The Number of Zeros of a Polynomial in a Disk as a Consequence of Restrictions on the Coefficients

Robert Gardner
Department of Mathematics and Statistics
East Tennessee State University
Johnson City, Tennessee 37614 − 0663
E-mail: `gardnerr@etsu.edu`

Brett Shields
Department of Mathematics and Statistics
East Tennessee State University
Johnson City, Tennessee 37614 − 0663
E-mail: `shieldsb@goldmail.etsu.edu`

Abstract. In this paper, we put restrictions on the coefficients of polynomials and give bounds concerning the number of zeros in a specific region. The restrictions involve a monotonicity-type condition on the coefficients of the even powers of the variable and on the coefficients of the odd powers of the variable (treated separately). We present results by imposing the restrictions on the moduli of the coefficients, the real and imaginary parts of the coefficients, and the real parts (only) of the coefficients.

AMS Mathematics Subject Classification (2010): 30A10
Keywords: polynomials, counting zeros, monotone coefficients.

1 Introduction

The classical Eneström-Kakeya Theorem restricts the location of the zeros of a polynomial based on a condition imposed on the coefficients of the polynomial. Namely:

Eneström-Kakeya Theorem. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be such that $0 < a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_n$. Then all the zeros of P lie in $|z| \leq 1$.

There exists a huge body of literature on these types of results. For a brief survey, see Section 3.3.3 of *Topics in Polynomials: Extremal Problems, Inequalities, Zeros* by Milovanović, Mitrinović, and Rassias [8]. A more detailed and contemporary survey is to appear in [5].

In 1996, Aziz and Zargar [2] introduced the idea of imposing a monotonicity condition on the coefficients of the even powers of z and on the coefficients of the odd powers of z separately in order to get a restriction on the location of the zeros of a polynomial with positive real coefficients (see their Theorem 3). These types of hypotheses apply to more polynomials than a simple restriction of monotonicity on the coefficients as used in the Eneström-Kakeya Theorem. Of course if the hypotheses of the Eneström-Kakeya Theorem are satisfied by a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$, say $0 < a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_n$, then P also satisfies the hypotheses of Aziz and Zargar: $0 < a_0 \leq a_2 \leq a_4 \leq \ldots \leq a_{2\lfloor n/2 \rfloor}$ and $0 < a_1 \leq a_3 \leq \ldots \leq a_{2\lfloor (n+1)/2 \rfloor - 1}$. Notice that $2\lfloor n/2 \rfloor$ is the largest even subscript and $2\lfloor (n+1)/2 \rfloor - 1$ is the largest odd subscript, regardless of the parity of n. However, a polynomial such as $p(z) = 2 + z + 3z^2 + 2z^3 + 4z^4 + 3z^5 + 5z^6 + 4z^7$ does not satisfy the monotonicity condition of Eneström-Kakeya, but does satisfy the Aziz and Zargar hypotheses. Therefore Aziz and Zargar's hypotheses are less restrictive and are satisfied by a larger class of polynomials.

Cao and Gardner [3] generalized this idea of imposing a monotonicity condition on even and odd indexed coefficients separately and imposed the following conditions on the coefficients of a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$:

$$\alpha_0 \leq \alpha_2 t^2 \leq \alpha_4 t^4 \leq \cdots \leq \alpha_{2k} t^{2k} \geq \alpha_{2k+2} t^{2k+2} \geq \cdots \geq \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor},$$

$$\alpha_1 \leq \alpha_3 t^2 \leq \alpha_5 t^4 \leq \cdots \leq \alpha_{2\ell-1} t^{2\ell-2} \geq \alpha_{2\ell+1} t^{2\ell} \geq \cdots \geq \alpha_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor},$$

$$\beta_0 \leq \beta_2 t^2 \leq \beta_4 t^4 \leq \cdots \leq \beta_{2s} t^{2s} \geq \beta_{2s+2} t^{2s+2} \geq \cdots \geq \beta_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor},$$

$$\beta_1 \leq \beta_3 t^2 \leq \beta_5 t^4 \leq \cdots \leq \beta_{2q-1} t^{2q-2} \geq \beta_{2q+1} t^{2q} \geq \cdots \geq \beta_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor},$$

for some k, ℓ, s, q between 0 and n. Again, Cao and Gardner presented results restricting the location of the zeros of P.

