \cos t' \cos t''' \sin t \sin t'' \\
&\qquad - \cos t \cos t''' \sin t' \sin t'' - \cos t' \cos t'' \sin t \sin t''') \\
&\quad = \cos A(\cos t \sin t' - \sin t \cos t')(\cos t'' \sin t''' - \sin t'' \cos t''') \\
&\quad = \cos A.\sin(t - t).\sin(t''' - t'') \\
&\quad = \cos A.\sin LL'.\sin L''L'''
\end{aligned}$$

Ceterum quum inde a puncto A bini rami utriusque circuli maximi proficiscantur, duo quidem ibi anguli formantur, quorum alter alterius complementum ad $180°$: sed analysis nostra monstrat, eos ramos adoptandos esse, quorum directiones cum sensu progressionis a puncto L ad L', et a puncto L'' ad L''' consentiunt: quibus intellectis simul patet, quum circuli maximi duobis punctis concurrant, arbitrarium esse, utrum eligatur. Loco anguli A etiam arcus inter polos circolorum maximorum, quorum partes sunt arcus LL', $L''L'''$, adhiberi potest: manifesto iacent, puta vel uterque polus ad dextram iacens, dum a L versus L' atque av L'' versus L''' procedimus, vel uterque ad laevam.

VII. Sint L, L', L'' tria puncta in superficie sphaerica, statuamusque brevitatis caussa

$$\begin{array}{lll} \cos(1)L \;= x, & \cos(2)L \;= y, & \cos(3)L \;= z \\ \cos(1)L' = x', & \cos(2)L' = y', & \cos(3)L' = z' \\ \cos(1)L'' = x'', & \cos(2)L'' = y'', & \cos(3)L'' = z'' \end{array}$$

nec non

$$xy'z'' + x'y''z + x''yz' - xy''z' - x'yz'' - x''y'z = \Delta$$

Designet λ polum circuli maximi, cuius pars est arcus LL', et quidem eum, qui respectu huius arcus similiter iacet, ac punctum (1) respectu arcus (2)(3). Tunc erit, ex theoremate praecedente, $yz' - y'z = \cos(1)\lambda.\sin(2)(2).\sin LL'$, sive, propter $(2)(3) = 90°$,

$$\begin{aligned} yz' - y'z &= \cos(1)\lambda.\sin LL', \text{ et perinde} \\ zx' - z'x &= \cos(2)\lambda.\sin LL' \\ xy' - x'y &= \cos(1)\lambda.\sin LL' \end{aligned}$$

Multiplicando has aequationes resp. per x'', y'', z'' et addendo, obtinemus adiumento theorematis secundi in V prolati.

$$\Delta = \cos\lambda L''.\sin LL'$$

Iam tres casus sunt distinguendi. *Primo,* quoties L'' iacet in eodem circulo maximo, cuius pars est arcus LL', erit $\lambda L'' = 90°$, adeoque $\Delta = 0$. Quoties vero L'' iacet extra circulum illum maximum, aderit casus *secundus*, si est ab eadem parte, a qua est λ, *tertius*, si ab opposita: in his casibus puncta L, L', L'' formabunt triangulum sphaericum, et quidem iacebunt in casu secundo eodem ordine quo puncta (1), (2), (3), in casu tertio vero ordine opposito. Denotando angulos illius trianguli simpliciter per L, L', L'', atque perpendiculum in superficie sphaerica a puncto L'' ad latus LL' ductum per p, erit

$$\sin p = \sin L.\sin LL'' = \sin L'.\sin L'L'', \quad \text{atque} \quad \lambda L'' = 90° \mp p$$

valente signo superiori pro casu secundo, inferiori pro tertio. Hinc itaque colligimus

$$\pm\Delta = \sin L . \sin LL' . \sin LL'' = \sin L' . \sin LL' . \sin L'L'' = \sin L'' . \sin LL'' . \sin L'L''$$

Ceterum manifesto casus primus in secundo vel tertio comprehendi censeri potest, nulloque negotio perspicitur, $\pm\Delta$ exhibere sextuplum soliditatis pyramidis inter puncta L, L', L'' atque centrum sphaerae formatae. Denique hinc facillime colligitur, eandem expressionem $\pm\frac{1}{6}\Delta$ generaliter exprimere soliditatem cuiusvis pyramidis inter initium coordinatarum atque puncta quorum coordinatae sunt x, y, z; x', y', z'; x'', y'', z'', contentae.

3.

Superficies curva apud punctum A in ipsa situm curvatura continua gaudere dicitur, si directiones omnium rectarum ab A ad omnia puncta superficiei ad A infinite parum distantia ductarum infinite parum ab uno eodemque plano per A transiente deflenctuntur: hoc planum superficiem curvam in puncto A *tangere* dicitur. Quodsi hiuc conditioni in aliquo puncto satisfieri nequit, continuitas curvaturae hic interrumpitur, uti e.g. evenit in cuspide coni. Disquisitiones praesentes ad tales superficies curvas, vel ad tales superficiei partes, restringentur, in quibus continuitas curvaturae nullibi interrumpitur. Hic tantummodo observamus, methodos, quae positioni plani tangentis determinandae inserviunt, pro puncits singularibus, in quibus continuitas curvaturae interrumpitur, vim suam perdere, et ad indeterminata perducere debere.

4.

Situs plani tangentis commodissime e situ rectae ipsi in puncto A normalis cognoscitur, quae etiam ipsi superficiei curvae normalis dicitur. Directionem huius normalis punctum L in superficie sphaerae auxiliaris repraesentabimus, atque statuemus

$$\cos(1)L = X, \quad \cos(2)L = Y, \quad \cos(3)L = Z$$

coordinatas puncti A per x, y, z denotamus. Sint porro $x + \mathrm{d}x$, $y + \mathrm{d}y$, $z + \mathrm{d}z$ coordinatae alius puncti in superficie curva A'; $\mathrm{d}s$ ipsium distantia infinite parva ab A; denique λ punctum superficiei sphaericae repraesentans directionem elementi AA'. Erit itaque

$$\mathrm{d}x = \mathrm{d}s . \cos(1)\lambda, \quad \mathrm{d}y = \mathrm{d}s . \cos(2)\lambda, \quad \mathrm{d}z = \mathrm{d}s . \cos(3)\lambda$$

et, quum esse debeat $\lambda L = 90°$,

$$X \cos(1)\lambda + Y \cos(2)\lambda + Z \cos(3)\lambda = 0$$

E combinatione harum aequationum derivamus

$$X \,\mathrm{d}x + Y \,\mathrm{d}y + Z \,\mathrm{d}s = 0$$

Duae habentur methodi generales ad exhibendam indolem superficiei curvae. Methodus *prima* utitur aequatione inter coordinatas x, y, x quam reductam esse supponemus ad formam $W = 0$, ubi W erit functio indeterminatarum x, y, z. Sit differentiale completum functionis W

$$\mathrm{d}W = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

eritque in superficie curva

$$P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = 0$$

et proin

$$P\cos(1)\lambda + Q\cos(2)\lambda + R\cos(3)\lambda = 0$$

Quum haec aequatio, perinde ut ea quam supra stabilivimus, valere debat pro directionibus omnium elementorum $\mathrm{d}s$ in superficie curva, facile perspiciemus, X, Y, Z proportionales esse debere ipsis P, Q, R et proin, quum fiat

$$XX + YY + ZZ = 1$$

erit vel

$$X = \frac{P}{\sqrt{(PP+QQ+RR)}}, \quad Y = \frac{Q}{\sqrt{(PP+QQ+RR)}}, \quad Z = \frac{R}{\sqrt{(PP+QQ+RR)}}$$

vel

$$X = \frac{-P}{\sqrt{(PP+QQ+RR)}}, \quad Y = \frac{-Q}{\sqrt{(PP+QQ+RR)}}, \quad Z = \frac{-R}{\sqrt{(PP+QQ+RR)}}$$

Methodus *secunda* sistit coordinatas in forma functionum duarum variabilium p, q. Supponamus per differentiationem harum functionum prodire

$$\begin{aligned} \mathrm{d}x &= a\mathrm{d}p + a'\mathrm{d}q \\ \mathrm{d}y &= b\mathrm{d}p + b'\mathrm{d}q \\ \mathrm{d}z &= c\mathrm{d}p + c'\mathrm{d}q \end{aligned}$$

quibus valoribus in formula supra data substitutis, obtinemus

$$(aX + bY + cZ)\mathrm{d}p + (a'X + b'Y + c'Z)\mathrm{d}q = 0$$

Quum haec aequatio locum habere debeat independenter a valoribus differentialium $\mathrm{d}p$, $\mathrm{d}q$, manifesto esse debebit

$$aX + bY + cZ = 0, \qquad a'X + b'Y + c'Z = 0$$

unde colligimus, X, Y, Z proportionales esse debere quantitatibus

$$bc' - cb', \qquad ca' - ac', \qquad ab' - ba'$$

Statuendo itaque brevitatis caussa

$$\sqrt{\left((bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2\right)} = \Delta$$

erit vel

$$X = \tfrac{bc'-cb'}{\Delta}, \qquad Y = \tfrac{ca'-ac'}{\Delta}, \qquad Z = \tfrac{ab'-ba'}{\Delta}$$

vel

$$X = \tfrac{cb'-bc'}{\Delta}, \qquad Y = \tfrac{ac'-ca'}{\Delta}, \qquad Z = \tfrac{ba'-ab'}{\Delta}$$

His duabus methodis generalibus accredir *tertia*, ubi una coordinatarum e.g. z exhibetur in forma functionis reliquarum x, y: haec methodus manifesto nihil aliud est, nisi casus specialis vel methodi primae, vel secundae. Quodsi hic statuitur

$$\mathrm{d}z = t\,\mathrm{d}x + u\,\mathrm{d}y$$

erit vel

$$X = \tfrac{-t}{\sqrt{(1+tt+uu)}}, \qquad Y = \tfrac{-u}{\sqrt{(1+tt+uu)}}, \qquad Z = \tfrac{-1}{\sqrt{(1+tt+uu)}}$$

vel

$$X = \tfrac{t}{\sqrt{(1+tt+uu)}}, \qquad Y = \tfrac{u}{\sqrt{(1+tt+uu)}}, \qquad Z = \tfrac{1}{\sqrt{(1+tt+uu)}}$$

5.

Duae solutiones in art. praec. inventae manifesto ad puncta superficiei sphaericae opposita, sive ad directiones oppositas referuntur, quod cum rei natura quadrat, quum normalem ad utramvis plagam superficiei curvae ducere liceat. Quosi duas plagas, superficiei contiguas, inter se distinguere, alteramque exteriorem alteram interiorem vocare placet, etiam utrique normali suam solutionem rite tribuere licebit adiumento theorematis in art, 2 (VII) evoluti, simulatque criterium stabilitum est ad plagam alteram ab altera distinguendam.

In methodo prima tale criterium petendum erit a signo valoris quantitatis W. Scilicet generaliter loquendo superficies curva eas spatii partes, in quibus W valorem positivum obtinet, ab iis dirimet, in quibus valor ipsius W fit negativus. E theoremate illo vero facile colligitur, si W valorem positivum obtineat versus plagam exteriorem, normalisque extrorsum ducta concipiatur, solutionem priorem adoptandam esse. Ceterum in quovis casu facile diiudicabitur, utrum per superficiem integram eadem regula respectu signi ipsius W valeat, an pro diversis partibus diversae: quamdiu coëfficientes P, Q, R valores finitos habent, nec simul omnes tres evanescunt, lex continuitatis vicissitudinem vetabit.

Si methodum secundam sequimur, in superficie curva duo systemata linearum curvarum concipere possumus, alterum, pro quo p est variabilis, q constans; alterum, pro quo q variabilis, p constans: situs mutuus harum linearum respectu plagae exterioris decidere debet, utram solutionem adoptare oporteat. Scilicet quoties tres lineae, puta ramus lineae prioris systematis a puncto A proficiscens crescente p, ramus posterioris systematis a puncto A egrediens crescente q, atque normalis versus plagam exteriorem ducta *similiter* iacent, ut, inde ab origine abscissarum, axes ipsarum x, y, z resp. (e.g. si tum e tribus lineis illis, tum et tribus his, prima sinistrorsum, secunda dextrorsum, tertia sursum directa concipi potest), solution prima adoptari debet; quoties autem situs mutuus trium linearum oppositus est situi mutuo axium ipsarum x, y, z, solutio secunda valebit.

In methodo tertia dispiciendum est, utrum, dum z incrementum positivum accipit, manentibus x et y invariatis, transitus fiat versus plagam exteriorem an interiorem. In casu priore, pro normali extrorsum directa, solutio prima valet, in posteriore secunda.

6.

Sicuti, per translatam directionem normalis in superficiem curvam ad superficiem sphaerae, cuius puncto determinato prioris superficiei respondet punctum determinatum in posteriore, ita etiam quaevis linea, vel quaevis figura in illa repraesentabitur per lineam vel figuram correspondentem in hac. In comparatione duarum figurarum hoc modo sibi mutuo correspondentium, quarum altera quasi imago alterius erit, duo momenta sunt respicienda, alterum, quatenus sola quantitas consideratur, alterum, quaetenus abstrahendo a relationibus quantitativis solum situm contemplamur.

Momentum primum basis erit quarundam notionum, quas in doctrinam de superficiebus curvis recipere utile videtur. Scilicet cuilibet parti superficiei curvae limitibus determinatis cinctae *curvaturam totalem* seu *integram* adscribemus, quae per aream figurae illi in superficie sphaerica respondentem exprimetur. Ab hac curvatura integra probe distinguenda est curvatura quasi specifica, quam nos *mensuram curvaturae* vocabimus: haec posterior ad *punctum* superficiei refertur, et denotabit quotientem qui oritur, dum curvatura integra elementi superficialis puncto adiacentis per aream ipsius elementi dividitur, et proin indicat rationem arearum infinite parvarum in superficie curva et in superficie sphaerica sibi mutuo respondentium. Utilitas harum innovationum per ea, quae in posterum a nobis explicabuntur, abunde, ut speramus, sancietur. Quod vero attinet ad terminologiam, imprimis prospiciendum esse duximus, ut omnis ambiguitas arceatur, quapropter haud congruum putavimus, analogiam terminologiae in doctrina de lineis curvis planis vulgo receptam (etsi non omnibus probatam) stricte sequi, secundum quam mensura curvaturae simpliciter audire debuisset curvatura, curvatura integra autem amplitudo. Sed quidni in verbis faciles esse liceret, dummodo res non sint inanes, neque dictio interpretationi erroneae obnoxia?

