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Exercises on Banach Spaces

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1. Let V be the Banach space of continuous functions on $[-1,1]$ with sup norm. Let E be the set of continuous functions f with

$$\int_0^1 f\,dx - \int_{-1}^0 f\,dx = 1$$

Show that E is closed and convex in V, but has *no* element of minimal norm (in contrast to the Hilbert space situation).

2. Let E be the set of functions f in $L^1[0,1]$ such that $\int_0^1 f\,dx = 1$. Show that E is a closed convex subset with infinitely many elements of minimal norm.

3. Let c_0 be sequences which go to zero at infinity. Let ℓ^1 be absolutely integrable sequences, and let ℓ^∞ be bounded sequences. Show that $c_0^* = \ell^1$ by

$$\lambda_{\{a_n\}}(\{b_n\}) = \sum_n a_n \cdot b_n$$

Show that $(\ell^1)^* = \ell^\infty$ by the same pairing. Show that ℓ^1 imbeds in $(\ell^\infty)^*$ but does not give all of it, since the latter dual space contains elements which are 0 on the closed subspace c_0 of ℓ^∞. Observe that c_0 and ℓ^1 are separable, but ℓ^∞ is not.

4. Let X, d be a complete metric space or locally compact Hausdorff space with no isolated points. (That is, there are no points x such that there is a uniform lower bound $b > 0$ such that $d(x,y) \geq b$ for all $y \neq x$.) Show that any dense G_δ (countable intersection of opens) is *uncountable*.

5. Let T_n be a sequence of continuous linear maps $T_n : X \to Y$ for normed X and Banach space Y. Suppose that there is a uniform bound for all the operator norms: $|T_n| < M$ for all n. Suppose that there is a dense subset E of X such that for all $x \in E$ the sequence $T_n x$ converges in Y. Show that $T_n x$ converges for *all* $x \in X$.

6. Given $\{a_n\}$, suppose that $\sum_n a_n \cdot b_n$ converges (not necessarily absolutely) for every sequence $\{b_n\}$ with $b_n \to 0$. Show that $\{a_n\}$ is in ℓ^1.

7. Given $\{a_n\}$, suppose that $\sum_n a_n \cdot b_n$ converges (not necessarily absolutely) for every sequence $\{b_n\}$ with $b_n \in 0\ell^q$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Show that $\{a_n\}$ is in ℓ^p.

8. Construct a dense G_δ in $C^o[a,b]$ consisting of nowhere differentiable functions, as follows. Let U_n be the set of f such that there exists a $t \in [a,b]$ such that

$$|f(s) - f(t)| > n \cdot |s - t|$$

for all $s \in [a,b]$. Show that U_n is dense open. (*Hint:* Approximate continuous functions in sup norm by piecewise linear functions with needlessly large slopes.) Show that the intersection of the U_n is dense and consists of nowhere differentiable functions.