In this paper, we impose these monotonicity-type conditions on the coefficients of a polynomial and then give a restriction on the number of zeros of the polynomial in a disk of a specific radius which is centered at zero. The first result relevant to this study which involves counting zeros appears in Titchmarsh's 1939 *The Theory of Functions* [11]:

Theorem 1.1 *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in the disk $|z| \leq R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of F in the disk $|z| \leq \delta R$ does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|F(0)|}.$$

Notice that in order to apply Theorem 1.1, we must find $\max_{|z|\leq R} |F(z)|$. Since this can be, in general, very complicated, it is desirable to present results which give bounds on the number of zeros in terms of something more tangible, such as the coefficients in the series expansion of F. With the same hypotheses as the Eneström-Kakeya Theorem, Mohammad used a special case of Theorem 1.1 to prove the following [9].

Theorem 1.2 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be such that $0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n$. Then the number of zeros in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \left(\frac{a_n}{a_0} \right).$$

Theorem 1.2 was extended from polynomials with real coefficients to polynomials with complex coefficients by K. K. Dewan [4, 8]. Using hypotheses related to those of Theorem 1.2, Dewan imposed a monotonicity condition on the moduli and then on the real parts of the coefficients. Her results were recently generalized by Pukhta [10] who proved the following two theorems.

Theorem 1.3 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be such that $|arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for $0 \leq j \leq n$ and for some real α and β, and let $0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{n-1}| \leq |a_n|$. Then for $0 < \delta < 1$ the number of zeros of P in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Theorem 1.4 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j) - \beta| \leq \alpha \leq \pi/2$ *for* $0 \leq j \leq n$ *and for some real* α *and* β, *and let* $0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$. *Then for* $0 < \delta < 1$ *the number of zeros of* P *in* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{\alpha_n + \sum_{k=0}^{n} |\beta_k|}{|a_0|}.$$

In a result similar to the Eneström-Kakeya Theorem, but for analytic functions as opposed to polynomials, Aziz and Mohammad [1] imposed the condition $0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots$ on the real parts (and a similar condition on the imaginary parts) of the coefficients of analytic function $F(z) = \sum_{j=0}^{\infty} a_j z^j$. Gardner and Shields [6] used this type of hypothesis to prove the following three results.

Theorem 1.5 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 \leq k \leq n$,

$$0 < |a_0| \leq t|a_1| \leq t^2|a_2| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^{n-1}|a_{n-1}| \geq t^n|a_n|$$

and $|arg(a_j) - \beta| \leq \alpha \leq \pi/2$ *for* $1 \leq j \leq n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* P *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|a_0|t + |a_n|t^{n+1})(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sin\alpha \sum_{j=0}^{n} |a_j|t^{j+1}.$$

Theorem 1.6 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* $0 \leq k \leq n$ *we have*

$$0 \neq \alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* P *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where $M = (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=0}^{n} |\beta_j|t^{j+1}$.

Theorem 1.7 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some* $t > 0$, *for some* $0 \leq k \leq n$ *we have*

$$0 \neq \alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n,$$

and for some $0 \leq \ell \leq n$ *we have*

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^{\ell-1}\beta_{\ell-1} \leq t^\ell\beta_\ell \geq t^{\ell+1}\beta_{\ell+1} \geq \cdots \geq t^{n-1}\beta_{n-1} \geq t^n\beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* P *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where $M = (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + (|\beta_0| - \beta_0)t + 2\beta_\ell t^{\ell+1} + (|\beta_n| - \beta_n)t^{n+1}$.

The purpose of this paper is to put the monotonicity-type condition of Cao and Gardner on (1) the moduli, (2) the real and imaginary parts, and (3) the real parts only of the coefficients of polynomials. In this way, we give results related to Theorems 1.5, 1.6, and 1.7, but with more flexible hypotheses and hence applicable to a larger class of polynomials.

2 Results

We now present three major theorems, as described above, and several corollaries. First, we put the monotonicity-type condition on the moduli of the coefficients.