Situs figurae in superficie sphaerica vel similis esse potest sutui figurae respondentis in superficie curva, vel oppositus (inversus); casus prior locum habet, ubi binae lineae in superficie curva ab eodem puncto directionibus inaequalibus sed non oppositis proficiscentes repraesentantur in superficie sphaerica per lineas similiter iacentes, puta ubi imago lineae ad dextram iacentis ipsa est ad dextram; casus posterior, ubi contratium valet. Hos duos casus per *signum* mensurae curvaturae vel positivum vel negativum distinguemus. Sed manifesto haec distinctio eatenus tantum locum habere potest, quatenus in utraque superficie plagam determinatam eligimus, in qua figura concipi debet. In sphaera auxiliari sempre plagam exteriorem, a centro aversam, adhibebimus: in superficie curva etiam plaga exterior sive quae tamquam exterior consideratur, adoptari potest, vel potius plaga eadem, a qua normalis erecta concipitur; manifesto enim respectu similitudinis figurarum nihil mutatur, si in superficie curva tum figura ad plagam oppositam transfertur, tum normalis, dummodo ipsius imago semper in eadem plaga superficiei sphaericae depingatur.

Signum positivum vel negativum, quod pro situ figurae infinite parvae *mensurae* curvaturae adscribimus, etiam ad curvaturam integram figurae finitae in superficie curva extendimus. Attamen si argumentum omni generalitate amplecti suscipimus, quaedam dilucidationes requiruntur, quas hic breviter tantum attingemus. Quamdiu figura in superficie curva ira comparata est, ut singulis punctis intra ipsam puncta *diversa* in superficie sphaerica respondeant, definition ulteriore explicatione non indiget. Quoties autem conditio ista locum non habet, necesse erit, quasdam partes figurae in superficie sphaerica bis vel pluries in computum ducere, unde, pro situ simili vel opposito, vel accumulatio vel destructio oriri poterit. Simplicissimum erit in tali casu, figuram in superficie curva in partes tales divisam concipere, quae singulae per se spectatae conditioni illi satisfaciant, singulis tribuere curvaturam suam integram, quantitate per aream figurae in superficie sphaerica respondentis, signo per situm determinatis, ac denique figurae toti adscribere curvaturam integram ortam per additionem curvaturarum integrarum, quae singulis partibus respondent. Generaliter itaque curvatura integra figurae est $= \int k \, \mathrm{d}\sigma$, denotante $\mathrm{d}\sigma$ elementum areae figurae, k mensuram curvaturae in quovis puncto. Quod vero attinet ad repraesentationem geometricam huius integralis, praecipua huius rei momenta ad sequentia redeunt. Peripheriae figurae in superficie curva (sub restrictione art. 3) semper respondebit in superficie sphaerica linea in se ipsam rediens. Quea si se ipsam nullibi intersecat, totam superficiem sphaericam in duas partes dirimet, quarum altera respondebit figurae in superficie curva, et cuius area, positive vel negative accipienda, prout respectu peripheriae suae similiter iacet ut figura in superficie curva respectu suae, vel inverse, exhibebit posterioris curvaturam integram. Quoties vero linea ista se ipsam semel vel pluries secat, exhibebit figuram complicatam, cui tamen area certa aeque legitime tribui potest, ac figuris absque nodis, haecque area, rite intellecta, semper valorem iustum curvaturae integrae exhibebit. Attamen uberiorem huius argumenti de figuris generalissime concepris expositionem ad aliam occasionem nobis reservare debemus.

7.

Investigemus iam formulam ad exprimendam mensuram curvaturae pro quovis puncto superficiei curvae. Denotante $\mathrm{d}\sigma$ aream elementi huius superficiei, $Z\,\mathrm{d}\sigma$ erit area proiectionis huius elementi in planum coordinatarum x, y; et perinde, si $\mathrm{d}\Sigma$ est area elementi respondentis in superficie sphaerica, erit $Z\,\mathrm{d}\sigma$ area proiectonis ad idem planum: signum positivum vel negativum ipsius Z vero indicabit situm proiectionis similem vel oppositum situi elementi proiecti: manifesto itaque illae proiectiones eandem rationem quoad quantitatem, simulque eandem relationem quoad situm, inter se tenent, ut elementa ipsa. Consideremus iam elementum triangulare in superficie curva, supponamusque coordinatas trium punctorum, quea formant ipsius proiectionem, esse

$$\begin{array}{ll} x, & y \\ x + \mathrm{d}x, & y + \mathrm{d}y \\ x + \delta x, & y + \delta y \end{array}$$

Duplex area huius trianguli exprimetur per formulam

$$\mathrm{d}x.\delta y - \mathrm{d}y.\delta x$$

et quidem in forma positiva vel negativa, prout situs lateris a puncto primo ad tertium respectu lateris a puncto primo ad secundum similis vel oppositus est situi axis coordinatarum y respectu axis coordinatarum x.

Perinde si coordinatae trium punctorum, quae formant proiectionem elementi respondentis in superficiei sphaerica, a centro sphaerae inchoatae, sun

$$\begin{array}{ll} X, & Y \\ X + \mathrm{d}X, & Y + \mathrm{d}Y \\ X + \delta X, & Y + \delta Y \end{array}$$

Duplex area huius proiectionis exprimetur per

$$\mathrm{d}X.\delta Y - \mathrm{d}Y.\delta X$$

de cuius expressionis signo eadem valent quae supra. Quocirca mensura curvaturae in hoc loco superficiei curvae erit

$$k = \tfrac{\mathrm{d}X.\delta Y - \mathrm{d}Y.\delta X}{\mathrm{d}x.\delta y - \mathrm{d}y.\delta x}$$

Quodsi iam supponimus, indolem superficiei curvae datam esse secundum modum terium in art. 4 consideratum, habebuntur X et Y in forma functionum quantitatum x, y, unde erit

$$\begin{aligned} \mathrm{d}X &= (\tfrac{\mathrm{d}X}{\mathrm{d}x})\,\mathrm{d}x + (\tfrac{\mathrm{d}X}{\mathrm{d}y})\,\mathrm{d}y \\ \delta X &= (\tfrac{\mathrm{d}X}{\mathrm{d}x})\,\delta x + (\tfrac{\mathrm{d}X}{\mathrm{d}y})\,\delta y \\ \mathrm{d}Y &= (\tfrac{\mathrm{d}Y}{\mathrm{d}x})\,\mathrm{d}x + (\tfrac{\mathrm{d}Y}{\mathrm{d}y})\,\mathrm{d}y \\ \delta X &= (\tfrac{\mathrm{d}Y}{\mathrm{d}x})\,\delta x + (\tfrac{\mathrm{d}Y}{\mathrm{d}y})\,\delta y \end{aligned}$$

Substitutis his valoribus, expressio praecedens transit in hanc

$$k = (\tfrac{\mathrm{d}X}{\mathrm{d}x})(\tfrac{\mathrm{d}Y}{\mathrm{d}y}) - (\tfrac{\mathrm{d}X}{\mathrm{d}y})(\tfrac{\mathrm{d}Y}{\mathrm{d}x})$$

Statuendo ut supra

$$\frac{\mathrm{d}\,z}{\mathrm{d}\,x} = t, \qquad \frac{\mathrm{d}\,z}{\mathrm{d}\,y} = u$$

atque insuper

$$\frac{\mathrm{d}\,\mathrm{d}\,z}{\mathrm{d}\,x^2} = T, \qquad \frac{\mathrm{d}\,\mathrm{d}\,z}{\mathrm{d}\,x.\mathrm{d}\,y} = U, \qquad \frac{\mathrm{d}\,\mathrm{d}\,z}{\mathrm{d}\,y^2} = V$$

sive

$$\mathrm{d}t = T\,\mathrm{d}x + U\,\mathrm{d}y, \qquad \mathrm{d}u = U\,\mathrm{d}x + V\,\mathrm{d}y$$

habemus ex formulis supra datis

$$X = -tZ, \qquad Y = -uZ, \qquad (1+tt+uu)ZZ = 1$$

atque hinc

$$\begin{aligned}
\mathrm{d}X &= -Z\,\mathrm{d}t - t\,\mathrm{d}Z\\
\mathrm{d}Y &= -Z\,\mathrm{d}u - u\,\mathrm{d}Z\\
(1+tt+uu)\,\mathrm{d}Z &+ Z(t\,\mathrm{d}t + u\,\mathrm{d}u) = 0
\end{aligned}$$

sive

$$\begin{aligned}
\mathrm{d}Z &= -Z^3(t\,\mathrm{d}t + u\,\mathrm{d}u)\\
\mathrm{d}X &= -Z^3(1+uu)\,\mathrm{d}t + Z^3tu\,\mathrm{d}u\\
\mathrm{d}Y &= +Z^3tu\,\mathrm{d}t - Z^3(1+tt)\,\mathrm{d}u
\end{aligned}$$

adeoque

$$\begin{aligned}
\frac{\mathrm{d}\,X}{\mathrm{d}\,x} &= Z^3(-(1+uu)T + tuU)\\
\frac{\mathrm{d}\,X}{\mathrm{d}\,y} &= Z^3(-(1+uu)U + tuV)\\
\frac{\mathrm{d}\,Y}{\mathrm{d}\,x} &= Z^3(tuT - (1+tt)U)\\
\frac{\mathrm{d}\,Y}{\mathrm{d}\,y} &= Z^3(tuT - (1+tt)V)
\end{aligned}$$

quibus valoribus in expressione praecedente substitutis, prodit

$$k = Z^6(TV-UU)(1+tt+uu) = Z^4(TV-UU) = \frac{TV-UU}{(1+tt+uu)^2}$$

8.

Per idoneam electionem initii et axium coordinatarum facile effici potest, ut pro puncto determinato A valores quantitatum t, u, U evanescant. Scilicet duae priores conditiones iam adimplentur, si planum tangens in hoc puncto pro plano coordinatarum x, y adoptatur. Quarum initium si insuper in puncto A ipso collocatur, manifesto expressio coordinatarum z adipiscitur formam talem

$$z = \tfrac{1}{2}T^0xx + U^0xy + \tfrac{1}{2}V^0yy + \Omega$$

ubi Ω erit ordinis altioris quam secundi. Mutando dein situm axium ipsarum x, y angulo M tali ut habeatur

$$\tan 2M = \frac{2U^0}{T^0 - V^0}$$

facile perspicitur, prodituram esse aequationem huius formae

$$z = \tfrac{1}{2}Txx + \tfrac{1}{2}Vyy + \Omega$$

quo pacto etiam tertiae conditioni satisfactum est. Quibus ita factis, patet

I. Si superficies curva secetur plano ipsi normali et per axem coordinatarum x transeunte, oriri curvam planam, cuius radius curvaturae in puncto A fiat $= \frac{1}{T}$, signo positivo vel negativo indicante concavitatem vel convexitatem versus plagam eam, versus quam coordinatae z sunt positivae.

II. Simili modo $\frac{1}{V}$ erit in puncto A radius curvaturae curvae planae, quae oritur per sectionem superficiei curvae cum plano per axes ipsarum y, z transeunte.

III. Statuendo $x = r\cos\varphi$, $y = r\sin\varphi$, fit

$$z = \tfrac{1}{2}\left(T\cos\varphi^2 + V\sin\varphi^2\right) rr + \Omega$$

unde colligitur, si sectio fiat per planum superficiei in A normale et cum axe ipsarum x angulum φ efficiens, oriri curvam planam, cuius radius curvaturae in puncto A sit

$$= \frac{1}{T\cos\varphi^2 + V\sin\varphi^2}$$

IV. Quoties itaque habetur $T = V$, radii curvaturae in *cunctis* planis normalibus aequales erunt. Si vero T et V sunt inaequales, manifestum est, quum $T\cos\varphi^2 + V\sin\varphi^2$ pro quovis anguli φ cadat intra T et V, radios curvaturae in sectionibus principalibus, in I et II consideratis, referri ad curvaturas extremas, puta alterum ad curvaturam maximam, alterum ad minimam, si T et V eodem signo affectae sint, contra alterum ad maximam convexitatem, alterum ad maximam concavitatem, si T et V signis oppositis gaudeant. Hae conclusiones omnia fere continent, quae ill.EULER de curvatura superficierum curvarum primus docuit.

V. Mensura curvaturae superficiei curvae in puncto A qutem nanciscitur expressionem simplicissimam $k = TV$, unde habemus

THEOREMA. *Mensura curvaturae in quovis superficiei puncto aequalis est fractioni, cuious numerator unitas, denominator autem productum duorum radiorum curvaturae extremorum in sectionibus per plana normalia.*

Simul patet, mensuram curvaturae fieri positivam pro superficiebus concavo-concavis vel convexo-convexis (quod discrimen non est essentiale), negativam vero pro concavo-convexis. Si superficies constat e partibus utriusque generis, in earum confiniis mensura curvaturae evanescens esse debebit. De indole superficierum curvarum talium, in quibus mensura curvaturae ubique evanescit, infra pluribus agetur.

9.

