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Operators on Hilbert spaces

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- Kernels, boundedness, continuity
- Adjoints of maps on Hilbert spaces
- Stable subspaces and complements
- Spectrum, eigenvalues

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1. Kernels, boundedness, continuity

Definition: A linear (not necessarily continuous) map $T : X \to Y$ from one Hilbert space to another is **bounded** if, for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in X$ with $|x|_X < \delta$ we have $|Tx|_Y < \varepsilon$. The following simple result is used constantly.

Proposition: Let $T : X \to Y$ be a linear (not necessarily continuous) map. Then the following three conditions are equivalent:
(i) T is continuous
(ii) T is continuous at 0
(iii) T is bounded

Proof: Suppose T is continuous as 0. Given $\varepsilon > 0$ and $x \in X$, let $\delta > 0$ be small enough such that for $|x' - 0|_X < \delta$ we have $|Tx' - 0|_Y < \varepsilon$. Then for $|x" - x|_X < \delta$, using the linearity, we have

$$|Tx" - Tx|_X = |T(x" - x) - 0|_X < \delta$$

That is, continuity at 0 implies continuity.

Since $|x| = |x - 0|$, continuity at 0 is immediately equivalent to boundedness. ///

Definition: The **kernel** $\ker T$ of a linear (not necessarily continuous) linear map $T : X \to Y$ from one Hilbert space to another is

$$\ker T = \{x \in X : Tx = 0 \in Y\}$$

Proposition: The kernel of a continuous linear map $T : X \to Y$ is closed.

Proof: For T continuous

$$\ker T = T^{-1}\{0\} = X - T^{-1}(Y - \{0\}) = X - T^{-1}(\text{open}) = X - \textit{open} = \text{closed}$$

since the inverse images of open sets by a continuous map are continuous. ///

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2. Adjoints of maps on Hilbert spaces

Definition: An **adjoint** T^* of a continuous linear map $T : X \to Y$ from a pre-Hilbert space X to a pre-Hilbert space Y (if T^* exists) is a continuous linear map $T^* : Y^* \to X^*$ such that

$$\langle Tx, y\rangle_Y = \langle x, T^*y\rangle_X$$

Remark: Without an assumption that a pre-Hilbert space X is complete, hence a Hilbert space, we do not know that an operator $T : X \to Y$ has an adjoint.

Theorem: A continuous linear map $T : X \to Y$ of a *Hilbert* space X to a pre-Hilbert space Y *has* a unique adjoint T^*.

Remark: Note that the target space of T need not be a Hilbert space, that is, need not be complete.

Proof: For each fixed $y \in Y$, the map

$$\lambda_y : X \to \mathbf{C}$$

given by

$$\lambda_y(x) = \langle Tx, y \rangle$$

is a continuous linear functional on X. Thus, by the Riesz-Fischer theorem, there is a unique $x_y \in X$ so that

$$\langle Tx, y \rangle = \lambda_y(x) = \langle x, x_y \rangle$$

Take

$$T^* y = x_y$$

This is a perfectly well-defined map from Y to X, and has the crucial property $\langle Tx, y \rangle_Y = \langle x, T^* y \rangle_X$.

To prove that T^* is continuous, prove that it is bounded. From Cauchy-Schwarz-Bunyakowsky

$$|T^* y|^2 = |\langle T^* y, T^* y \rangle_X| = |\langle y, TT^* y \rangle_Y| \leq |y| \cdot |TT^* y| \leq |y| \cdot |T| \cdot |T^* y|$$

where $|T|$ is the *uniform* operator norm of T. If $T^* y \neq 0$, then we divide by it to find

$$|T^* y| \leq |y| \cdot |T|$$

Thus, $|T^*| \leq |T|$. In particular, T^* is bounded. Since $(T^*)^* = T$, we obtain $|T| = |T^*|$.