Theorem 2.1 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$. *Suppose that for some* $t > 0$, *and for some* $0 \leq k \leq n$ *and* $1 \leq s \leq n$ *we have*

$$0 \neq |a_0| \leq |a_2|t^2 \leq |a_4|t^4 \leq \cdots \leq |a_{2k}|t^{2k} \geq |a_{2k+2}|t^{2k+2} \geq \cdots \geq |a_{2\lfloor n/2 \rfloor}|t^{2\lfloor n/2 \rfloor},$$

$$|a_1| \leq |a_3|t^2 \leq |a_5|t^4 \leq \cdots \leq |a_{2s-1}|t^{2s-2} \geq |a_{2s+1}|t^{2s} \geq \cdots \geq |a_{2\lfloor (n+1)/2 \rfloor - 1}|t^{2\lfloor n/2 \rfloor}$$

Then for $0 < \delta < 1$ *the number of zeros of* P *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M=(|a_0|t^2+|a_1|t^3+|a_{n-1}|t^{n+1}+|a_n|t^{n+2})(1-\cos\alpha-\sin\alpha)$$
$$+2\cos\alpha(|a_{2k}|t^{2k+2}+|a_{2s-1}|t^{2s+1})+2\sin\alpha\sum_{j=0}^{n}|a_j|t^{j+2}.$$

When $t=1$, Theorem 2.1 yields the following.

Corollary 2.2 *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$. *Suppose that for some* $0\leq k\leq n$ *and* $1\leq s\leq n$ *we have*

$$0\neq|a_0|\leq|a_2|\leq|a_4|\leq\cdots\leq|a_{2k}|\geq|a_{2k+2}|\geq\cdots\geq|a_{2\lfloor n/2\rfloor}|,$$

$$|a_1|\leq|a_3|\leq|a_5|\leq\cdots\leq|a_{2s-1}|\geq|a_{2s+1}|\geq\cdots\geq|a_{2\lfloor(n+1)/2\rfloor-1}|.$$

Then for $0<\delta<1$ *the number of zeros of* P *in the disk* $|z|\leq\delta$ *does not exceed*

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where

$$M=(|a_0|+|a_1|+|a_{n-1}|+|a_n|)(1-\cos\alpha-\sin\alpha)+2\cos\alpha(|a_{2k}|+|a_{2s-1}|)+2\sin\alpha\sum_{j=0}^{n}|a_j|.$$

Next, we put the monotonicity-type condition on the real and imaginary parts of the coefficients.

Theorem 2.3 *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *where* $Re(a_j)=\alpha_j$, $Im(a_j)=\beta_j$ *for* $0\leq j\leq n$. *Suppose that for some* $t>0$, *some* $0\leq k\leq n$, $1\leq\ell\leq n$, $0\leq s\leq n$, *and* $1\leq q\leq n$ *we have*

$$0\neq\alpha_0\leq\alpha_2t^2\leq\alpha_4t^4\leq\cdots\leq\alpha_{2k}t^{2k}\geq\alpha_{2k+2}t^{2k+2}\geq\cdots\geq\alpha_{2\lfloor n/2\rfloor}t^{2\lfloor n/2\rfloor},$$

$$\alpha_1\leq\alpha_3t^2\leq\alpha_5t^4\leq\cdots\leq\alpha_{2\ell-1}t^{2\ell-2}\geq\alpha_{2\ell+1}t^{2\ell}\geq\cdots\geq\alpha_{2\lfloor(n+1)/2\rfloor-1}t^{2\lfloor n/2\rfloor},$$

$$\beta_0\leq\beta_2t^2\leq\beta_4t^4\leq\cdots\leq\beta_{2s}t^{2s}\geq\beta_{2s+2}t^{2s+2}\geq\cdots\geq\beta_{2\lfloor n/2\rfloor}t^{2\lfloor n/2\rfloor},$$

$$\beta_1\leq\beta_3t^2\leq\beta_5t^4\leq\cdots\leq\beta_{2q-1}t^{2q-2}\geq\beta_{2q+1}t^{2q}\geq\cdots\geq\beta_{2\lfloor(n+1)/2\rfloor-1}t^{2\lfloor n/2\rfloor}.$$

Then for $0 < \delta < 1$ the number of zeros of P in the disk $|z| \leq \delta t$ does not exceed

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0)t^2 + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1)t^3 + 2(\alpha_{2k}t^{2k+2} + \alpha_{2\ell-1}t^{2\ell+1} + \beta_{2s}t^{2s+2} + \beta_{2q-1}t^{2q+1})$$

$$+(|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1})t^{n+1} + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+2}.$$

With $t = 1$ in Theorem 2.3, we have:

Corollary 2.4 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $0 \leq k \leq n$, $1 \leq \ell \leq n$, $0 \leq s \leq n$, and $1 \leq q \leq n$ we have*

$$0 \neq \alpha_0 \leq \alpha_2 \leq \alpha_4 \leq \cdots \leq \alpha_{2k} \geq \alpha_{2k+2} \geq \cdots \geq \alpha_{2\lfloor n/2 \rfloor},$$

$$\alpha_1 \leq \alpha_3 \leq \alpha_5 \leq \cdots \leq \alpha_{2\ell-1} \geq \alpha_{2\ell+1} \geq \cdots \geq \alpha_{2\lfloor (n+1)/2 \rfloor - 1},$$

$$\beta_0 \leq \beta_2 \leq \beta_4 \leq \cdots \leq \beta_{2s} \geq \beta_{2s+2} \geq \cdots \geq \beta_{2\lfloor n/2 \rfloor},$$

$$\beta_1 \leq \beta_3 \leq \beta_5 \leq \cdots \leq \beta_{2q-1} \geq \beta_{2q+1} \geq \cdots \geq \beta_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ the number of zeros of P in the disk $|z| \leq \delta$ does not exceed

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0) + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1) + 2(\alpha_{2k} + \alpha_{2\ell-1} + \beta_{2s} + \beta_{2q-1})$$

$$+(|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

By manipulating the parameters k, ℓ, s, and q we easily get eight corollaries from Corollary 2.4. For example, with $k = s = 2\lfloor n/2 \rfloor$ and $\ell = q = 2\lfloor (n+1)/2 \rfloor - 1$ we have:

Corollary 2.5 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that:*

$$0 \neq \alpha_0 \leq \alpha_2 \leq \alpha_4 \leq \cdots \leq \alpha_{2\lfloor n/2 \rfloor}, \quad \alpha_1 \leq \alpha_3 \leq \alpha_5 \leq \cdots \leq \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

$$\beta_0 \leq \beta_2 \leq \beta_4 \leq \cdots \leq \beta_{2\lfloor n/2 \rfloor}, \quad \beta_1 \leq \beta_3 \leq \beta_5 \leq \cdots \leq \beta_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ *the number of zeros of* P *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0) + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1) + (|\alpha_{n-1}| + \alpha_{n-1} + |\beta_{n-1}| + \beta_{n-1}) + (|\alpha_n| + \alpha_n + |\beta_n| + \beta_n).$$

With $k = 2\lfloor n/2 \rfloor$, $\ell = 1$, and each a_j real in Corollary 2.4 we have:

Corollary 2.6 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $a_j \in \mathbb{R}$ *for* $0 \leq j \leq n$. *Suppose that:*

$$0 \neq a_0 \leq a_2 \leq a_4 \leq \cdots \leq a_{2\lfloor n/2 \rfloor}, \quad a_1 \geq a_3 \geq a_5 \geq \cdots \geq a_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ *the number of zeros of* P *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|} \text{ where } M = |a_0| + a_0 + |a_{2\lfloor n/2 \rfloor}| + a_{2\lfloor n/2 \rfloor}.$$

Example. Consider the polynomial $P(z) = 1 + 10z + 2z^2 + 0z^3 + 3z^4 + 0z^5 + 4z^6 + 0z^7 + 8z^8$. The zeros of P are approximately -0.102119, -0.872831, $-0.629384 \pm 0.855444i$, $0.22895 \pm 1.05362i$, and $0.887908 \pm 0.530244i$. Applying Corollary 2.6 with $\delta = 0.15$ we see that it predicts no more than 1.888926 zeros in $|z| \leq 0.15$. In other words, Corollary 2.6 predicts at most one zero in $|z| \leq 0.15$. In fact, P does have exactly one zero in $|z| \leq 0.15$, and Corollary 2.6 is sharp for this example.

Theorem 2.7 *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *where* $Re(a_j)=\alpha_j$, $Im(a_j)=\beta_j$ *for* $0\le j\le n$. *Suppose that for some* $t>0$, *some* $0\le k\le n$ *and* $1\le \ell\le n$ *we have*

$$0\neq\alpha_0\le\alpha_2t^2\le\alpha_4t^4\le\cdots\le\alpha_{2k}t^{2k}\ge\alpha_{2k+2}t^{2k+2}\ge\cdots\ge\alpha_{2\lfloor n/2\rfloor}t^{2\lfloor n/2\rfloor},$$

$$\alpha_1\le\alpha_3t^2\le\alpha_5t^4\le\cdots\le\alpha_{2\ell-1}t^{2\ell-2}\ge\alpha_{2\ell+1}t^{2\ell}\ge\cdots\ge\alpha_{2\lfloor (n+1)/2\rfloor-1}t^{2\lfloor n/2\rfloor}$$