Formula generalis pro mensura curvaturae in fine art. 7 proposita, omnium simplicissima est, quippe quae quinque tantum elementa implicat; ad magis complicatam, scilicet novem elementa involventem, deferimur, si adhibere volumus modum primum indolem superficiei curvae exprimendi. Retinendo notationes art. 4 insuper statuemus:

$$\frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}x^2} = P', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}y^2} = Q', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}z^2} = R'$$
$$\frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}y\,\mathrm{d}z} = P'', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}x\,\mathrm{d}z} = Q'', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}x\,\mathrm{d}y} = R''$$

ita ut fiat

$$\begin{aligned} \mathrm{d}P &= P'\,\mathrm{d}x + R''\,\mathrm{d}y + Q''\,\mathrm{d}z \\ \mathrm{d}Q &= R''\,\mathrm{d}x + Q'\,\mathrm{d}y + P''\,\mathrm{d}z \\ \mathrm{d}R &= Q''\,\mathrm{d}x + P''\,\mathrm{d}y + R'\,\mathrm{d}z \end{aligned}$$

Iam quum habeatur $t = -\frac{P}{R}$, invenimus per differentiationem

$$RR\,\mathrm{d}t = -R\,\mathrm{d}P + P\,\mathrm{d}R = (PQ'' - RP')\,\mathrm{d}x + (PP'' - RR'')\,\mathrm{d}y + (PR' - RQ'')\,\mathrm{d}z$$

sive, eliminata $\mathrm{d}z$ adiumento aequationis $P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$,

$$R^3\,\mathrm{d}t = (-RRP' + 2PRQ'' - PPR')\,\mathrm{d}x + (PRP'' + QRQ'' - PQR' - RRR'')\,\mathrm{d}y$$

Prorsus simili modo obtinemus

$$R^3\,\mathrm{d}t = (-PRP'' + QRQ'' - PQR' - RRR'')\,\mathrm{d}x + (-RRQ' + 2QRP'' - QQR')\,\mathrm{d}y$$

Hinc itaque colligimus

$$\begin{aligned} R^3T &= -RRP' + 2PRQ'' - PPR' \\ R^3U &= \quad PRP'' + \quad QRQ'' - PQR' - RRR'' \\ R^3V &= -RRQ' + 2QRP'' - QQR' \end{aligned}$$

Substituendo hos valores in formula art. 7, obtinemus pro mensura curvaturae k expressionem symmetricam sequentem:

$$\begin{aligned} &(PP+QQ+RR)^2k \\ &\qquad = PP(Q'R' - P''P'') + QQ(P'R' - Q''Q'') + RR(P'Q' - R''R'') \\ &\qquad = \quad +2QR(Q''R'' - P'P'') + 2PR(P''R'' - Q'Q'') + 2PQ(P''Q'' - R'R'') \end{aligned}$$

10.

Formulam adhuc magis complicatam, puta e quindecim elementis conflatam, obtinemus, si methodum generalem secundam, indolem, superficierum curvarum exprimendi, sequimur. Magni tamen momenti est, hanc quoque elaborare. Retinendo signa art. 4, insuper statuemus

$$\begin{array}{lll} \frac{\mathrm{d}\,\mathrm{d}x}{\mathrm{d}p^2} = \alpha, & \frac{\mathrm{d}\,\mathrm{d}x}{\mathrm{d}p.\mathrm{d}q} = \alpha', & \frac{\mathrm{d}\,\mathrm{d}x}{\mathrm{d}q^2} = \alpha'' \\ \frac{\mathrm{d}\,\mathrm{d}y}{\mathrm{d}p^2} = \beta, & \frac{\mathrm{d}\,\mathrm{d}y}{\mathrm{d}p.\mathrm{d}q} = \beta', & \frac{\mathrm{d}\,\mathrm{d}y}{\mathrm{d}q^2} = \beta'' \\ \frac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}p^2} = \gamma, & \frac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}p.\mathrm{d}q} = \gamma', & \frac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}q^2} = \gamma'' \end{array}$$

Praeterea brevitatis caussa faciemus

$$\begin{aligned} bc' - cb' &= A \\ ca' - ac' &= B \\ ab' - ba' &= C \end{aligned}$$

Primo observamus, haberi $A\mathrm{d}x + B\mathrm{d}y + C\mathrm{d}z = 0$, sive $\mathrm{d}z = -\frac{A}{C}\mathrm{d}x - \frac{B}{C}\mathrm{d}y$; quatenus itaque z spectatur tamquam functio ipsarum x, y fit

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} &= t = -\frac{A}{C} \\ \frac{\mathrm{d}z}{\mathrm{d}y} &= u = -\frac{B}{C} \end{aligned}$$

Porro deducimus, ex $\mathrm{d}x = a\mathrm{d}p + a'\mathrm{d}q$, $\mathrm{d}y = b\mathrm{d}p + b'\mathrm{d}q$,

$$\begin{aligned} C\mathrm{d}p &= b'\mathrm{d}x - a'\mathrm{d}y \\ C\mathrm{d}q &= -b\mathrm{d}x + a\mathrm{d}y \end{aligned}$$

Hinc obtinemus differentialia completa ipsarum t, u

$$\begin{aligned} C^3\mathrm{d}t &= (A\tfrac{\mathrm{d}C}{\mathrm{d}p} - C\tfrac{\mathrm{d}A}{\mathrm{d}p})(b'\mathrm{d}x - a'\mathrm{d}y) + (C\tfrac{\mathrm{d}A}{\mathrm{d}q} - A\tfrac{\mathrm{d}C}{\mathrm{d}q})(b\mathrm{d}x - a\mathrm{d}y) \\ C^3\mathrm{d}u &= (B\tfrac{\mathrm{d}C}{\mathrm{d}p} - C\tfrac{\mathrm{d}B}{\mathrm{d}p})(b'\mathrm{d}x - a'\mathrm{d}y) + (C\tfrac{\mathrm{d}B}{\mathrm{d}q} - B\tfrac{\mathrm{d}C}{\mathrm{d}q})(b\mathrm{d}x - a\mathrm{d}y) \end{aligned}$$

Iam si in his formulis substituimus

$$\begin{aligned} \tfrac{\mathrm{d}A}{\mathrm{d}p} &= c'\beta + b\gamma' - c\beta' - b'\gamma \\ \tfrac{\mathrm{d}A}{\mathrm{d}q} &= c'\beta' + b\gamma'' - c\beta'' - b'\gamma' \\ \tfrac{\mathrm{d}B}{\mathrm{d}p} &= a'\gamma + c\alpha' - a\gamma' - c'\alpha \\ \tfrac{\mathrm{d}B}{\mathrm{d}q} &= a'\gamma' + c\alpha'' - a\gamma'' - c'\alpha' \\ \tfrac{\mathrm{d}C}{\mathrm{d}p} &= b'\alpha + a\beta' - b\alpha' - a'\beta \\ \tfrac{\mathrm{d}C}{\mathrm{d}Q} &= b'\alpha' + a\beta'' - b\alpha'' - a'\beta' \end{aligned}$$

atque perpendimus, valores differentialium $\mathrm{d}t$, $\mathrm{d}u$ sic prodeuntium, aequales esse debere, independenter a differentialibus $\mathrm{d}x$, $\mathrm{d}y$, quantitatibus $T\,\mathrm{d}x + U\,\mathrm{d}y$, $U\,\mathrm{d}x + V\,\mathrm{d}y$ resp. inveniemus, post quasdam transformationes satis obvias:

$$\begin{aligned}
C^3T = {} & \alpha Ab'b + \beta Bb'b' + \gamma Cb'b' \\
& - 2\alpha' Abb' - 2\beta' Bbb' - 2\gamma' Cbb' \\
& + \alpha'' Abb + \beta'' Bbb + \gamma'' Cbb \\
C^3U = {} & -\alpha Aa'b' - \beta Ba'b' - \gamma Ca'b' \\
& + \alpha' A(ab' + ba') + \beta' B(ab' + ba') + \gamma' C(ab' + ba') \\
& - \alpha'' Aab - \beta'' Bab - \gamma'' Cab \\
C^3V = {} & \alpha Aa'a' + \beta Ba'a' + \gamma Ca'a' \\
& - 2\alpha' Aaa' - 2\beta' Baa' - 2\gamma' Caa' \\
& + \alpha'' Aaa + \beta'' Baa + \gamma'' Caa
\end{aligned}$$

Si itaque brevitatis caussa statuimus

$$A\alpha + B\beta + C\gamma = D \quad \ldots\ldots\ldots\ldots \quad (1)$$
$$A\alpha' + B\beta' + C\gamma' = D' \quad \ldots\ldots\ldots\ldots \quad (2)$$
$$A\alpha'' + B\beta'' + C\gamma'' = D' \quad \ldots\ldots\ldots\ldots \quad (3)$$

fit

$$\begin{aligned}
C^3T &= Db'b' - 2D'bb' + D''bb \\
C^3U &= -Da'b' + D'(ab' + ba') - D''ab \\
C^3V &= Da'a - 2D'aa' + D''aa
\end{aligned}$$

Hinc invenimus, evolutione facta,

$$C^6(TV - UU) = (DD'' - D'D')(ab' - ba')^2 = (DD'' - D'D')CC$$

et proin formulam pro mensura curvaturae

$$k = \tfrac{DD'' - D'D'}{(AA+BB+CC)^2}$$

11.

Formulae modo inventae iam aliam superstruemus, quae inter fertilissima theoremata in doctrina de superficiebus curvis referenda est. Introducamus sequentes notationes:

$$\begin{aligned}
aa + bb + cc &= E \\
aa' + bb' + cc' &= F \\
a'a' + b'b' + c'c' &= G \\
a\alpha + b\beta + c\gamma &= m \qquad \ldots\ldots\ldots\ldots \quad (4) \\
a\alpha' + b\beta' + c\gamma' &= m' \qquad \ldots\ldots\ldots\ldots \quad (5) \\
a\alpha'' + b\beta'' + c\gamma'' &= m'' \qquad \ldots\ldots\ldots\ldots \quad (6) \\
a'\alpha + b'\beta + c'\gamma &= n \qquad \ldots\ldots\ldots\ldots \quad (7) \\
a'\alpha' + b'\beta' + c'\gamma' &= n' \qquad \ldots\ldots\ldots\ldots \quad (8) \\
a'\alpha'' + b'\beta'' + c'\gamma'' &= n'' \qquad \ldots\ldots\ldots\ldots \quad (9) \\
AA + BB + CC = EG - FF &= \Delta
\end{aligned}$$

Eliminemus ex aequationibus 1, 4, 7 quantitates β, γ, quod fit multiplicando illas per $bc - cb'$, $b'C - c'B$, $cB - bC$, et addendo: ita oritur

$$\begin{aligned}
&(A(bc' - cb') + a(b'C - c'B) + a'(cB - bC))\,\alpha \\
&= D(bc' - cb') + m(b'C - c'B) + n(cB - bC)
\end{aligned}$$

quam aequationem facile transformamus in hanc:

$$AD = \alpha\Delta + a(nF - mG) + a'(mF - nE)$$

Simili modo eliminatio quantitatum α, γ vel α, β ex iisdem aequationibus suppeditat

$$\begin{aligned}
BD &= \beta\Delta + b(nF - mG) + b'(mF - nE) \\
CD &= \gamma\Delta + c(nF - mG) + c'(mF - nE)
\end{aligned}$$

Multiplicando has tres aequationes per α'', β'?, γ'' et addendo obtinemus

$$DD' = (\alpha\alpha''+\beta\beta''+\gamma\gamma'')\Delta+m''(nF-mG)+n''(mF-nE) \ldots\ldots\ldots\ldots (10)$$

Si perinde tractamus aequationes 2, 5, 8, prodit

$$\begin{aligned}
AD' &= \alpha'\Delta + a(n'F - m'G) + a'(m'F - n'E) \\
BD' &= \beta'\Delta + b(n'F - m'G) + b'(m'F - n'E) \\
CD' &= \gamma'\Delta + c(n'F - m'G) + c'(m'F - n'E)
\end{aligned}$$

quibus aequationibus per α', β', γ' multiplicatis, additio suppeditat:

$$D'D' = (\alpha'\alpha' + \beta'\beta' + \gamma'\gamma')\Delta + m'(n'F - m'G) + n'(m'F - n'E)$$

Combinatio huius aequationis cum aequatione (10) producit

$$\begin{aligned}DD'' - D'D' = {} & (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'\alpha' - \beta'\beta' - \gamma'\gamma')\Delta \\ & + E(n'n' - nn'') + F(nm'' - 2m'n' + mn'') + G(m'm' - mm'')\end{aligned}$$

Iam patet esse

$$\frac{\mathrm{d}E}{\mathrm{d}p} = 2m,\ \frac{\mathrm{d}E}{\mathrm{d}q} = 2m',\ \frac{\mathrm{d}F}{\mathrm{d}p} = m' + n,\ \frac{\mathrm{d}F}{\mathrm{d}q} = m'' + n',\ \frac{\mathrm{d}G}{\mathrm{d}p} = 2n',\ \frac{\mathrm{d}G}{\mathrm{d}q} = 2n''$$

sive

$$\begin{array}{llll} m = \frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}p}, & m' & = \frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}q}, & m'' = \frac{\mathrm{d}F}{\mathrm{d}q} - \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}p} \\ n = \frac{\mathrm{d}F}{\mathrm{d}p} - \frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}q}, & n' & = \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}p}, & n'' = \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}q} \end{array}$$

Porro facile confirmatur haberi

$$\begin{aligned}\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'\alpha' - \beta' - \beta' - \gamma'\gamma' &= \frac{\mathrm{d}n}{\mathrm{d}q} - \frac{\mathrm{d}n'}{\mathrm{d}p} = \frac{\mathrm{d}m''}{\mathrm{d}p} - \frac{\mathrm{d}m'}{\mathrm{d}q} \\ &= -\frac{1}{2}\frac{\mathrm{d}\,\mathrm{d}E}{\mathrm{d}q^2} + \frac{\mathrm{d}\,\mathrm{d}F}{\mathrm{d}p.\mathrm{d}q} - \frac{1}{2}\frac{\mathrm{d}\,\mathrm{d}G}{\mathrm{d}p^2}\end{aligned}$$

Quodsi iam has expressiones diversas in formula pro mensura curvaturae in fine art. praec. eruta substituimus, pervenimus ad formulam sequentem, e solis quantitatibus E, F, G atque earum quotientibus differentialibus primi et secundi ordinis concinnatam:

$$\begin{aligned}4(EG - FF)^2 k = {} & E\left(\frac{\mathrm{d}E}{\mathrm{d}p}.\frac{\mathrm{d}G}{\mathrm{d}q} - 2\frac{\mathrm{d}F}{\mathrm{d}p}\frac{\mathrm{d}G}{\mathrm{d}q} + \left(\frac{\mathrm{d}G}{\mathrm{d}p}\right)^2\right) \\ & + F\left(\frac{\mathrm{d}E}{\mathrm{d}p}.\frac{\mathrm{d}G}{\mathrm{d}q} - \frac{\mathrm{d}E}{\mathrm{d}q}\frac{\mathrm{d}G}{\mathrm{d}p} - 2 + \frac{\mathrm{d}E}{\mathrm{d}q}\frac{\mathrm{d}F}{\mathrm{d}q} + 4\frac{\mathrm{d}F}{\mathrm{d}p}\frac{\mathrm{d}F}{\mathrm{d}p} - 2\frac{\mathrm{d}F}{\mathrm{d}p}\frac{\mathrm{d}G}{\mathrm{d}p}\right) \\ & + G\left(\frac{\mathrm{d}E}{\mathrm{d}p}.\frac{\mathrm{d}G}{\mathrm{d}p} - 2\frac{\mathrm{d}E}{\mathrm{d}p}\frac{\mathrm{d}F}{\mathrm{d}q} + \left(\frac{\mathrm{d}E}{\mathrm{d}q}\right)^2\right) \\ & - 2(EG - FF)\left(\frac{\mathrm{d}\,\mathrm{d}E}{\mathrm{d}q^2} - 2\frac{\mathrm{d}\,\mathrm{d}F}{\mathrm{d}p.\mathrm{d}q} + \frac{\mathrm{d}\,\mathrm{d}G}{\mathrm{d}p^2}\right)\end{aligned}$$

12.