The linearity is easy. ///

Corollary: For a continuous linear map $T : X \to Y$ of Hilbert spaces, $T^{**} = T$. ///

An operator $T \in \mathrm{End}(X)$ is **normal** if it *commutes with its adjoint*, that is, if

$$TT^* = T^* T$$

This definition only makes sense in application to operators from a Hilbert space *to itself*. An operator T is **self-adjoint** or **hermitian** if $T = T^*$. That is, T is hermitian if

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for all $x, y \in X$. An operator T is **unitary** if

$$TT^* = T^* T = \text{ identity map } 1_X \text{ on } X$$

There are simple examples in infinite-dimensional spaces where $TT^* = 1$ does not imply $T^* T = 1$, and vice-versa. Thus, it does *not* suffice to check something like $\langle Tx, Tx \rangle = \langle x, x \rangle$ in order to prove unitariness. Obviously hermitian operators are normal. With this more careful definition of *unitary* operators, it is also immediate that unitary operators are normal.

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3. Stable subspaces and complements

Let $T : X \to X$ be a continuous linear operator on a Hilbert space X. A vector subspace is T-**stable** or T-**invariant** if $Ty \in Y$ for all $y \in Y$. Often one is most interested in the case that the subspace be *closed* in addition to being *invariant*.

Proposition: Let $T : X \to X$ be a continuous linear operator on a Hilbert space X, and let Y be a T-stable subspace. Then $Y^\perp$ is T^*-stable.

Proof: Take $z \in Y^\perp$ and $y \in Y$. Then

$$\langle T^* z, y \rangle = \langle z, T^{**} y \rangle = \langle z, Ty \rangle$$

since $T^{**} = T$, from above. Since Y is T-stable, $Ty \in Y$, and this inner product is 0. Thus, $T^* z \in Y^\perp$. ///

Corollary: Let T be a continuous linear operator on a Hilbert space X, and let Y be a *closed* T-stable subspace. For T *self-adjoint* both Y and $Y^\perp$ are T-stable. ///

Remark: The hypothesis of *normality* is insufficient to assure the conclusion of the corollary, in general. For example, with the two-sided ℓ^2 space

$$X = \{\{c_n : n \in \mathbf{Z}\} : \sum_{n \in \mathbf{Z}} |c_n|^2 < \infty\}$$

let T be the right shift operator

$$(Tc)_n = c_{n-1}$$

Then T^* is the left shift operator

$$(T^* c)_n = c_{n+1}$$

and

$$T^* T = T T^* = 1_X$$

So this T is not merely normal, but unitary. However, the T-stable subspace

$$Y = \{\{c_n\} \in X : c_k = 0 \text{ for } k < 0\}$$

is certainly not T^*-stable, and the orthogonal complement is not T-stable. On the other hand, if we look at adjoint-stable collections of operators, we recover a good stability result, as in the following proposition.

Proposition: Let A be a set of bounded linear operators on a Hilbert space V, and suppose that for $T \in A$ also the adjoint T^* is in A. Then for an A-stable closed subspace W of V, the orthogonal complement $W^\perp$ is also A-stable.

Proof: Let y be in $W^\perp$, and $T \in A$. Then for $x \in W$

$$\langle x, Ty \rangle = \langle T^* x, y \rangle \in \langle W, y \rangle = \{0\}$$

since $T^* \in A$. ///

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4. Spectrum, eigenvalues

For a continuous linear operator $T \in \mathrm{End}(X)$, the **λ-eigenspace** of T is

$$X_\lambda = \{x \in X : Tx = \lambda x\}$$

If this space is not simply $\{0\}$, then λ is an **eigenvalue**.

Proposition: An eigenspace X_λ for a continuous linear operator T on X is a *closed* and T-stable subspace of X. Further, for *normal* T the adjoint T^* acts by the scalar $\overline{\lambda}$ on X_λ.

