Decompositions, Packings, and Coverings of the Complete Digraph with Orientations of $K_3 \bigcup \{e\}$

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Abstract. There are eight orientations of the complete graph on three vertices with a pendant edge, $K_3 \cup \{e\}$. Two of these are 3-circuits with a pendant arc and the other six are transitive triples with a pendant arc. Necessary and sufficient conditions are given for decompositions, packings, and coverings of the complete digraph with each of these eight orientations of $K_3 \cup \{e\}$.

1 Introduction

A *G-decomposition of* a graph H is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of H (called *blocks*) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \bigcap E(g_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^n E(g_i) = E(H)$. A G-decomposition of H where G and H are digraphs is similarly defined (with arc sets replacing edge sets). Several decompositions of the complete graph K_v and the complete digraph D_v have been explored. In particular, a Steiner triple system of order v is equivalent to a K_3-decomposition of K_v and such systems exist if and only if $v \equiv 1$ or 3 (mod 6) [12]. A Mendelsohn triple system is equivalent to a 3-circuit (C_3) decomposition of D_v and exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [9]. A directed triple system is equivalent to a transitive triple (T, see Figure 1) decomposition of D_v and exists if and only if $v \equiv 0$ or 1 (mod 3) [8]. Also of relevance to our results are decompositions of K_v into copies of K_3 with a pendant edge (the graph L of Figure 1). Bermond and Schönheim showed that such decompositions exist if and only if $v \equiv 0$ or 1 (mod 8) [2].

A *maximum G-packing* of graph H is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of H (called *blocks*) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, $\cup_{i=1}^n g_i \subset H$, and $|E(H) \setminus \cup_{i=1}^n E(g_i)|$ is minimum. The *leave* of the packing is the set $E(H) \setminus \cup_{i=1}^n E(g_i)$. A maximum G-packing of H

where G and H are digraphs is similarly defined (with arc sets replacing edge sets). Maximum K_3-packings of K_v were explored by Schönheim [10]. Maximum 3-circuit and transitive triple packings of D_v were addressed in [5].

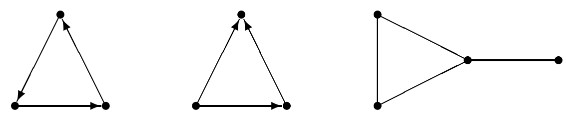

Figure 1. The 3-circuit C_3, transitive triple T, and K_3 with a pendant edge L.

A *minimum G-covering* of graph H is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of H (called *blocks*) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $H \subset \cup_{i=1}^n g_i$, and $|\cup_{i=1}^n E(g_i) \setminus E(H)|$ is minimum (the graph $\cup_{i=1}^n g_i$ may not be simple and $\cup_{i=1}^n E(g_i)$ may be a multiset). A minimum G-covering of H where G and H are digraphs is similarly defined (with arc sets replacing edge sets). The *padding* of the covering is the multiset $\cup_{i=1}^n E(g_i) \setminus E(H)$. Minimum K_3-coverings of K_v were explored by Fort and Hedlund [3]. Minimum 3-circuit and transitive triple coverings of D_v were addressed in [5].

We note that K_3-decompositions of K_v were followed by decompositions of D_v with orientations of K_3. Thus, a natural follow-up to the the work of [2] would be to consider orientations of graphs of order four or less. Because of this, we are motivated to consider decompositions, packings, and coverings of D_v with copies of digraph G where G an orientation of $L = K_3 \bigcup \{e\}$ (see Figure 2).
We denote the orientations of $L = K_3 \cup \{e\}$ given in Figure 2 as $[a, b, c; d]_{m1}$, $[a, b, c; d]_{m2}$, $[a, b, c; d]_{d1}$, ..., $[a, b, c; d]_{d6}$, respectively. The purpose of this paper is to give necessary and sufficient conditions for decompositions, packings, and coverings of D_v with each of the eight orientations of $L = K_3 \cup \{e\}$.

2 Decompositions

We note that since each of these orientations has four arcs, it is necessary that $|A(D_v)| \equiv 0 \pmod 4$ for the existence of a decomposition of D_v into one of the digraphs of Figure 2. Hence $v \equiv 0$ or $1 \pmod 4$ is necessary in all cases.

The *wheel*, denoted W_n is the graph containing a cycle on n vertices such that every vertex in the cycle is adjacent to a center vertex, ∞. We will denote the wheel W_n with center ∞ and cycle $(0, a, 2a, \ldots, (n-1)a)$ by $W_n(\infty : a)$. Note that $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$. This can be extended to a digraph by replacing each edge with a forward arc and a backward arc.