Quum indefinite habeatur

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = E\mathrm{d}p^2 + 2F\mathrm{d}p.\mathrm{d}q + G\mathrm{d}q^2$$

patet $\surd(E\mathrm{d}p^2 + 2F\mathrm{d}p.\mathrm{d}q + G\mathrm{d}q^2)$ esse expressionem generalem elementi linearis in superficie curva. Docet itaque analysis in art. praec. explicata, ad inveniendam mensuram curvaturae haud opus esse formulis finitis, quae coordinatas x, y, z tamquam functiones indeterminatarum p, q exhibeant, sed sufficere expressionem generalem pro magnitudine cuiusvis elementi linearis. Progrediamur ad aliquot applicationes huius gravissimi theorematis.

Supponamus, superficiem nostram curvam explicari posse in aliam superficiem, curvam seu planam, ita ut cuivis puncto prioris superficiei per coordinatas x, y, z determinato respondeat punctum determinatum superficiei posterioris, cuius coordinatae sint x', y', z'. Manifesto itaque x', y', z' quoque considerari possunt tamquam functiones indeterminatarum p, q, unde pro elemento $\sqrt{}(\mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2)$ prodibit expressio talis

$$\sqrt{}\left(E'\mathrm{d}p^2 + 2F'\mathrm{d}p.\mathrm{d}q + G'\mathrm{d}q^2\right)$$

denotantibus etiam E', F', G' functiones ipsarum p, q. At per ipsam notionem *explicationis* superficiei in superficiem patet, elementa at utraque superficie correspondentia necessario aequalia esse, adeoque identice fieri

$$E = E', \quad F = F', \quad G = G'$$

Formula itaque art. praec. sponte perducit ad egregium

THEOREMA. *Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet.*

Manifesto quoque *quaevis pars finita superficiei curvae post explicationem in aliam superficiem eandem curvaturam integram retinebit.*

Casum specialem, ad quem geometrae hactenus investigationes suas restrinxerunt, sistunt superficies in planum explicabiles. Theoria nostra sponte docet, talium superficierum mensuram curvaturae in quovis puncto fieri $= 0$, quocirca, si earum indoles secundum modum tertium exprimitur, ubique erit

$$\tfrac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}x^2}\cdot\tfrac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}y^2} - \left(\tfrac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}x.\mathrm{d}y}\right)^2 = 0$$

quod criterium, dudum quidem notum, plerumque nostro saltem iudicio haud eo rigore qui desiderari posset demonstratur.

13.

Quae in art. praec. exposuimusm, coharent cum modo peculiari superficies considerandi, summopere digno, qui a geometris diligenter excolatur. Scilicet quatenus superficies consideratur non tamquam limes solidi, sed tamquam solidum, cuius dimensio una pro evanescente habetur, flexile quidem, sed non extensibile, qualitates superficiei partim a forma pendent, in quam illa reducta concipitur, partim absolutae sunt, atque invariatae manent, in quamcunque formam illa flectatur. Ad has posteriores, quarum investigatio campum geometriae novum fertilemque aperit, referendae sunt mensura curvaturae atque curvatura integra eo sensu, quo hae expressiones a nobis accipiuntur; porro huc pertinet doctrina de lineis brevissimis, pluraque alia, de quibus in posterum agere nobis reservamus. In hoc considerationis modo superficies plana atque superficies in planum explicabilis, e. g. cylindrica, conica etc. tamquam essentialiter identicae spectantur, modusque genuinus indolem superficiei ita consideratae generaliter exprimendi semper innititur formulae $\surd(E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p.\,\mathrm{d}q + G\,\mathrm{d}q^2)$, quae nexum elementi cum duabus indeterminatis p, q sistit. Sed antequam hoc argumentum ulterius prosequamur, principia theoriae linearum brevissimarum in superficie curva data praemittere oportet.

14.

Indoles lineae curvae in spatio generaliter ita datur, ut coordinatae x, y, z singulis illius puncits respondentes exhibeantur in forma functionum unius variabilis, quam per w denotabimus. Longitudo talis lineae a puncto initiali arbitrario usque ad punctum, cuius coordinatae sunt x, y, z, exprimitur per integrale

$$\int \mathrm{d}w.\surd\left(\left(\tfrac{\mathrm{d}x}{\mathrm{d}w}\right)^2 + \left(\tfrac{\mathrm{d}y}{\mathrm{d}w}\right)^2 + \left(\tfrac{\mathrm{d}z}{\mathrm{d}w}\right)^2\right)$$

Si supponimus, situm lineae curvae variationem infinite parvam pati, ita ut coordinatae singulorum punctorum accipiant variationes δx, δy, δz, variatio totius longitudinis invenitur

$$= \int \tfrac{\mathrm{d}x.\,\mathrm{d}\delta x + \mathrm{d}y.\,\mathrm{d}\delta y + \mathrm{d}z.\,\mathrm{d}\delta z}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}$$

quam expressionem in hanc formam transmutamus:

$$\tfrac{\mathrm{d}x.\,\mathrm{d}\delta x + \mathrm{d}y.\,\mathrm{d}\delta y + \mathrm{d}z.\,\mathrm{d}\delta z}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}$$
$$- \int \left(\delta x.\,\mathrm{d}\tfrac{\mathrm{d}x}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)} + \delta y.\,\mathrm{d}\tfrac{\mathrm{d}y}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)} + \delta z.\,\mathrm{d}\tfrac{\mathrm{d}z}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}\right)$$

In casu eo, ubi linea est brevissima inter puncta sua extrema, constat ea, quae hic sub signo integrali sunt, evanescere debere. Quatenus linea esse debet in superficie data, cuius indoles exprimitur per aequationem $P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$, etiam variationes δx, δy, δz satisfacere debent aequationi $P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$, unde per principia nota facile colligitur, differentialia

$$\mathrm{d}\tfrac{\mathrm{d}x}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}, \quad \mathrm{d}\tfrac{\mathrm{d}y}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}, \quad \mathrm{d}\tfrac{\mathrm{d}z}{\surd(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}$$

resp. quantitatibus P, Q, R proportionalia esse debere. Iam sit $\mathrm{d}r$ elementum lineae curvae, λ punctum in superficie sphaerica repraesentans directionem huius elementi, L punctum in superficie sphaerica repraesentans directionem normalis in superficiem curvam; denique sint ξ, η, ζ coordinatae puncti λ atque X, Y, Z coordinatae puncti L respectu centri sphaerae. Ita erit

$$\mathrm{d}x = \xi\,\mathrm{d}r, \quad \mathrm{d}y = \eta\,\mathrm{d}r, \quad \mathrm{d}z = \zeta\,\mathrm{d}r$$

unde colligimus, differentialia illa fieri $\mathrm{d}\xi$, $\mathrm{d}\eta$, $\mathrm{d}\zeta$. Et quum quantitates P, Q, R proportionales sint ipsis X, Y, Z, character lineae brevissimae consistit in aequationibus

$$\tfrac{\mathrm{d}\xi}{X} = \tfrac{\mathrm{d}\eta}{Z} = \tfrac{\mathrm{d}\zeta}{Y}$$

Ceterum facile perspicitur, $\sqrt{(\mathrm{d}\xi^2 + \mathrm{d}\eta^2 + \mathrm{d}\zeta^2)}$ aequari arculo in superficie sphaerica, qui mensurat angulum inter directiones tangentium in initio et fine elementi $\mathrm{d}r$, adeoque esse $= \frac{\mathrm{d}r}{\rho}$, si ρ denotet radium curvaturae in hoc loco curvae brevissimae; ita fiet

$$\rho\mathrm{d}\xi = X\mathrm{d}r, \quad \rho\mathrm{d}\eta = Y\mathrm{d}r, \quad \rho\mathrm{d}\zeta = Z\mathrm{d}r$$

15.

Supponamus, in superficie curva a puncto dato A proficisci innumeras curvas brevissimas, quas inter se distinguemus per angulum, quem constituit singularum elementum primum cum elemento primo unius ex his lineis pro prima assumtae: sit φ ille angulus, vel generalius functio illius anguli, nec non r longitudo talis lineae brevissimae a puncto A usque ad punctum, cuius coordinatae sunt x, y, z. Quum itaque valoribus determinatis variabilium r, φ respondeant puncta determinata superficiei, coordinatae x, y, z considerari possunt tamquam functiones ipsarum r, φ. Notationes λ, L, ξ, η, ζ, X, Y, Z in eadem significatione retinebimus, in qua in art. praec. acceptae fuerunt, modo indefinite ad punctum indefinitum cuiuslibet linearum brevissimarum referuntur.

Lineae brevissimae omnes, quae sunt aequalis longitudinis r, terminabuntur ad aliam lineam, cuius longitudinem ab initio arbitrario numeratam denotamus per v. Considerari poterir itaque v tamquam functio indeterminatarum r, φ, et si per λ' designamus punctum in superficie sphaerica respondens directioni elementi $\mathrm{d}v$, nec non per ξ', η', ζ' coordinatas huius puncti respectu centri sphaerae, habebimus:

$$\tfrac{\mathrm{d}x}{\mathrm{d}\varphi} = \xi' . \tfrac{\mathrm{d}v}{\mathrm{d}\varphi}, \quad \tfrac{\mathrm{d}y}{\mathrm{d}\varphi} = \eta' . \tfrac{\mathrm{d}v}{\mathrm{d}\varphi}, \quad \tfrac{\mathrm{d}z}{\mathrm{d}\varphi} = \zeta' . \tfrac{\mathrm{d}v}{\mathrm{d}\varphi}$$

Hinc et ex

$$\tfrac{\mathrm{d}x}{\mathrm{d}r} = \xi, \quad \tfrac{\mathrm{d}y}{\mathrm{d}r} = \eta, \quad \tfrac{\mathrm{d}z}{\mathrm{d}r} = \zeta$$

sequitur

$$\tfrac{\mathrm{d}x}{\mathrm{d}r} . \tfrac{\mathrm{d}x}{\mathrm{d}\varphi} + \tfrac{\mathrm{d}y}{\mathrm{d}r} . \tfrac{\mathrm{d}y}{\mathrm{d}\varphi} + \tfrac{\mathrm{d}z}{\mathrm{d}r} . \tfrac{\mathrm{d}z}{\mathrm{d}\varphi} = (\xi\xi' + \eta\eta' + \zeta\zeta') . \tfrac{\mathrm{d}v}{\mathrm{d}\varphi} = \cos\lambda\lambda' . \tfrac{\mathrm{d}v}{\mathrm{d}\varphi}$$

Membrum primum huius aequationis, quod etiam erit functio ipsarum r, φ, per S denotamus; cuius differentiatio secundum r suppeditat:

$$\begin{aligned} \tfrac{\mathrm{d}S}{\mathrm{d}r} &= \tfrac{\mathrm{d}\,\mathrm{d}x}{\mathrm{d}r^2} . \tfrac{\mathrm{d}x}{\mathrm{d}\varphi} + \tfrac{\mathrm{d}\,\mathrm{d}y}{\mathrm{d}r^2} . \tfrac{\mathrm{d}y}{\mathrm{d}\varphi} + \tfrac{\mathrm{d}\,\mathrm{d}z}{\mathrm{d}r^2} . \tfrac{\mathrm{d}z}{\mathrm{d}\varphi} + \tfrac{1}{2} \frac{\mathrm{d}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right)}{\mathrm{d}\varphi} \\ &= \tfrac{\mathrm{d}\xi}{\mathrm{d}r} . \tfrac{\mathrm{d}x}{\mathrm{d}\varphi} + \tfrac{\mathrm{d}\eta}{\mathrm{d}r} . \tfrac{\mathrm{d}y}{\mathrm{d}\varphi} + \tfrac{\mathrm{d}\zeta}{\mathrm{d}r} . \tfrac{\mathrm{d}z}{\mathrm{d}\varphi} + \tfrac{1}{2} \tfrac{\mathrm{d}(\xi\xi + \eta\eta + \zeta\zeta)}{\mathrm{d}\varphi} \end{aligned}$$

Sed $\xi\xi + \eta\eta + \zeta\zeta = 1$, adeoque ipsius differentiale $= 0$; et per art. praec. habemus, si etiam hic ρ denotat radium curvaturae in linea r,

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} = \frac{X}{\rho}, \quad \frac{\mathrm{d}\eta}{\mathrm{d}r} = \frac{Y}{\rho}, \quad \frac{\mathrm{d}\zeta}{\mathrm{d}r} = \frac{Z}{\rho}$$

Ita obtinemus

$$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{1}{\rho}.(X\xi' + Y\eta' + Z\zeta').\frac{\mathrm{d}v}{\mathrm{d}\varphi} = \frac{1}{\rho}.\cos L\lambda'.\frac{\mathrm{d}v}{\mathrm{d}\varphi} = 0$$

quoniam manifesto λ' iacet in circulo maximo, cuius polus L. Hinca itaque concludimus, S independentem esse ab r et proin functionem solius φ. At pro $r = 0$ manifesto fit $v = 0$, et proin etiam $\frac{\mathrm{d}v}{\mathrm{d}\varphi} = 0$, nec non $S = 0$, independenter a φ. Necessario itaque generaliter esse debebit $S = 0$, adeoque $\cos\lambda\lambda' = 0$, i. e. $\lambda\lambda' = 90°$. Hinc colligimus

THEOREMA. *Ductis in superficie curva ab eodem puncto initiali innumeris lineis brevissimis aequalis longitudinis, linea earum extremitates iungens ad illas singulas erit normalis.*

Operae pretium esse duximus, hoc theorema e proprietate fundamentali linearum brevissimarum deducere: ceterum eius veritas etiam absque calculo per sequens ratiocinium intelligi potest. Sint AB, AB' duae lineae brevissimae eiusdem longitudinis, angulum infinite parvum ad A includentes, supponamusque alterutrum angulorum elementi BB' cum lineis BA, $B'A$ differre quantitate finita ab angulo recto, unde per legem continuitatis alter maior alter minor erit angulo recto. Supponamus, angulum ad B esse $= 90° - \omega$, capiamusque in linea BA punctum C, ita ut sit $BC = BB'.\operatorname{cosec}\omega$: hinc quum triangulum infinite parvum $BB'C$ tamquam planum tractare liceat, erit $CB' = BC\cos\omega$ et proin

$$AC + CB' = AC + BC.\cos\omega = AB - BC.(1 - \cos\omega) = AB' - BC(1 - \cos\omega)$$

i. e. transitus a puncto A ad B' per punctum C brevior linea brevissima. Q.E.A.