Proof: The λ-eigenspace is the kernel of the continuous linear map $T-\lambda$, so is closed. The stability is clear, since the operator restricted to the eigenspace is a scalar operator. For $v \in X_\lambda$, using normality,

$$(T-\lambda)T^*v = T^*(T-\lambda)v = T^*\cdot 0 = 0$$

Thus, X_λ is T^*-stable. For $x,y\in X_\lambda$,

$$\langle (T^*-\overline{\lambda})x, y\rangle = \langle x, (T-\lambda)y\rangle = \langle x, 0\rangle$$

Thus, $(T^*-\overline{\lambda})x = 0$. ///

Proposition: For T *normal*, for $\lambda\neq\mu$, and for $x\in X_\lambda, y\in X_\mu$, always $\langle x,y\rangle = 0$. For T *self-adjoint*, if $X_\lambda \neq 0$ then $\lambda\in\mathbf{R}$. For T *unitary*, if $X_\lambda\neq 0$ then $|\lambda| = 1$.

Proof: Let $x\in X_\lambda$, $y\in X_\mu$, with $\mu\neq\lambda$. Then

$$\lambda\langle x,y\rangle = \langle Tx,y\rangle = \langle x,T^*y\rangle = \langle x,\overline{\mu}y\rangle = \mu\langle x,y\rangle$$

invoking the previous result. Thus,

$$(\lambda-\mu)\langle x,y\rangle = 0$$

which gives the result. For T self-adjoint and x a non-zero λ-eigenvector,

$$\lambda\langle x,x\rangle = \langle x,T^*x\rangle = \langle x,Tx\rangle = \langle x,\lambda x\rangle = \overline{\lambda}\langle x,x\rangle$$

Thus, $(\lambda-\overline{\lambda})\langle x,x\rangle = 0$. Since x is non-zero, the result follows. For T unitary and x a non-zero λ-eigenvector,

$$\langle x,x\rangle = \langle T^*Tx,x\rangle = \langle Tx,Tx\rangle = |\lambda|^2\cdot\langle x,x\rangle$$

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In what follows, for a complex scalar λ instead of the more cumbersome notation $\lambda\cdot 1_X$ for the scalar multiplication by λ on X we may write simply λ.

Definition: The **spectrum** $\sigma(T)$ of a continuous linear operator $T: X\to X$ on a Hilbert space X is the collection of complex numbers λ such that $T-\lambda$ has no (continuous linear) inverse.

Definition: The **discrete spectrum** $\sigma_{\rm disc}(T)$ is the collection of complex numbers λ such that $T-\lambda$ fails to be *injective*. (In other words, the discrete spectrum is the collection of eigenvalues.)

Definition: The **continuous spectrum** $\sigma_{\rm cont}(T)$ is the collection of complex numbers λ such that $T-\lambda\cdot 1_X$ *is* injective, *does* have dense image, but fails to be *surjective*.

Definition: The **residual spectrum** $\sigma_{\rm res}(T)$ is everything else: neither discrete nor continuous spectrum. That is, the residual spectrum of T is the collection of complex numbers λ such that $T-\lambda\cdot 1_X$ *is* injective, and *fails* to have dense image (so is certainly not surjective).

Proposition: A normal operator $T: X\to X$ has empty residual spectrum.

Proof: The adjoint of $T-\lambda$ is $T^*-\overline{\lambda}$, so we may as well consider $\lambda = 0$, to lighten the notation. Suppose that T does *not* have dense image. Then there is a non-zero vector z in the orthogonal complement to the image TX. Thus, for every $x\in X$,

$$0 = \langle z, Tx\rangle = \langle T^*z, x\rangle$$

Therefore $T^*z = 0$. Thus, the 0-eigenspace for T^* is non-zero. From just above, $T = T^{**}$ stabilizes the 0-eigenspace Z of T^*. Thus, Z is both T and T^*-stable. Therefore, from above, the orthogonal complement $Z^\perp$ of Z is both T and T^*-stable. Then for $z, z'\in Z$

$$\langle Tz, z'\rangle = \langle z, T^*z'\rangle = \langle z, 0\rangle = 0$$

This holds for all $z'\in Z$, so by the T-stability of Z we see that $Tz = 0$ for $z\in Z$. That is, T fails to be injective, having 0-eigenvectors Z. In other words, there is no residual spectrum. ///