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Hilbert spaces

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Hilbert spaces are possibly-infinite-dimensional analogues of the finite-dimensional Euclidean spaces familiar to us. In particular, Hilbert spaces have *inner products*, so notions of *perpendicularity* (or *orthogonality*), and *orthogonal projection* are available. Reasonably enough, in the infinite-dimensional case we must be careful not to extrapolate too far based only on the finite-dimensional case.

Perhaps strangely, few naturally-occurring spaces of functions are Hilbert spaces. Given the intuitive geometry of Hilbert spaces, this fact is a little disappointing, as it suggests that our physical intuition is a little distant from the behavior of spaces of functions, for example. [1]

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1. Pre-Hilbert spaces: definition

Let V be a complex vector space. A complex-valued function

$$\langle , \rangle : V \times V \to \mathbf{C}$$

of two variables on V is a **(hermitian) inner product** if

$$\begin{cases} \langle x, y\rangle &=& \overline{\langle y, x\rangle} & \text{(the } \textit{hermitian-symmetric} \text{ property)} \\ \langle x+x', y\rangle &=& \langle x, y\rangle + \langle x', y\rangle & (\textit{additivity in} \text{ first argument}) \\ \langle x, y+y'\rangle &=& \langle x, y\rangle + \langle x, y'\rangle & (\textit{additivity} \text{ in second argument}) \\ \langle x, x\rangle &\geq& 0 & \text{(and equality only for } x=0\text{: } \textit{positivity}) \\ \langle \alpha x, y\rangle &=& \alpha\langle x, y\rangle & (\textit{linearity} \text{ in first argument}) \\ \langle x, \alpha y\rangle &=& \bar{\alpha}\langle x, y\rangle & (\textit{conjugate-linearity} \text{ in second argument}) \end{cases}$$

Then V equipped with such a $\langle , \rangle$ is a **pre-Hilbert space**. Among other easy consequences of these requirements, for all $x, y \in V$

$$\langle x, 0\rangle = \langle 0, y\rangle = 0$$

where inside the angle-brackets the 0 is the zero-*vector*, and outside it is the zero-*scalar*.

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[1] However, a little later we will see that suitable *families* of Hilbert spaces *may* capture what we want. Such ideas originate with Sobolev in the 1930's. Sobolev's ideas were not widely known in the West when Schwartz formulated his notions of *distributions*, so were not directly incorporated. Certainly Sobolev's ideas fit into Schwartz' general scheme, but they do also offer some useful specifics, as we will see.

The **associated norm** $|\,|$ on V is defined by

$$|x| = \langle x, x\rangle^{1/2}$$

with the non-negative square-root. Even though we use the same notation for the norm on V as for the usual complex value $||$, context should always make clear which is meant.

Sometimes such spaces V with $\langle,\rangle$ are called *inner product spaces* or *hermitian inner product spaces.*

For two vectors v, w in a pre-Hilbert space, if $\langle v, w\rangle = 0$ then v, w are **orthogonal** or **perpendicular**, sometimes written $v \perp w$. A vector v is a **unit vector** if $|v| = 1$.

There are several essential algebraic identities, variously and ambiguously called **polarization identities**. First, there is

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

which is obtained simply by expanding the left-hand side and cancelling where opposite signs appear. In a similar vein,

$$|x + y|^2 - |x - y|^2 = \langle x, y\rangle + \langle y, x\rangle$$

Therefore,

$$(|x + y|^2 - |x - y|^2) + i(|x + iy|^2 - |x - iy|^2) = 2\langle x, y\rangle$$

These and closely-related identites are of frequent use.

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2. Cauchy-Schwarz-Bunyakowski inequality

This inequality is fundamental. It is necessary to prove that the triangle inequality holds for the norm, from which we get the associated *metric*, as indicated below.

The **Cauchy-Schwarz-Bunyakowsky inequality** in a pre-Hilbert space asserts that

$$|\langle x, y\rangle| \le |x| \cdot |y|$$

with strict inequality unless x, y *are collinear*, i.e., unless one of x, y is a multiple of the other.