16.

Theorematì art. praec. associamus aliud, quod ita enunciamus. *Si in superficie curva concipitur linea qualiscunque, a cuius punctis singulis profiscantur sub angulis rectis et versus eandem plagam innumerae lineae brevissimae aequalis longitudinis, curva, quae earum extremitates alteras iungit, illas singulas sub angulis rectis secabit.* Ad demonstrationem nihil in analysi praecedente mutandum est, nisi quod φ designare debet longitudinem curvae *datae* inde a puncto arbitrario numeratam, aut si mavis functionem huius longitudinis; ita omnia ratiocinia etiamnum valebunt, ea modificatione, quod veritas aequationis $S = 0$ pro $r = 0$ nunc iam in ipsa hypothesi implicatur. Ceterum hoc alterum theorema generalius est praecedente, quod adeo in illo comprehendi censeri potest, dum pro linea data adoptamus circulum infinite parvum circa centrum A descriptum. Denique monemus, hic quoque considerationes geometricas analyseos vice fungi posse, quibus tamen, quum satis obviae sint, hic non immoramur.

17.

Revertimur ad formulam $\sqrt{}(E\,\mathrm{d}p^2+2F\,\mathrm{d}p.\mathrm{d}q+G\,\mathrm{d}q^2)$, quae indefinite magnitudinem elementi linearis in superficie curva exprimit, atque ante omnia significationem geometricam coefficientium E, F, G examinamus. Iam in art. 5 monuimus, in superficie curva concipi posse duo systemata linearum, alterum, in quibus singulis sola p sit variabilis, q constans; alterum, in quibus sola q variabilis, p constans. Quodlibet punctum superficiei considerari potest tamquam intersectio lineae primi systematis cum linea secundi: tuncque elementum lineae primae huic puncto adiacens et variationi $\mathrm{d}p$ respondens erit $=\sqrt{E}.\mathrm{d}p$, nec non elementum lineae secundae respondens variationi $\mathrm{d}q$ erit $=\sqrt{G}.\mathrm{d}q$; denique denotanto per ω angulum inter haec elementa, facile perspicitur, fieri $\cos\omega=\frac{F}{\sqrt{EG}}$. Area autem elementi parallelogrammatici in superficie curva inter duas lineas primi systematis, quibus respondent q, $q+\mathrm{d}q$, atque duas lineas systematis secundi, quibus respondent p, $p+\mathrm{d}p$, erit $\sqrt{}(EG-FF)\,\mathrm{d}p.\mathrm{d}q$.

Linea quaecunque in superficie curva ad neutrum illorum systematum pertinens, oritur, dum p et q concipiuntur esse functiones unius variabilis novae, vel altera illarum functio alterius. Sit s longitudo talis curvae ab initio arbitrario numerata et versus directionem utramvis pro positiva habita. Denotemus per θ angulum, quem efficit elementum $\mathrm{d}s=\sqrt{}(E\,\mathrm{d}p^2+2F\,\mathrm{d}p.\mathrm{d}q+G\,\mathrm{d}q^2)$ cum linea primi systematis per initium elementi ducta, et quidem ne ulla ambiguitas remaneat, hunc angulum sempre ab eo ramo illius lineae, in quo valores ipsius p crescunt, inchoari, et versus eam plagam positive accipi supponemus, versus quam valores ipsius q crescunt. His ita intellectis facile perspicitur haberi

$$\cos\theta.\mathrm{d}s=\sqrt{E}.\mathrm{d}p+\sqrt{G}.\cos\omega\,\mathrm{d}q=\frac{E\,\mathrm{d}p+F\,\mathrm{d}q}{\sqrt{E}}$$
$$\sin\theta.\mathrm{d}s=\sqrt{G}.\sin\omega.\mathrm{d}q=\frac{\sqrt{}(EG-FF).\mathrm{d}q}{\sqrt{E}}$$

18.

Investigabimus nunc, quaenam sit conditio, ut haec linea sit brevissima. Quum ipsius longitudo s expressa sit per integrale

$$s=\int\sqrt{}\left(E\,\mathrm{d}p^2+2F\,\mathrm{d}p.\mathrm{d}q+G\,\mathrm{d}q^2\right)$$

conditio minimi requirit, ut variatio huius integralis a mutatione infinite parva tractus lineae oriunda fiat $=0$. Calculus ad propositum nostrum in hoc casu commodius absolvitur, si p tamquam functionem ipsius q consideramus. Quo pacto, si variatio per characteristicam δ denotatur, habemus

$$\delta s=\int\frac{\left(\frac{\mathrm{d}E}{\mathrm{d}p}.\mathrm{d}p^2+\frac{2\,\mathrm{d}F}{\mathrm{d}p}.\mathrm{d}p.\mathrm{d}q+\frac{\mathrm{d}G}{\mathrm{d}p}\mathrm{d}q^2\right)\delta p+(2E\,\mathrm{d}p+2F\,\mathrm{d}q)\,\mathrm{d}\delta p}{2\,\mathrm{d}s}$$
$$=\frac{E\,\mathrm{d}p+F\,\mathrm{d}q}{\mathrm{d}s}.\delta p+\int\delta p.\Big\{\frac{\frac{\mathrm{d}E}{\mathrm{d}p}.\mathrm{d}p^2+\frac{2\,\mathrm{d}F}{\mathrm{d}p}.\mathrm{d}p.\mathrm{d}q+\frac{\mathrm{d}G}{\mathrm{d}p}.\mathrm{d}q^2}{2\,\mathrm{d}s}-\mathrm{d}.\frac{E\,\mathrm{d}p+F\,\mathrm{d}q}{\mathrm{d}s}\Big\}$$

constatque, quae hic sunt sub signo integrali, independenter a δp evanescere debere. Fit itaque

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}p}.\mathrm{d}p^2 + \frac{2\mathrm{d}F}{\mathrm{d}p}.\mathrm{d}p.\mathrm{d}q + \frac{\mathrm{d}G}{\mathrm{d}p}.\mathrm{d}q^2 &= 2\mathrm{d}s.\mathrm{d}.\frac{E\,\mathrm{d}p+F\,\mathrm{d}q}{\mathrm{d}s}\\ &= 2\mathrm{d}s.\mathrm{d}.\sqrt{E}.\cos\theta = \frac{\mathrm{d}s.\mathrm{d}E.\cos\theta}{\sqrt{E}} - 2\mathrm{d}s.\mathrm{d}\theta.\sqrt{E}.\sin\theta\\ &= \frac{(E\,\mathrm{d}p+F\,\mathrm{d}q)\,\mathrm{d}E}{E} - \sqrt{(EG-FF)}.\mathrm{d}q.\mathrm{d}\theta\\ &= \left(\frac{(E\,\mathrm{d}p+F\,\mathrm{d}q)}{E}\right).\left(\frac{\mathrm{d}E}{\mathrm{d}p}.\mathrm{d}p + \frac{\mathrm{d}E}{\mathrm{d}q}.\mathrm{d}q\right) - 2\sqrt{(EG-FF)}.\mathrm{d}q.\mathrm{d}\theta\end{aligned}$$

Hinc itaque nanciscimur aequationem conditionalem pro linea brevissima sequentem:

$$\sqrt{(EG-FF)}.\mathrm{d}\theta = \frac{1}{2}\frac{F}{E}.\frac{\mathrm{d}E}{\mathrm{d}p}.\mathrm{d}p + \frac{1}{2}.\frac{F}{E}.\frac{\mathrm{d}E}{\mathrm{d}q}.\mathrm{d}q + \frac{1}{2}.\frac{\mathrm{d}E}{\mathrm{d}q}.\mathrm{d}p - \frac{\mathrm{d}F}{\mathrm{d}p}.\mathrm{d}p. - \frac{1}{2}.\frac{\mathrm{d}G}{\mathrm{d}p}.\mathrm{d}q$$

quam etiam ita scribere licet

$$\sqrt{(EG-FF)}.\mathrm{d}\theta = \frac{1}{2}\frac{F}{E}.\mathrm{d}E + \frac{1}{2}.\frac{\mathrm{d}E}{\mathrm{d}q}.\mathrm{d}p - \frac{\mathrm{d}F}{\mathrm{d}p}.\mathrm{d}p - \frac{1}{2}.\frac{\mathrm{d}G}{\mathrm{d}p}.\mathrm{d}q$$

Ceterum adiumento aequationis

$$\operatorname{cotg}\theta = \frac{E}{\sqrt{(EG-FF)}}.\frac{\mathrm{d}p}{\mathrm{d}q} + \frac{F}{\sqrt{(EG-FF)}}$$

ex illa aequatione angulus θ eliminari, atque sic aequatio differentio-differentialis inter p et q evolvi potest, quae tamen magis complicata et ad applicationes minus utilis evaderet, quam praecedens.

19.

Formulae generales, quas pro mensura curvaturae et pro variatione directionis lineae brevissimae in artt. 11, 18 eruimus, multo simpliciores fiunt, si quantitates p, q ita sunt electae, ut lineae primi systematis lineas secundi systematis ubique orthogonaliter secent, i. e. ut generaliter habeatur $\omega = 90°$, sive $F = 0$. Tunc scilicet fit, pro mensura curvaturae,

$$4EEGGk = E.\frac{\mathrm{d}E}{\mathrm{d}q}.\frac{\mathrm{d}G}{\mathrm{d}q} + E\left(\frac{\mathrm{d}G}{\mathrm{d}p}\right)^2 + G.\frac{\mathrm{d}E}{\mathrm{d}p}.\frac{\mathrm{d}G}{\mathrm{d}p} + G\left(\frac{\mathrm{d}E}{\mathrm{d}q}\right)^2 - 2EG\left(\frac{\mathrm{d}\,\mathrm{d}E}{\mathrm{d}q^2} + \frac{\mathrm{d}\,\mathrm{d}G}{\mathrm{d}p^2}\right)$$

et pro variatione anguli θ

$$\sqrt{EG}.\mathrm{d}\theta = \frac{1}{2}.\frac{\mathrm{d}E}{\mathrm{d}q}.\mathrm{d}p - \frac{1}{2}.\frac{\mathrm{d}G}{\mathrm{d}p}.\mathrm{d}q$$

Inter variops casus, in quibus haec conditio orthogonalitatis valet, primarium locum tenet is, ubi lineae omnes, alterutrius systematis, e. g. primi, sunt lineae brevissimae. Hic itaque pro valore constante ipsius q, angulus θ fit $= 0$, unde aequatio pro variatione anguli θ modo tradita docet, fieri debere $\frac{\mathrm{d}E}{\mathrm{d}q} = 0$, sive coëfficientem E a q

independentem, i. e. E esse debet vel constans vel functio solius p. Simplicissimum erit, pro p adoptare longitudinem ipsam cuiusque lineae primi systematis, et quidem, quoties omnes lineae primi systematis in uno puncto concurrunt, ab hoc puncto numeratam, vel si communis intersectio non adest, a qualibet linea secundi systematis. Quibus ita intellectis patet, p et q idam eadem denotare, quae in artt. 15, 16 per r et φ expresseramus, atque fieri $E = 1$. Ita duae formulae praecedentes iam transeunt in has:

$$4GGk = \left(\frac{\mathrm{d}G}{\mathrm{d}p}\right)^2 - 2G\frac{\mathrm{d}\,\mathrm{d}G}{\mathrm{d}p^2}$$

$$\sqrt{G}.\mathrm{d}\theta = -\tfrac{1}{2}.\frac{\mathrm{d}G}{\mathrm{d}p}.\mathrm{d}q$$

vel statuendo $\sqrt{G} = m$,

$$k = -\frac{1}{m}.\frac{\mathrm{d}\,\mathrm{d}m}{\mathrm{d}p^2}, \qquad \mathrm{d}\theta = -\frac{\mathrm{d}m}{\mathrm{d}p}.\mathrm{d}q$$

Generaliter loquendo m erit functio ipsarum p, q atque $m\mathrm{d}q$ expressio elementi cuiusvis lineae secundi systematis. In casu speciali autem, ubi omnes lineae p ad eodem puncto proficiscuntur, manifesto pro $p = 0$ esse debet $m = 0$; porro si in hoc casu pro q adoptamus angulum ipsum, quem elementum primum cuiusvis lineae primi systematis facit cum elemento alicuius ex ipsis ad arbitrarium electae, quum pro valore infinite parvo ipsius p, elementum lineae secundi systematis (quae considerari potest tamquam circulus radio p descriptus), sit $= p\mathrm{d}q$, erito pro valore infinite parvo ipsius $p.m = p$, adeoque, pro $p = 0$ simul $m = 0$ et $\frac{\mathrm{d}m}{\mathrm{d}p} = 1$.

20.