Then for $0<\delta<1$ *the number of zeros of* P *in the disk* $|z|\le\delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where

$$M=(|\alpha_0|-\alpha_0)t^2+(|\alpha_1|-\alpha_1)t^3+2\alpha_{2k}t^{2k+2}+\alpha_{2\ell-1}t^{2\ell+1}$$
$$+(|\alpha_{n-1}|-\alpha_{n-1})t^{n+1}+(|\alpha_n|-\alpha_n)t^{n+2}+2\sum_{j=0}^{n}|\beta_j|t^{j+2}.$$

With $t=1$ in Theorem 2.7, we get the following:

Corollary 2.8 *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *where* $Re(a_j)=\alpha_j$, $Im(a_j)=\beta_j$ *for* $0\le j\le n$. *Suppose that for some* $0\le k\le n$ *and* $1\le \ell\le n$ *we have*

$$0\neq\alpha_0\le\alpha_2\le\alpha_4\le\cdots\le\alpha_{2k}\ge\alpha_{2k+2}\ge\cdots\ge\alpha_{2\lfloor n/2\rfloor},$$

$$\alpha_1\le\alpha_3\le\alpha_5\le\cdots\le\alpha_{2\ell-1}\ge\alpha_{2\ell+1}\ge\cdots\ge\alpha_{2\lfloor (n+1)/2\rfloor-1}$$

Then for $0<\delta<1$ *the number of zeros of* P *in the disk* $|z|\le\delta$ *does not exceed*

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where

$$M=(|\alpha_0|-\alpha_0)+(|\alpha_1|-\alpha_1)+2\alpha_{2k}+\alpha_{2\ell-1}+(|\alpha_{n-1}|-\alpha_{n-1})+(|\alpha_n|-\alpha_n)+2\sum_{j=0}^{n}|\beta_j|.$$

3 Proofs of the Results

The following is due to Govil and Rahman and appears in [7].

Lemma 3.1 *Let $z, z' \in \mathbb{C}$ with $|z| \geq |z'|$. Suppose $|arg\, z^* - \beta| \leq \alpha \leq \pi/2$ for $z^* \in \{z, z'\}$ and for some real α and β. Then*

$$|z - z'| \leq (|z| - |z'|)\cos\alpha + (|z| + |z'|)\sin\alpha.$$

We now give proofs of our results.

Proof of Theorem 2.1. Define

$$G(z) = (t^2 - z^2)P(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n}(a_j t^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_n z^{n+2}.$$

For $|z| = t$ we have

$$\begin{aligned}
G(z) &\leq |a_0|t^2 + |a_1|t^3 + \sum_{j=2}^{n} |a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&= |a_0|t^2 + |a_1|t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |a_j t^2 - a_{j-2}|t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2s-1} |a_j t^2 - a_{j-2}|t^j + \sum_{\substack{j=2s+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&\leq |a_0|t^2 + |a_1|t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \{(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \{(|a_{j-2}| - |a_j|t^2)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2s-1} \{(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + \sum_{\substack{j=2s+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \{(|a_{j-2}| - |a_j|t^2)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j
\end{aligned}$$