Proof: Suppose that x is not a scalar multiple of y, and that neither x nor y is 0. Then $x - \alpha y$ is not 0 for any complex α. Consider

$$0 < |x - \alpha y|^2$$

We know that the inequality is indeed *strict* for all α since x is not a multiple of y. Expanding this,

$$0 < |x|^2 - \alpha\langle x, y\rangle - \bar{\alpha}\langle x, y\rangle + \alpha\bar{\alpha}|y|^2$$

Let

$$\alpha = \mu t$$

with real t and with $|\mu| = 1$ so that

$$\mu\langle x, y\rangle = |\langle x, y\rangle|$$

Then

$$0 < |x|^2 - 2t|\langle x, y\rangle| + |t^2|y|^2$$

The *minimum* of the right-hand side, viewed as a function of the real variable t, occurs when the derivative vanishes, i.e., when

$$0 = -2|\langle x, y\rangle| + 2t|y|^2$$

Using this minimization as a *mnemonic* for the value of t to substitute, we indeed substitute

$$t = \frac{|\langle x, y\rangle|}{|y|^2}$$

into the inequality to obtain

$$0 < |x|^2 + \left(\frac{|\langle x, y\rangle|}{|y|^2}\right)^2 \cdot |y|^2 - 2\frac{|\langle x, y\rangle|}{|y|^2} \cdot |\langle x, y\rangle|$$

which simplifies to

$$|\langle x, y\rangle|^2 < |x|^2 \cdot |y|^2$$

as desired. ///

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3. Example: spaces ℓ^2

Before any further abstract discussion of Hilbert spaces, we can note that, up to isomorphism,[2] there is just one *infinite-dimensional* Hilbert space occurring in practice,[3] namely the space ℓ^2 constructed as follows. Proof that *most* Hilbert spaces are isomorphic to this one will be given later.

Let ℓ^2 be the collection of sequences $a = \{a_i : 1 \le i < \infty\}$ of complex numbers meeting the constraint

$$\sum_{i=1}^{\infty} |a_i|^2 < +\infty$$

For two such sequences $a = \{a_i\}$ and $b = \{b_i\}$, the *inner product* is

$$\langle a, b\rangle = \sum_i a_i \overline{b_i}$$

The associated *norm* is[4]

$$|a| = \langle a, a\rangle^{1/2}$$

We can immediately generalize this construction in one fashion by replacing the countable set $\{1, 2, 3, \ldots\}$ by an arbitrary set A.[5] Let A be an arbitrary index set, and let $\ell^2(A)$ be the collection of complex-valued functions f on A such that

$$\sum_\alpha |f(\alpha)|^2 < +\infty$$

[2] And, lest anyone be fooled, often the description of such isomorphisms is where any subtlety lies.

[3] Most infinite-dimensional Hilbert spaces occurring in practice have a countable dense subset, and this itself is because the Hilbert spaces are completions of spaces of continuous functions on topological spaces with a countable basis to the topology. This will be amplified subsequently.

[4] That the triangle inequality holds is not immediate, needing the Cauchy-Schwarz-Bunyakowsky inequality. We will give the proof shortly.

[5] Replacement of $\{1, 2, \ldots\}$ by an arbitrary set A is mildly pointless except as an exercise in technique, since, as noted already, in practice we will rarely encounter Hilbert spaces not isomorphic to ℓ^2, if not already isomorphic to the finite-dimensional spaces $\mathbf{C}^n$.

For two such functions f, φ, define an inner product by

$$\langle f, \varphi \rangle = \sum_\alpha f(\alpha)\overline{\varphi(\alpha)}$$

For any *finite* subset A_o of A_o we can apply the *Cauchy-Schwarz-Bunyakowsky inequality* to obtain

$$|\sum_{\alpha \in A_o} f(\alpha)\overline{\varphi(\alpha)}| \le (\sum_{\alpha \in A_o} |f(\alpha)|^2)^{1/2}\ (\sum_{\alpha \in A_o} |\varphi(\alpha)|^2)^{1/2}$$

Thus, the net of all partial sums $\sum_{\alpha \in A_o} f(\alpha)\overline{\varphi(\alpha)}$ has a limit, so is necessarily *Cauchy*.