Immoremur adhuc eidem suppositioni, puta p designare indefinite longitudinem lineae brevissimae a puncto determinato A ad punctum quodlibet superficiei ductum, atque q angulum, quem primum elementum huius lineae efficit cum elemento primo alicuius lineae brevissimae ex A proficiscentis datae. Sit B punctum determinatum in hac linea pro qua $q = 0$, atque C aliud punctum determinatum superficiei pro quo valorem ipsius q simpliciter per A designabimus. Supponamus, puncta B, C per lineam brevissimam iuncta, cuius partes, inde a puncto B numeratas, indefinite ut in art. 18 per s denotabimus, nec non perinde ut illie, per θ angulum, quem quodvis elementum $\mathrm{d}s$ facit cum elemento $\mathrm{d}p$: denique sint $\theta°$, θ' valores anguli θ in punctis B, C. Habemus itaque in superficie curva triangulum lineis brevissimis inclusum, eiusque anguli ad B et C, per has ipsas literas simpliciter designandi aequales erunt ille complemento anguli $\theta°$ ad $180°$, hic ipsi angulo θ'. Sed quum analysis nostram inspicienti facile pateat, omnes angulos non per gradus se per numeros expressos concipi, ita ut angulus $57°17'45''$, cui respondet arcus radio aequalis, pro unitate habeatur, statuere oportet, denotanto per 2π perpheriam circuli

$$\theta^0 = \pi - B, \qquad \theta' = C$$

Inquiramus nunc in curvaturam integram huius trianguli, quae fit $= \int k\,\mathrm{d}\sigma$, denotante $\mathrm{d}\sigma$ elementum superficiale trianguli; quare quum hoc elementum exprimatur per $m\,\mathrm{d}p.\mathrm{d}q$, eruere oportet integrale $\iint km\,\mathrm{d}p.\mathrm{d}q$ supra totam trianguli superficiem. Incipiamus ab integratione secundum p, quae propter $k = -\frac{1}{m}.\frac{\mathrm{d}\,\mathrm{d}m}{\mathrm{d}p^2}$, suppeditat $\mathrm{d}q.(\mathrm{Const.} - \frac{\mathrm{d}m}{\mathrm{d}p})$ pro curvatura integra areae iacentis inter lineas primi systematis, quibus respondent valores indeterminatae secundae q, $q + \mathrm{d}q$: quum haec curvatura pro $p = 0$ evanescere debeat, quantitas constans per integrationem introducta aequalis esse debet valori ipsius $\frac{\mathrm{d}m}{\mathrm{d}p}$ pro $p = 0$, i. e. unitati. Habemus itaque $\mathrm{d}q(1 - \frac{\mathrm{d}m}{\mathrm{d}p})$, ubi pro $\frac{\mathrm{d}m}{\mathrm{d}p}$ accipere oportet valorem respondentem fini illius areae in linea CB. In hac linea vero fit per art. praec. $\frac{\mathrm{d}m}{\mathrm{d}p}.\mathrm{d}q = -\mathrm{d}\theta$, unde expressio nostra mutatur in $\mathrm{d}q + \mathrm{d}\theta$. Accedente iam integratione altera a $q = 0$ usque ad $q = A$ extendenda, obtinemus curvaturam integram trianguli $= A + \theta' - \theta^\circ = A + B + C - \pi$.

Curvatura integra aequalis est areae eius partis superficiei sphaericae, quae respondet triangulo signo positivo vel negativo affectae, prout superficies curva, in qua triangulum iacet, est concavo-concava vel concavo-convexa: pro unitate areae accipiendum est quadratum, cuius latus est unitas (radius sphaerae), quo pacto superficies tota sphaerae fit $= 4\pi$. Est itaque pars superficiei sphaericae triangulo respondens ad sphaerae superficiem integram ut $\pm(A + B + C - \pi)$ ad 4π. Hoc theorema, quod ni fallimur ad elegantissima in theoria superficierum curvarum referentum esse videtur, etiam sequenti modo enuntiari potest:

Excessus summae angulorum trianguli a lineis brevissimis in superficie curva concavo-concava formati ultra 180°, *vel defectus summae angulorum trianguli a lineis brevissimis in superficie curva concavo-convexa formati a* 180° *mensuratur per aream partis superficiei sphaericae, quae illi triangulo per directiones normalium respondet, si superficies integra* 720 *gradibus aequiparatur.*

Generalius in quovis polygono n laterum, quade singula formantur per lineas brevissimas, excessus summae angulorum supra $2n - 4$ rectos, vel defectus a $2n - 4$ rectis (pro indole curvaturae superficiei), aequatur areae polygoni respondentis in superficie sphaerica, dum tota superficies sphaerae 720 gradibus aequiparatur, uti per discerptionem polygoni in triangula e theoremate praecedenti sponte demanat.

21.

Restituamus characteribus p, q, E, F, G, ω significationes generales, quibus supra accepti fuerant, supponamusque, indolem superficiei curvae praeterea alio simili modo per duas alias variabiles p', q' determinari, ubi elementum lineare indefinitum exprimatur per

$$\surd(E'\,\mathrm{d}p'^2 + 2F'\,\mathrm{d}p'.\mathrm{d}q' + G'\,\mathrm{d}q'^2)$$

Ita cuivis puncto superficiei per valores determinatos variabilium p, q definito respondebunt valores determinati variabilium p', q', quocirca hae erunt functiones ipsarum p,

q, e quarum differentiatione prodire supponemus

$$\begin{aligned} \mathrm{d}p' &= \alpha\,\mathrm{d}p + \beta\,\mathrm{d}q \\ \mathrm{d}q' &= \gamma\,\mathrm{d}p + \delta\,\mathrm{d}q \end{aligned}$$

Iam proponimus nobis investigare significationem geometricam horum coëfficientium α, β, γ, δ.

Quatuor itaque nunc systemata linearum in superficie curva concipi possunt, pro quibus resp. q, p, q', p' sint constantes. Si per punctum determinatum, cui respondent variabilium valores p, q, p', q' quatuor lineas ad singula illa systemata pertintentes ductas supponimus, harum elementa, variationibus positivis $\mathrm{d}p$, $\mathrm{d}q$, $\mathrm{d}p'$, $\mathrm{d}q'$, respondentes erunt

$$\sqrt{E}.\mathrm{d}p, \quad \sqrt{G}.\mathrm{d}q, \quad \sqrt{E'}.\mathrm{d}q', \quad \sqrt{G}.\mathrm{d}q'$$

Angulos, quos horum elementorum directiones faciunt cum directione fixa arbitraria, denotabimus per M, N, M', N', numerando eo sensu, quo iacet secunda respectu primae, ita sunt $\sin(N-M)$ fiat quantitas positiva: eodem sensu iacere supponemus (quod licet) quartam respectu tertiae, ita ut etiam $\sin(N'-M')$ sit quantitas positiva. His ita intellectis, si consideramus punctum aliud, a priore infinite parum distans, cui respondeant valores variabilium

$$p + \mathrm{d}p, \quad q + \mathrm{d}q, \quad p' + \mathrm{d}p', \quad q' + \mathrm{d}q'$$

levi attentione adhibita cognoscemus, fieri generaliter, i. e. independenter a valoribus variationum $\mathrm{d}p$, $\mathrm{d}q$, $\mathrm{d}p'$, $\mathrm{d}q'$,

$$\sqrt{E}.\mathrm{d}p.\sin M + \sqrt{G}.\mathrm{d}q.\sin N = \sqrt{E'}.\mathrm{d}p'.\sin M' + \sqrt{G'}.\mathrm{d}q'.\sin N'$$

quum utraque expressio nihil aliud sit, nisi distantia puncti novi a linea, a qua anguli directionum incipiunt. Sed habemus, per notationem iam supra introductam $N-M=\omega$ et per analogiam statuemus $N', M' = \omega'$, nec non insuper $N - M' = \psi$. Ita aequatio modo inventa exhiberi potest in forma sequenti

$$\begin{aligned} &\sqrt{E}.\mathrm{d}p.\sin(M' - \omega + \psi) + \sqrt{G}.\mathrm{d}q.\sin(M' + \psi) \\ &\qquad = \sqrt{E'}.\mathrm{d}p'.\sin M' + \sqrt{G'}.\mathrm{d}q'.\sin(M' + \omega) \end{aligned}$$

vel ita

$$\begin{aligned} &\sqrt{E}.\mathrm{d}p.\sin(N' - \omega - \omega' + \psi) + \sqrt{G}.\mathrm{d}q.\sin(N' - \omega' + \psi) \\ &\qquad = \sqrt{E'}.\mathrm{d}p'.\sin(N' - \omega') + \sqrt{G'}.\mathrm{d}q'.\sin N' \end{aligned}$$

Et quum aequatio manifesto independens esse debeat a directione initiali, hanc ad lubitum accipere licet. Statuendo itaque in forma secunda $N' = 0$, vel in prima $M' = 0$, obtinemus aequationes sequentes:

$$\sqrt{E'}.\sin\omega'.\mathrm{d}p' = \sqrt{E}.\sin(\omega+\omega'-\psi).\mathrm{d}p + \sqrt{G}.\sin(\omega'-\psi).\mathrm{d}q$$
$$\sqrt{G'}.\sin\omega'.\mathrm{d}q' = \sqrt{E}.\sin(\psi-\omega).\mathrm{d}p + \sqrt{G}.\sin\psi.\mathrm{d}q$$

quae aequationes quum identicae esse debeant sum his

$$\mathrm{d}p' = \alpha\,\mathrm{d}p + \beta\,\mathrm{d}q$$
$$\mathrm{d}q' = \gamma\,\mathrm{d}p + \delta\,\mathrm{d}q$$

suppeditabunt determinationem coëfficientium α, β, γ, δ. Erit scilicet

$$\alpha = \sqrt{\tfrac{E}{E'}}.\tfrac{\sin(\omega+\omega'-\psi)}{\sin\omega'}, \qquad \beta = \sqrt{\tfrac{G}{E'}}.\tfrac{\sin(\omega'-\psi)}{\sin\omega'}$$
$$\gamma = \sqrt{\tfrac{E}{G'}}.\tfrac{\sin(\psi-\omega)}{\sin\omega'}, \qquad \delta = \sqrt{\tfrac{G}{G'}}.\tfrac{\sin\psi}{\sin\omega'}$$

Adiungi debent aequationes

$$\cos\omega = \tfrac{F}{\sqrt{EG}}, \quad \cos\omega' = \tfrac{F'}{\sqrt{E'G'}}, \quad \sin\omega = \sqrt{\tfrac{EG-FF}{EG}}, \quad \sin\omega' = \sqrt{\tfrac{E'G'-F'F'}{E'G'}}$$

unde quatuor aequationes ita quoque exhiberi possunt

$$\alpha\sqrt{(E'G'-F'F')} = \sqrt{EG'}.\sin(\omega+\omega'-\psi)$$
$$\beta\sqrt{(E'G'-F'F')} = \sqrt{GG'}.\sin(\omega'-\psi)$$
$$\gamma\sqrt{(E'G'-F'F')} = \sqrt{EE'}.\sin(\psi-\omega)$$
$$\delta\sqrt{(E'G'-F'F')} = \sqrt{GE'}.\sin\psi$$

Quum per substitutiones $\mathrm{d}p' = \alpha\,\mathrm{d}p+\beta\,\mathrm{d}q$, $\mathrm{d}q' = \gamma\,\mathrm{d}p+\delta\,\mathrm{d}q$ trinomium $E'\,\mathrm{d}p'^2 + 2F'\,\mathrm{d}p'.\mathrm{d}q' + G'\,\mathrm{d}q'^2$ transire debeat in $E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p.\mathrm{d}q + G\,\mathrm{d}q^2$, facile obtinemus

$$EG - FF = (E'G' - F'F')(\alpha\delta - \beta\gamma)^2$$

et quum vice versa trinomium posterius rursus transire debeat in prius per substitutionem

$$(\alpha\delta-\beta\gamma)\,\mathrm{d}p = \delta\,\mathrm{d}p' - \beta\,\mathrm{d}q', \quad (\alpha\delta-\beta\gamma)\,\mathrm{d}q = -\gamma\,\mathrm{d}p' + \alpha\,\mathrm{d}q'$$

invenimus

$$E\delta\delta - 2F\gamma\delta + G\gamma\gamma = \tfrac{EG-FF}{E'G'-F'F'}.E'$$
$$E\beta\delta - F(\alpha\delta+\beta\gamma) + G\alpha\gamma = -\tfrac{EG-FF}{E'G'-F'F'}.F'$$
$$E\beta\beta - 2F\alpha\beta + G\alpha\alpha = \tfrac{EG-FF}{E'G'-F'F'}.G'$$

22.

A disquisitione generali art. praec. descendimus ad applicationem latissime patentem, ubi, dum p et q etiam significatione generalissima accipiuntur, pro p', q', adoptamus quantitates in art. 15 per r, φ denotatas, quibus characteribus etiam hic utemur, scilicet ut pro quovis puncto superficiei r sit distantia minima a puncto determinato, atque φ angulus in hoc puncto inter elementum primum ipsius r atque directionem fixam. Ita habemus $E' = 1$, $F' = 0$, $\omega' = 90°$: statuemus insuper $\surd G' = m$, ita ut elementum lineare quodcunque fiat $= \surd(\mathrm{d}r^2 + mm\,\mathrm{d}\varphi^2)$. Hinc quatuor aequationes in art. praec. pro α, β, γ, δ, erutae, suppeditant:

$$\surd E.\cos(\omega - \psi) = \frac{\mathrm{d}r}{\mathrm{d}p} \quad \ldots\ldots \quad (1)$$

$$\surd G.\cos\psi = \frac{\mathrm{d}r}{\mathrm{d}q} \quad \ldots\ldots \quad (2)$$

$$\surd E.\sin(\psi - \omega) = m.\frac{\mathrm{d}\varphi}{\mathrm{d}p} \quad \ldots\ldots \quad (3)$$

$$\surd G.\sin\psi = m.\frac{\mathrm{d}\varphi}{\mathrm{d}q} \quad \ldots\ldots \quad (4)$$

Ultima et penultima vero has

$$EG - FF = E\left(\frac{\mathrm{d}r}{\mathrm{d}q}\right)^2 - 2F.\frac{\mathrm{d}r}{\mathrm{d}p}.\frac{\mathrm{d}r}{\mathrm{d}q} + G\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 \quad \ldots\ldots \quad (5)$$

$$\left(E.\frac{\mathrm{d}r}{\mathrm{d}q} - F.\frac{\mathrm{d}r}{\mathrm{d}p}\right) = \left(F.\frac{\mathrm{d}r}{\mathrm{d}q} - G.\frac{\mathrm{d}r}{\mathrm{d}p}\right).\frac{\mathrm{d}\varphi}{\mathrm{d}p} \quad \ldots\ldots \quad (6)$$

Ex his aequationibus petenda est determinatio quantitatum r, φ, ψ et (si opus videatur) m, per p et q: scilicet integratio aequationis (5) dabit r, qua inventa integratio aequationis (6) dabit φ, atque alterutra aequationum (1), (2) ipsam ψ: denique m habebitur per alterutram aequationum (3), (4).