$$
\begin{aligned}
& +|a_{n-1}|t^{n+1}+|a_n|t^{n+2} \text{ by Lemma 3.1} \\
= \;& |a_0|t^2+|a_1|t^3-|a_0|t^2\cos\alpha+|a_{2k}|t^{2k+2}\cos\alpha+|a_0|t^2\sin\alpha+|a_{2k}|t^{2k+2}\sin\alpha \\
& +2\sin\alpha \sum_{\substack{j=2 \\ j \text{ even}}}^{2k-2} |a_j|t^{j+2}+|a_{2k}|t^{2k+2}\cos\alpha-|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\cos\alpha \\
& +|a_{2k}|t^{2k+2}\sin\alpha+|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\sin\alpha+2\sin\alpha \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor-2} |a_j|t^{j+2} \\
& -|a_1|t^3\cos\alpha+|a_{2s-1}|t^{2s+1}\cos\alpha+|a_1|t^3\sin\alpha+|a_{2s-1}|t^{2s+1}\sin\alpha+2\sin\alpha \sum_{\substack{j=3 \\ j \text{ odd}}}^{2s-3} |a_j|t^{j+2} \\
& +|a_{2s-1}|t^{2s+1}\cos\alpha-|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}\cos\alpha+|a_{2s-1}|t^{2s+1}\sin\alpha \\
& +|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}\sin\alpha+2\sin\alpha \sum_{\substack{j=2s+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3} |a_j|t^{j+2}+|a_{n-1}|t^{n+1}+|a_n|t^{n+2} \\
= \;& |a_0|t^2(1-\cos\alpha+\sin\alpha)+|a_1|t^3(1-\cos\alpha+\sin\alpha)+2|a_{2k}|t^{2k+2}(\cos\alpha+\sin\alpha) \\
& +2|a_{2s-1}|t^{2s+1}(\cos\alpha+\sin\alpha)+2\sin\alpha \sum_{\substack{j=2 \\ j \text{ even}}}^{2k-2} |a_j|t^{j+2}+2\sin\alpha \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor-2} |a_j|t^{j+2} \\
& +2\sin\alpha \sum_{\substack{j=3 \\ j \text{ odd}}}^{2s-2} |a_j|t^{j+2}+2\sin\alpha \sum_{\substack{j=2s+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3} |a_j|t^{j+2} \\
& +|a_{n-1}|t^{n+1}(1-\cos\alpha+\sin\alpha)+|a_n|t^{n+2}(1-\cos\alpha+\sin\alpha) \\
= \;& (|a_0|t^2+|a_1|t^3+|a_{n-1}|t^{n+1}+|a_n|t^{n+2})(1-\cos\alpha-\sin\alpha) \\
& +2\cos\alpha(|a_{2k}|t^{2k+2}+|a_{2s-1}|t^{2s-1})+2\sin\alpha\sum_{j=0}^{n}|a_j|t^{j+2} \\
= \;& M.
\end{aligned}
$$

Now $G(z)$ is analytic in $|z| \leq t$, and $|G(z)| \leq M$ for $|z| = t$. So by Titchmarsh's theorem (Theorem 1.1) and the Maximum Modulus Theorem, the number of zeros of G (and hence of P) in $|z| \leq \delta t$ does not exceed

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. ∎

Proof of Theorem 2.3. Define $G(z) = (t^2 - z^2)P(z)$. For $|z| = t$ we have

$$\begin{aligned}
|G(z)| \le\ & (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 + \sum_{j=2}^{n} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{j=2}^{n} |\beta_j t^2 - \beta_{j-2}|t^j \\
& + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
=\ & (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |\alpha_j t^2 - \alpha_{j-2}|t^j \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\beta_j t^2 - \beta_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |\beta_j t^2 - \beta_{j-2}|t^j \\
& + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
=\ & (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} (\alpha_j t^2 - \alpha_{j-2})t^j + \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} (\alpha_{j-2} - \alpha_j t^2)t^j \\
& + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell - 1} (\alpha_j t^2 - \alpha_{j-2})t^j + \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} (\alpha_{j-2} - \alpha_j t^2)t^j + \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} (\beta_j t^2 - \beta_{j-2})t^j \\
& + \sum_{\substack{j=2s+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} (\beta_{j-2} - \beta_j t^2)t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2q-1} (\beta_j t^2 - \beta_{j-2})t^j + \sum_{\substack{j=2q+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} (\beta_{j-2} - \beta_j t^2)t^j \\
& + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
=\ & (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
& + \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_j t^{j+2} - \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \alpha_{j-2} t^j - \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \alpha_j t^{j+2} \right) \\
& + \left(\sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_j t^{j+2} - \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \alpha_{j-2} t^j - \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \alpha_j t^{j+2} \right) \\
& + \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \beta_j t^{j+2} - \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \beta_{j-2} t^j \right) + \left(\sum_{\substack{j=2s+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \beta_{j-2} t^j - \sum_{\substack{j=2s+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \beta_j t^{j+2} \right) \\
& + \left(\sum_{\substack{j=3 \\ j \text{ odd}}}^{2q-1} \beta_j t^{j+2} - \sum_{\substack{j=3 \\ j \text{ odd}}}^{2q-1} \beta_{j-2} t^j \right) + \left(\sum_{\substack{j=2q+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \beta_{j-2} t^j - \sum_{\substack{j=2q+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \beta_j t^{j+2} \right) \\
& + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2}
\end{aligned}$$

$$
\begin{aligned}
&= (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0)t^2 + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1)t^3 \\
&\quad +2(\alpha_{2k}t^{2k+2} + \alpha_{2\ell-1}t^{2\ell+1} + \beta_{2s}t^{2s+2} + \beta_{2q-1}t^{2q+1}) \\
&\quad +(|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1})t^{n+1} + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+2} \\
&= M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. ∎