In fact, the sum $\langle f, \varphi \rangle = \sum_\alpha f(\alpha)\overline{\varphi(\alpha)}$ is *absolutely convergent:* for each α let μ_α be a complex number of absolute value 1 so that

$$\mu_\alpha f(\alpha)\varphi(\alpha) = |f(\alpha)\varphi(\alpha)|$$

and let

$$F(\alpha) = \mu_\alpha f(\alpha)$$

Then F is still in $\ell^2(A)$, and

$$\langle F, \varphi \rangle = \sum_\alpha |f(\alpha)\overline{\varphi(\alpha)}|$$

We just saw that the partial sums of the latter infinite sum form a Cauchy net, so we have the asserted absolute convergence.

Remark: The more general spaces $L^2(X, \mu)$ for abstract measure spaces X, μ have a similar treatment, but need somewhat greater preparation in terms of *integration theory*.

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4. Triangle inequality, associated metric, continuity issues

As corollary of the Cauchy-Schwarz-Bunyakowsky inequality, we have a *norm* and associated *metric* topology on a pre-Hilbert space:

Again, the **(associated) norm** on a pre-Hilbert space V is

$$|x - y| = \langle x - y, x - y \rangle^{1/2}$$

and the **associated metric** is

$$d(x, y) = |x - y|$$

The reflexivity, symmetry, and positivity of this alleged distance function are clear from the definitional properties of $\langle , \rangle$, but the **triangle inequality**

$$d(x, z) \le d(x, y) + d(y, z)$$

needs proof. That is, we want

$$|x - z| \le |x - y| + |y - z|$$

Assuming for the moment that this triangle inequality holds, we *do* have a metric on the pre-Hilbert space V, and we can show that the map

$$\langle , \rangle : V \times V \to \mathbf{C}$$

is *continuous* as a function of two variables. Indeed, suppose that $|x - x'| < \varepsilon$ and $|y - y'| < \varepsilon$ for $x, x', y, y' \in V$. Then

$$\langle x, y \rangle - \langle x', y' \rangle = \langle x - x', y \rangle + \langle x', y \rangle - \langle x', y' \rangle = \langle x - x', y \rangle + \langle x', y - y' \rangle$$

Using the triangle inequality for the ordinary absolute value, and then the Cauchy-Schwarz-Bunyakowsky inequality, we obtain

$$|\langle x,y\rangle - \langle x',y'\rangle| \le |\langle x-x',y\rangle| + |\langle x',y-y'\rangle| \le |x-x'||y| + |x'||y-y'|$$

$$< \varepsilon(|y|+|x'|)$$

This proves the continuity of the inner product.

Further, scalar multiplication and vector addition are readily seen to be continuous. In particular, it is easy to check that for any fixed $y \in V$ and for any fixed $\lambda \in \mathbf{C}^\times$ both maps

$$x \to x+y$$

$$x \to \lambda x$$

are *homeomorphisms* of V to itself.

Now we prove the desired inequality

$$|x-z| \le |x-y| + |y-z|$$

which is equivalent to the triangle inequality for the alleged metric $d(x,y) = |x-y|$. The appearance of this can be simplified a bit. Replacing x, z by $x+y, z+y$ in this, we see that we want

$$|x-z| \le |x| + |z|$$

We have

$$|x-z|^2 = |x|^2 - \langle x,z\rangle - \langle z,x\rangle + |z|^2 \le |x|^2 + 2|x||z| + |z|^2$$

by the Cauchy-Schwarz-Bunyakowsky inequality. The right-hand side is the square of $|x|+|z|$, as desired. *Done.*

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5. Hilbert spaces, completions, infinite sums

If a pre-Hilbert space is *complete* with respect to the metric arising from its inner product (and norm), then it is called a **Hilbert space**.

An arbitrary pre-Hilbert space can be *completed* as metric space. Since metric spaces have *countable local bases* for their topology (e.g., open balls of radii $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$) all points in the completion are limits of Cauchy *sequences* (rather than being limits of more complicated Cauchy *nets*). The completion inherits an inner product defined by a limiting process

$$\langle \lim_m x_m, \lim_n y_n\rangle = \lim_{m,n} \langle x_m, y_n\rangle$$

It is not hard to verify that the indicated limit exists (for Cauchy sequences $\{x_m\}, \{y_n\}$), and gives a hermitian inner product on the completion. The completion process *does nothing* to a space which is already complete.

In a Hilbert space, we can consider *infinite* sums

$$\sum_{\alpha \in A} v_\alpha$$

for sets $\{v_\alpha : \alpha \in A\}$ of vectors in V. Not wishing to have a notation