Integratio generalis aequationum (5), (6) necessario duas functiones arbitrarias introducere debet, ques quid sibi velint facile intelligemus, si perpendimus, illa aequationes ad casum eum quem hic consideramus non limitari, sed perinde valere, si r et φ accipiantur in significatione generaliore art. 16, ita ut sit r longitudo lineae brevissimae ad lineam arbitrariam determinatam normaliter ductae, atque φ functio arbitraria longitudinis eius partis lineae, quae inter lineam brevissimam indefinitam et punctum arbitrarium determinatum intercipitur. Solutio itaque genearlis haec omnia indefinite amplecti debet, functionesque arbitrariae tunc demum in definitas abibunt, quando linea illa arbitraria atque functio prtium, quam φ exhibere debet, praescriptae sunt. In casu nostro circulus infinite parvus adoptari potest, centrum in eo punctum gabens, a quo distantiae r numerantur, et φ denotabit partes huius circuli ipsas per radium divisas, unde facile colligetur, aequationes (5), (6) pro casu nostro complete sufficere, dummodo ea, quae indefinita relinquunt, ei conditioni accommodentur, ut r et φ pro puncto illo initiali atque punctis ad eo infinite parum distantibus quadrent.

Ceterum quod attinet ad integrationem ipsam aequationum (5), (6), constat, eam reduci posse ad integrationem aequationum differentialium vulgarium, quae tamen plerumque tam intricata evadunt, ut parum lucri inde redundet. Contra evolutio in series, quae ad usus practicos, quoties de partibus superficiei modicis agitur, abunde sufficiunt, nullis difficultatibus obnoxia est, atque sic formulae allatae fontem uberem aperiunt, ad multa problemata gravissima solvenda. Hoc vero loco exemplum unicum ad methodi indolem monstrandam evolvemus.

23.

Considerabimus casus eum, ubi omnes lineae, pro quibus p constans est, sunt lineae brevissimae orthogonaliter secantes lineam, pro qua $\varphi = 0$, et quam tamquam lineam abscissarum contemplari possumus. Sit A punctum, pro quo $r = 0$, D punctum indefinitum in linea abscissarum, $AD = p$, B punctum indefinitum in linea brevissima ipsi AD in D normali, atque $BD = q$, ita ut p considerari possit tamquam abscissa, q tamquam ordinata puncti B; abscissas positivas assumimus in eo ramo lineae abscissarum, cui respondet $\varphi = 0$, dum r sempre tamquam quantitatem positivam spectamus; ordinatas positivas statuimus in plaga ea, ubi φ numeratur inter 0 et $180°$.

Per theorema art. 16 habebimus $\omega = 90°$, $F = 0$, nec non $G = 1$; statuemus insuper $\sqrt{E} = n$. Erit itaque n functio ipsarum p, q, et quidem talis, quae pro $q = 0$ fieri debet $= 1$. Applicatio formulae in art. 18 allatae ad casum nostrum docet, in *quavis* linea brevissima esse debere $\mathrm{d}\theta = -\frac{\mathrm{d}n}{\mathrm{d}q}.\mathrm{d}p$, denotante θ angulum inter elementum huius linear atque elementum lineae, pro qua q constans: iam quum linea abscissarum ipsa sit brevissima, atque pro ea ubique $\theta = 0$, patet, pro $q = 0$ ubique fieri debere $\frac{\mathrm{d}n}{\mathrm{d}q} = 0$. Hinc igitur colligimus, si n in seriem secundum potestates ipsius q progredientem evolvatur, hanc habere debere formam sequentem

$$n = 1 + fqq + gq^3 + hq^4 + \text{etc.}$$

ubi f, g, h, etc. erunt functiones ipsius p, et quidem statuemus

$$\begin{aligned} f &= f^0 + f'p + f''pp + \text{etc.} \\ g &= g^0 + g'p + g''pp + \text{etc.} \\ h &= h^0 + h'p + h''pp + \text{etc.} \end{aligned}$$

etc. sive

$$\begin{aligned} n = 1 + f^0qq &+ f'pqq + f''ppqq + \text{etc.} \\ &+ g^0q^3 \;\; + g'pq^3 + \text{etc.} \\ &\qquad\qquad + h^0q^4 + \text{etc. etc.} \end{aligned}$$

24.

Aequationes art. 22 in casu nostro suppeditant

$$n\sin\psi = \frac{\mathrm{d}r}{\mathrm{d}p}, \quad \cos\psi = \frac{\mathrm{d}r}{\mathrm{d}q}, -n\cos\psi = m.\frac{\mathrm{d}\varphi}{\mathrm{d}p}, \quad \sin\psi = m.\frac{\mathrm{d}\varphi}{\mathrm{d}q}$$

$$nn = nn\left(\frac{\mathrm{d}r}{\mathrm{d}q}\right)^2 + \frac{\mathrm{d}r}{\mathrm{d}p}^2, \qquad nn.\frac{\mathrm{d}r}{\mathrm{d}q}\frac{\mathrm{d}\varphi}{\mathrm{d}q} + \frac{\mathrm{d}r}{\mathrm{d}p}\frac{\mathrm{d}\varphi}{\mathrm{d}p} = 0$$

Adiumento harum aequationum, quarum quinta et sexta iam in reliquis continentur, series evolvi poterunt pro r, φ, ψ, m, vel pro quibuslibet functionibus harum quantitatum, e quibus eas, quae imprimis attenrione sunt dignae, hic sistemus.

Quum pro valoribus infinite parvis ipsarum p, q fieri debeat $rr = pp+qq$, series pro rr incipiet a terminis $pp + qq$: terminos altiorum ordinum obtinemus per methodum coëfficientium indeterminatorum*) adiumento aequationis

$$\left(\frac{1}{n}.\frac{\mathrm{d}rr}{\mathrm{d}p}\right)^2 + \left(\frac{\mathrm{d}rr}{\mathrm{d}q}\right)^2 = 4rr$$

scilicet

$$[1] \qquad \begin{aligned} rr = pp &+ \tfrac{2}{3}f^0ppqq + \tfrac{1}{2}f'p^3qq + \left(\tfrac{2}{3}f'' - \tfrac{4}{45}f^0f^0\right)p^4qq \quad \text{etc.} \\ +qq &+ \tfrac{1}{2}g^0ppq^2 + \tfrac{2}{5}g'p^3q^3 \\ &+ \left(\tfrac{2}{5}h^0 - \tfrac{7}{45}f^0f^0\right)ppq^4 \end{aligned}$$

Dein habemus, ducente formula $r\sin\psi = \frac{1}{2n}.\frac{\mathrm{d}rr}{\mathrm{d}p}$,

$$[2] \qquad \begin{aligned} r\sin\psi = p &- \tfrac{1}{5}f^0pqq - \tfrac{1}{4}f'ppqq - \left(\tfrac{1}{5}f'' + \tfrac{8}{45}f^0f^0\right)p^3qq \quad \text{etc.} \\ &- \tfrac{1}{2}g^0pq^3 - \tfrac{2}{5}g'ppq^3 \\ &- \left(\tfrac{2}{5}h^0 - \tfrac{8}{45}f^0f^0\right)pq^4 \end{aligned}$$

nec non per formulam $r\cos\psi = \frac{1}{2}.\frac{\mathrm{d}rr}{\mathrm{d}q}$

$$[3] \qquad \begin{aligned} r\cos\psi = q &+ \tfrac{2}{3}f^0ppq + \tfrac{1}{2}f'p^3q + \left(\tfrac{2}{5}f'' - \tfrac{4}{45}f^0f^0\right)p^4q \quad \text{etc.} \\ &- \tfrac{3}{4}g^0ppqq - \tfrac{3}{5}g'p^3qq \\ &+ \left(\tfrac{4}{5}h^0 - \tfrac{14}{45}f^0f^0\right)ppq^3 \end{aligned}$$

Hinc simul innotescit angulus ψ. Perinde ad computum anguli φ concinnius evolvuntur series pro $r\cos\varphi$ atque $r\sin\varphi$, quibus inserviunt aequationes differentiales partiales

$$\frac{\mathrm{d}.r\cos\varphi}{\mathrm{d}p} = n\cos\varphi.\sin\psi - r\sin\varphi.\frac{\mathrm{d}\varphi}{\mathrm{d}p}$$

*) Calculum, qui per nonnulla artificia paullulum contrahi potest, hic adscribere superfluum duximus.

$$\begin{aligned}
\frac{\mathrm{d}\,.r\cos\varphi}{\mathrm{d}q} &= \quad \cos\varphi.\cos\psi - r\sin\varphi.\frac{\mathrm{d}\varphi}{\mathrm{d}q}\\
\frac{\mathrm{d}\,.r\sin\varphi}{\mathrm{d}p} &= n\sin\varphi.\sin\psi + r\cos\varphi.\frac{\mathrm{d}\varphi}{\mathrm{d}p}\\
\frac{\mathrm{d}\,.r\sin\varphi}{\mathrm{d}q} &= \quad \sin\varphi.\cos\psi + r\cos\varphi.\frac{\mathrm{d}\varphi}{\mathrm{d}q}
\end{aligned}$$

$$n\cos\psi.\frac{\mathrm{d}\varphi}{\mathrm{d}q} + \sin\psi.\frac{\mathrm{d}\varphi}{\mathrm{d}p} = 0$$

quarum combinatio suppeditat

$$\frac{r\sin\psi}{n}.\frac{\mathrm{d}\,.r\cos\varphi}{\mathrm{d}p} + r\cos\psi.\frac{\mathrm{d}\,.r\cos\varphi}{\mathrm{d}q} = r\cos\varphi$$

$$\frac{r\sin\psi}{n}.\frac{\mathrm{d}\,.r\sin\varphi}{\mathrm{d}p} + r\cos\psi.\frac{\mathrm{d}\,.r\sin\varphi}{\mathrm{d}q} = r\sin\varphi$$

Hinc facile evolvuntur series pro $r\cos\varphi$, $r\sin\varphi$, quarum termini primi manifesto esse debent p et q, puta

[4] $$\begin{aligned} r\cos\varphi = p + \tfrac{2}{3}f^0pqq + \tfrac{5}{12}f'ppqq + \left(\tfrac{3}{10}f'' - \tfrac{8}{45}f^0f^0\right)p^3qq \quad \text{etc.}\\
+\tfrac{1}{2}g^0pq^3 + \tfrac{7}{26}g'ppq^3\\
+\left(\tfrac{2}{5}h^0 - \tfrac{7}{45}f^0f^0\right)pq^4 \end{aligned}$$

[5] $$\begin{aligned} r\sin\varphi = q - \tfrac{1}{3}f^0ppq - \tfrac{1}{6}f'p^3q - \left(\tfrac{1}{10}f'' - \tfrac{7}{90}f^0f^0\right)p^4q \quad \text{etc.}\\
-\tfrac{1}{4}g^0ppqq - \tfrac{3}{26}g'p^3qq\\
+\left(\tfrac{1}{5}h^0 - \tfrac{13}{90}f^0f^0\right)ppq^3 \end{aligned}$$

E combinatione aequationum [2], [3], [4], [5] derivari posset serie pro $rr\cos(\psi+\varphi)$, atque hinc, dividendo per seriem [1], series pro $\cos(\psi+\varphi)$, a qua ad seriem pro ipso angulo $\psi+\varphi$ descendere liceret. Elegantius tamen eadem obtinetur sequenti modo. Differentiando aequationem primam et secundam ex iis, quae initio huius art. allatae sunt, obtinemus

$$\sin\psi.\frac{\mathrm{d}n}{\mathrm{d}q} + n\cos\psi.\frac{\mathrm{d}\psi}{\mathrm{d}q} + \sin\psi.\frac{\mathrm{d}\psi}{\mathrm{d}q} = 0$$

qua combinata cum hac

$$n\cos\psi.\frac{\mathrm{d}\varphi}{\mathrm{d}q} + \sin\psi.\frac{\mathrm{d}\varphi}{\mathrm{d}p} = 0$$

prodit

$$\frac{r\sin\psi}{n}.\frac{\mathrm{d}n}{\mathrm{d}q} + \frac{r\sin\psi}{n}.\frac{\mathrm{d}(\psi+\varphi)}{\mathrm{d}p} + r\cos\psi.\frac{\mathrm{d}(\psi+\varphi)}{\mathrm{d}q} = 0$$

Ex hac aequatione adiumento methodi coëfficientium indeterminatorum facile eliciemus seriem pro $\psi+\varphi$, si perpendimus, ipsius terminum primum esse debere $\frac{1}{2}\pi$, radio pro unitate accepto, atque denotante 2π peripheriam circuli,

[6] $$\psi + \varphi = \tfrac{1}{2}\pi - f^0pq - \tfrac{2}{3}f'ppq - \left(\tfrac{1}{2}f'' - \tfrac{1}{6}f^0f^0\right)p^3q \text{ etc.}$$

$$\begin{aligned}&-g^0 pqq - \tfrac{3}{4} g' ppqq\\ &\qquad -(h^0 - \tfrac{1}{3} f^0 f^0) pq^3\end{aligned}$$

Operae pretium videtur, etiam aream trianguli ABD in seriem evolvere. Huic evolutioni inservit aequatio conditionalis sequens, quae e considerationibus geometricis satis obviis facile derivatur, et in qua S aream quaesitam denotat:

$$\frac{r \sin\psi}{n}.\frac{\mathrm{d}S}{\mathrm{d}p} + r\cos\psi.\frac{\mathrm{d}S}{\mathrm{d}q} = \frac{r\sin\psi}{n}.\int n\,\mathrm{d}q$$

integratione a $q = 0$ incepta. Hinc scilicet obtinemus per methodum coëfficientium indeterminatorum

[7]
$$\begin{aligned}S = {}&\tfrac{1}{2}pq - \tfrac{1}{12} f^0 p^3 q - \tfrac{1}{20} f' p^4 q - \left(\tfrac{1}{30} f'' - \tfrac{1}{60} f^0 f^0\right) p^5 q \text{ etc.}\\ &-\tfrac{1}{2} f^0 pq^3 - \tfrac{2}{30} g^0 p^3 qq - \tfrac{1}{20} g' p^4 qq\\ &\quad -\tfrac{7}{120} f' ppq^3 - \left(\tfrac{1}{15} h^0 + \tfrac{2}{45} f'' + \tfrac{1}{60} f^0 f^0\right) p^3 q^3\\ &\quad -\tfrac{1}{10} g^0 pq^4 - \tfrac{3}{40} g' ppq^4\\ &\qquad -\left(\tfrac{1}{10} h^0 - \tfrac{1}{30} f^0 f^0\right) pq^5\end{aligned}$$

25.