Proof of Theorem 2.7. Define $G(z) = (t^2 - z^2)P(z)$. For $|z| = t$ we have

$$
\begin{aligned}
|G(z)| &\leq (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 + \sum_{j=2}^{n} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{j=2}^{n}(|\beta_j|t^2 + |\beta_{j-2}|)t^j \\
&\quad +(|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\alpha_j t^2 - \alpha_{j-2}|t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{j=2}^{n}(|\beta_j|t^2 + |\beta_{j-2}|)t^j \\
&\quad +(|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta|_0)t^2 + (|\alpha_1| + |\beta_1|)t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} (\alpha_j t^2 - \alpha_{j-2})t^j \\
&\quad + \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} (\alpha_{j-2} - \alpha_j t^2)t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} (\alpha_j t^2 - \alpha_{j-2})t^j + \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} (\alpha_{j-2} - \alpha_j t^2)t^j \\
&\quad + \sum_{j=2}^{n} |\beta_j|t^2 + \sum_{j=2}^{n} |\beta_{j-2}|t^j + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 \\
&\quad + \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_j t^{j+2} - \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \alpha_{j-2} t^j - \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \alpha_j t^{j+2} \right) \\
&\quad + \left(\sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_j t^{j+2} - \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \alpha_{j-2} t^j - \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \alpha_j t^{j+2} \right) \\
&\quad + \sum_{j=2}^{n} |\beta_j|t^2 + \sum_{j=0}^{n-2} |\beta_j|t^{j+2} \\
&= (|\alpha_0| - \alpha_0)t^2 + (|\alpha_1| - \alpha_1)t^3 + 2\alpha_{2k}t^{2k+2} + \alpha_{2\ell-1}t^{2\ell+1}
\end{aligned}
$$

$$
\begin{aligned}
& +(|\alpha_{n-1}| - \alpha_{n-1})t^{n+1} + (|\alpha_n| - \alpha_n)t^{n+2} + 2\sum_{j=0}^{n} |\beta_j| t^{j+2} \\
= \quad & M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. ■

References

[1] A. Aziz and Q. G. Mohammad, On the Zeros of a Certain Class of Polynomials and Related Entire Functions, *Journal of Mathematical Analysis and Applications*, **75** (1980), 495–502.

[2] A. Aziz and B. A. Zargar, Some Extensions of Eneström-Kakeya, *Glasnik Matematički*, **31**(51), 239–244 (1996).

[3] J. Cao and R. Gardner, Restrictions on the Zeros of a Polynomial as a Consequence of Conditions on the Coefficients of even Powers and Odd Powers of the Variable, *Journal of Computational and Applied Mathematics*, **155**, 153–163 (2003).

[4] K. K. Dewan, "Extremal Properties and Coefficient Estimates for Polynomials with Restricted Zeros and on Location of Zeros of Polynomials," Ph.D. Thesis, Indian Institutes of Technology, Delhi, 1980.

[5] R. Gardner and N. K. Govil, The Eneström-Kakeya Theorem and Some of Its Generalizations, *Current Topics in Pure and Computational Complex Analysis*, Springer-Verlag, to appear.

[6] R. Gardner and B. Shields, The Number of Zeros of a Polynomial in a Disk, *Journal of Classical Analysis*, **3**(2) (2013), 167–176.

[7] N. K. Govil and Q. I. Rahman, On the Eneström-Kakeya Theorem, *Tôhoku Mathematics Journal*, **20** (1968), 126–136.

[8] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, Singapore: World Scientific Publishing, 1994.

[9] Q. G. Mohammad, On the Zeros of the Polynomials, *American Mathematical Monthly*, **72**(6) (1965), 631–633.

[10] M. S. Pukhta, On the Zeros of a Polynomial, *Applied Mathematics*, **2** (2011), 1356–1358.

[11] E. C. Titchmarsh, *The Theory of Functions*, 2nd Edition, Oxford University Press, London, 1939.