A formulis art. praec., quae referuntur ad triangulum a lineis brevissimis formatum rectangulum, progredimur ad generalis. Sit C aliud punctum in eadem linea brevissima DB, pro quo, manente p, characteres q', r', φ', ψ', S' eadem designent, quae q, r, φ, ψ, S pro puncto B. Ita oritur triangulum inter puncta A, B, C, cuius angulos per A, B, C, latera opposita per a, b, c, aream per σ denotamus; mensuram curvaturae in punctis A, B, C resp. per α, β, γ exprimemus. Supponendo itaque (quod licet), quantitates p, q, $q - q'$ esse positivas, habemus

$$A = \varphi - \varphi', \quad B = \psi, \quad C = \pi - \psi', \quad a = q - q', \quad b = r', \quad c = r, \quad \sigma = S - S'$$

Ante omnia aream $sigma$ per seriem exprimemus. Mutando in [7] singulas quantitates ad B relatas in eas, quae ad C referuntur, prodit formula pro S', unde, usque ad quantitates sexti ordinis obtinemus

$$\begin{aligned}\sigma = \tfrac{1}{2}p(q - q')\{1 &- \tfrac{1}{6} f^0 (pp + qq + qq' + q'q')\\ &- \tfrac{1}{60} f' p(6pp + 7qq + 7qq' + 7q'q')\\ &- \tfrac{1}{20} g^0 (q + q')(3pp + 4qq + 4qq' + 4q'q')\}\end{aligned}$$

Haec formula, adiumento seriei [2] puta

$$c \sin B = p(1 - \tfrac{1}{3} f^0 qq - \tfrac{1}{4} f' pqq - \tfrac{1}{2} g^0 q^3 - \text{etc.})$$

transit in sequentem

$$\sigma = \tfrac{1}{2}ac\sin B\{1 - \tfrac{1}{6}f^0(pp - qq + qq' + q'q')$$
$$- \tfrac{1}{60}f'p(6pp - Sqq + 7qq' + 7q'q')$$
$$- \tfrac{1}{20}g^0(3pp + 3ppq' - 6q^3 + 4qqq' + 4qq'q' + 4q'^3)\}$$

Mensura curvaturae pro quovis superficiei puncto fit (per art. 19, ubi m, p, q erant quae hic sunt n, q, p)

$$= -\tfrac{1}{n}.\tfrac{\mathrm{d}\,\mathrm{d}n}{\mathrm{d}q^2} = -\tfrac{2f+5gq+12hqq+\text{etc.}}{1+fqq+\text{etc.}} = M - 2f - 6gq - (12h - 2ff)qq - \text{etc.}$$

Hinc fit, quatenus p, q ad punctum B referuntur.

$$\beta = -2f^0 - 2f'p - 6g^0q - 2f''pp - 6g'pq - (12h^0 - 2f^0f^0)qq - \text{etc.}$$

nec non

$$\gamma = -2f^0 - 2f'p - 6g^0q' - 2f''pp - 6g'pq' - (12h^0 - 2f^0f^0)q'q' - \text{etc.}$$
$$\alpha = -2f^0$$

Introducendo has mensuras curvaturae in serie pro σ, obtinemus expressionem sequentem, usque ad quantitates sexti ordinis (excl.) exactam:

$$\sigma = \tfrac{1}{2}ac\sin B\Big\{1 + \tfrac{1}{120}\alpha\,(4pp - 2qq + 3qq' + 3q'q')$$
$$+ \tfrac{1}{120}\beta\,(3pp - 6qq + 6qq' + 3q'q')$$
$$+ \tfrac{1}{120}\gamma\,(3pp - 2qq + qq' + 4q'q')\Big\}$$

Praecisio eadem manebit, si pro p, q, q' substituimus $c\sin B$, $c\cos B$, $c\cos B - a$, quo pacto prodit

[8] $$\sigma = \tfrac{1}{2}ac\sin B\Big\{1 + \tfrac{1}{120}\alpha\,(3aa + 4cc - 9ac\cos B)$$
$$+\tfrac{1}{120}\beta\,(3aa + 3cc - 12ac\cos B)$$
$$+\tfrac{1}{120}\gamma\,(4aa + 3cc - 9ac\cos B)\Big\}$$

Quum ex hac aequatione omnia, quae ad lineam AD normaliter ad BC ductam referuntur, evanuerint, etiam puncta A, B, C, cuum correlatis inter se permutare licebit, quapropter erit eadem praecisione

[9] $$\sigma = \tfrac{1}{2}bc\sin A\Big\{1 + \tfrac{1}{120}\alpha\,(3bb + 3cc - 12bc\cos A)$$

$$+\tfrac{1}{120}\beta\,(3bb+4cc-9bc\cos A)$$
$$+\tfrac{1}{120}\gamma\,(4bb+3cc-9bc\cos B)\Big\}$$

[10] $$\sigma=\tfrac{1}{2}ab\sin C\Big\{1+\tfrac{1}{120}\alpha\,(3aa+4bb-9ab\cos C)$$
$$+\tfrac{1}{120}\beta\,(4aa+3bb-9ab\cos C)$$
$$+\tfrac{1}{120}\gamma\,(3aa+3bb-12ab\cos C)\Big\}$$

26.

Magnam utilitatem affert consideratio trianguli plani rectilinei, cuius latera aequalia sunt ipsis a, b, c; angulis illius trianguli, quos per A^*, B^*, C^* designabimus, different ab angulis trianguli in superficie curva, puta ab A, B, C, quantitatibus secundi ordinis, operaeque pretium erit, has differentias accurate evolvere. Calculorum autem prolixiorum quam difficiliorum, primaria momenta apposuisse sufficiet.

Mutando in formulis [1], [4], [5], quantitates, quae referuntur ad B, in eas, quae referuntur ad C, nanciscemur formulas pro $r'r'$, $r'\cos\varphi'$, $r'\sin\varphi'$. Tunc evolutio expressionis $rr+r'r'-(q-q')^2-2r\cos\varphi.r'\cos\varphi'-2r\sin\varphi.r'\sin\varphi'$, quae fit $=bb+cc-aa-2bc\cos A=2bc(\cos A^*-\cos A)$, combinata cum evolutione expressionis $r\sin\varphi.r'\cos\varphi'-r\cos\varphi.r'\sin\varphi'$, quae fit $=bc\sin A$, suppeditat formulam sequentem

$$\begin{aligned}\cos A^*-\cos A=-(q-q')p\sin A\{&\tfrac{1}{3}f^0+\tfrac{1}{6}f'p+\tfrac{1}{4}g^0(q+q')\\&+\left(\tfrac{1}{10}f''-\tfrac{1}{45}f^0f^0\right)pp+\tfrac{3}{20}g'p(q+q')\\&+\left(\tfrac{1}{5}h^0-\tfrac{7}{90}f^0f^0\right)(qq+qq'+q'q')+\text{etc.}\}\end{aligned}$$

Hinc fit porro, usque ad quantitates quinti ordinis

$$\begin{aligned}A^*-A=-(q-q')p\{&\tfrac{1}{3}f^0+\tfrac{1}{6}f'p+\tfrac{1}{4}g^0(q+q')+\tfrac{1}{10}f''pp\\&+\tfrac{3}{20}g'p(q+q')+\tfrac{1}{5}h^0(qq+qq'+q'q')\\&-\tfrac{1}{90}f^0f^0\,(7pp+7qq+12qq'+7q'q')\}\end{aligned}$$

Combinando hanc formulam cum hac

$$2\sigma=ap\left(1-\tfrac{1}{6}f^0\,(pp+qq+qq'+q'q'+\text{etc.})\right)$$

atque cum valoribus quantitatum α, β, γ in art. praec. allatis, obtinemus usque ad quantitates quinti ordinis

[11] $$A^*=A-\sigma\{\tfrac{1}{6}\alpha+\tfrac{1}{12}\beta+\tfrac{1}{12}\gamma+\tfrac{2}{15}f''pp+\tfrac{1}{5}g'p(q+q')$$

$$+\tfrac{1}{5}h^0\,(3qq - 2qq' + 3q'q')$$
$$+\tfrac{1}{90}f^0f^0\,(4pp - 11qq + 14qq' - 11q'q')\}$$

Per operationes prorsus similes evolvimus

[12] $$B^* = B - \sigma\{\tfrac{1}{12}\alpha + \tfrac{1}{6}\beta + \tfrac{1}{12}\gamma + \tfrac{1}{10}f''pp + \tfrac{1}{10}g'p(2q+q')$$
$$+\tfrac{1}{5}h^0\,(4qq - 4qq' + 4q'q')$$
$$+\tfrac{1}{90}f^0f^0\,(2pp + 8qq - 8qq' + 11q'q')\}$$

[13] $$C^* = C - \sigma\{\tfrac{1}{12}\alpha + \tfrac{1}{12}\beta + \tfrac{1}{6}\gamma + \tfrac{1}{10}f''pp + \tfrac{1}{10}g'p(q+2q')$$
$$+\tfrac{1}{5}h^0\,(3qq - 4qq' + 4q'q')$$
$$+\tfrac{1}{90}f^0f^0\,(2pp + 11qq - 8qq' + 8q'q')\}$$

Hinc simul deducimus, quum summa $A^* + B^* + C^*$ duobus rectis aequalis sit, excessum summae $A + B + C$ supra duos angulos rectos, puta

[14] $$A + B + A = \pi + \sigma\{\tfrac{1}{3}\alpha + \tfrac{1}{3}\beta + \tfrac{1}{3}\gamma + \tfrac{1}{3}f''pp + \tfrac{1}{2}g'p(q+q')$$
$$+(2h^0 - \tfrac{1}{3}f^0f^0)(qq - qq' + q'q')\}$$

Haec ultima aequatio etiam formulae [6] superstrui potuisset.

27.

Si superficies curva est sphaera, cuius radium $= R$, erit

$$\alpha = \beta = \gamma = -2f^0 = \tfrac{1}{RR};\qquad f'' = 0,\quad g' = 0,\quad 6h^0 - f^0f^0 = 0 \quad \text{sive} \quad k^0 = \tfrac{1}{24R^4}$$

Hinc formula [14] fit

$$A + B + C = \pi + \tfrac{\sigma}{RR}$$

quae praecisione absoluta gaudet; formulae 11-13 autem suppeditant

$$A^* = A - \tfrac{\sigma}{3RR} - \tfrac{\sigma}{180R^*}\,(2pp - qq + 4qq' - q'q')$$
$$B^* = B - \tfrac{\sigma}{3RR} - \tfrac{\sigma}{180R^*}\,(pp - 2qq + 2qq' + q'q')$$
$$C^* = C - \tfrac{\sigma}{3RR} - \tfrac{\sigma}{180R^*}\,(pp + qq + 2qq' - 2q'q')$$

sive aeque exacte

$$A^* = A - \tfrac{\sigma}{3RR} - \tfrac{\sigma}{180R^*}\,(bb + cc - 2aa)$$
$$B^* = B - \tfrac{\sigma}{3RR} - \tfrac{\sigma}{180R^*}\,(aa + cc - 2bb)$$
$$C^* = C - \tfrac{\sigma}{3RR} - \tfrac{\sigma}{180R^*}\,(aa + bb - 2cc)$$

Neglectis quantitatibus quarti ordinis, prodit hinc theorema notum a clar. LEGENDRE primo propositum.

28.

Formulae nostrae generales, reiectis terminis quarti ordinis, presimplices evadunt scilicet

$$\begin{aligned} A^* &= A - \tfrac{1}{12}\sigma\,(2\alpha + \beta + \gamma) \\ B^* &= B - \tfrac{1}{12}\sigma\,(\alpha + 2\beta + \gamma) \\ C^* &= C - \tfrac{1}{12}\sigma\,(\alpha + \beta + 2\gamma) \end{aligned}$$

Angulis itaque A, B, C in superficie non sphaerica reductiones inaequales applicandae sunt, ut mutatorum sinus lateribus oppositis fiant proportionales. Inaequalitas generaliter loquendo erit tertii ordinis, at si superficies parum a sphaera discrepat, illa ad ordinem altiorem referenda erit: in triangulis vel maximis in superficie telluris, quorum quidem angulos dimetiri licet, differential semper pro insensibili haberi potest. Ita e.g. in triangulo maximo inter ea, quae annis praecedentibus dimensi sumus, puta inter puncta Hohehagen, Brocken, Inselsberg, ubi excessus summae angulorum fuit $= 14''85348$, calculus sequentes reductiones anguli applicandas prodidit:

Hohehagen..........	$-\,4''95113$
Brocken..........	$-\,4,95104$
Inselsberg..........	$-\,4,95131$

29.

Coronis caussa adhuc comparationem areae trianguli in superficie curva cum area trianguli rectilinei, cuius latera sunt a, b, c, adiiciemus. Aream posteriorem denotabimus per σ^*, quae fit $= \frac{1}{2}bc\sin A^* = \frac{1}{2}ac\sin B^* = \frac{1}{2}ab\sin C^*$.

Habemus, usque ad quantitates ordinis quarti

$$\sin A^* = \sin A - \tfrac{1}{12}\sigma\cos A.(2\alpha + \beta + \gamma)$$

sive aeque exacte

$$\sin A = \sin A^*.\left(+\tfrac{1}{24}bc\cos A.(2\alpha + \beta + \gamma)\right)$$

Substituto hoc valore in formula [9], erit usque ad quantitates sexti ordinis

$$\begin{aligned} \sigma = \tfrac{1}{2}bc\sin A^*.\{1 + \tfrac{1}{120}\alpha(3bb + 3cc - 2bc\cos A) + \tfrac{1}{120}\beta(3bb + 4cc - 4bc\cos A) \\ + \tfrac{1}{120}\gamma(4bb + 3cc - 4bc\cos A)\} \end{aligned}$$

sive aeque exacte

$$\begin{aligned} \sigma = \sigma^*\{1 + \tfrac{1}{120}\alpha(aa + 2bb + 2cc) + \tfrac{1}{120}\beta(2aa + bb + 2cc) \\ + \tfrac{1}{120}\gamma(2aa + 2bb + cc)\} \end{aligned}$$

Pro superficie sphaerica haec formula sequentem induit formam

$$\sigma = \sigma^*(1 + \tfrac{1}{24}\alpha(aa + bb + cc))$$

cuius loco etiam sequentem salva eadem praecisione adoptari posse facile confirmatur

$$\sigma = \sigma^* \sqrt{\tfrac{\sin A . \sin B . \sin C}{\sin A^* . \sin B^* . \sin C^*}}$$

Si eadem formula triangulis in superficie curva non sphaerica applicatur, error generaliter loquendo erit quinti ordinis, sed insensibilis in omnibus triangulis, qualia in superficie telluris dimetiri licet.