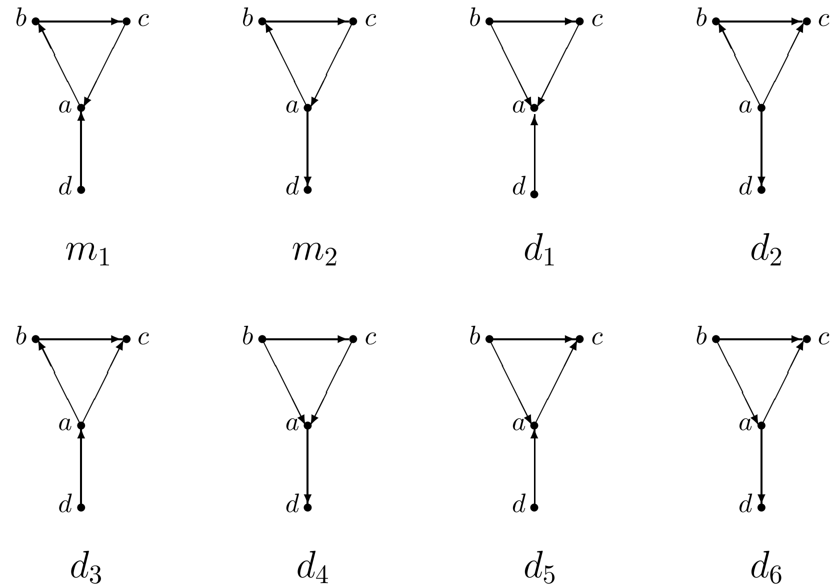

Figure 2. The eight orientations of $L = K_3 \cup \{e\}$.

The *circulant*, denoted $C_n(S)$, has vertex set $V(C_n(S)) = \{0, 1, \ldots, n-1\}$. Two vertices u and v are adjacent if and only if $|u - v|_n \in S$, where $|x|_n = \min\{x(\text{mod } n), n - x(\text{mod } n)\}$. The *directed circulant* will have a forward arc and a backward arc for each of these edges.

A *graceful labeling* on a graph G with q edges is an injective mapping f from $V(G)$ to $\{0, 1, \ldots, q\}$ such that the edge labels defined by $f'(u, v) = |f(u) - f(v)|$ satisfy $f'(E) = \{1, 2, \ldots, q\}$ [6, 11]. We note that wheels have graceful labelings [4, 7]. This being the case, there exists a W_p-decomposition of $C_n(1, 2, \ldots, 2p)$ where $n \geq 4p + 1$ [1].

Theorem 2.1 *An m_1-decomposition of D_v and an m_2-decomposition of D_v each exist if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. We note that $v \equiv 0$ or 1 (mod 4) is necessary by the above comments. Further note that there exists an m_1-decomposition of the directed wheel W_p, where $p \geq 3$. This decomposition is given by the set of blocks $\{[j, \infty, j+1; j-1]_{m1} \mid j = 0, 1, \ldots, p-1\}$ where the numerical vertex labels are reduced modulo p.

Case 1. Suppose $v \equiv 0$ (mod 4), say $v = 4k + 4$ where $k \geq 3$. We note that $D_{4k+4} = W_{4k+3}(\infty : 2k + 1) \cup C_{4k+3}(1, 2, \ldots, 2k)$ where $V(D_{4k+4}) =$

$\{0,1,2,\ldots,4k+2,\infty\}$. There exists an m_1-decomposition of W_{4k+3} and $C_{4k+3}(1,2,\ldots,2k)$ for $k \geq 3$ by the above comments.

For $v=4$, $D_4 \cong W_3$ and a decomposition of W_3 is given above.

For $v=8$, the decomposition is given by the set of blocks $\{[j,\infty,j+2;j+1]_{m1},[j,j+1,j+3;j+4]_{m_1} \mid j=0,1,\ldots,6\}$ where vertex labels are reduced modulo 7.

For $v=12$, the decomposition is given by the set of blocks $\{[j+5,\infty,j+10;j]_{m1},[j,j+1,j+3;j+7]_{m1},[j,j+3,j+1;j+4]_{m1} \mid j=0,1,\ldots,10\}$ where numerical vertex labels are reduced modulo 11.

Case 2. Suppose $v \equiv 1 \pmod 4$, say $v=4k+1$, where $k \geq 3$. Since W_k is graceful, there exists a decomposition of K_{4k+1} by the above comments. It follows that the directed wheel W_k decomposes the directed complete graph D_{4k+1}.

For $v=5$, the decomposition is given by the set of blocks $\{[4,0,1;3]_{m1}$, $[4,3,0;2]_{m1},[3,2,0;1]_{m1},[1,0,2;4]_{m1},[2,3,1;4]_{m1}\}$.

For $v=9$, the decomposition is given by the set of blocks $\{[j,j+1,j+3;j+5]_{m1},[j,j+3,j+1;j+4]_{m1} \mid j=0,1,\ldots,8\}$ where vertex labels are reduced modulo 9.

Since m_2 is the converse of m_1, the construction of an m_2-decomposition of D_v will similarly follow. ■

Theorem 2.2 *A d_1-decomposition of D_v and a d_2-decomposition of D_v each exist if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition follows as in Theorem 2.1. We now construct a d_1-decomposition of D_v for each $v \equiv 0$ or $1 \pmod 4$ and, since d_2 is the converse of d_1, the construction of a d_2-decomposition of D_v will similarly follow.

Case 1. Suppose $v \equiv 1 \pmod{12}$, say $v=12k+1$. Consider the set of blocks: $\{[j,6k-i+j,12k-2i+j;3k+1+i+j]_{d1},[j,5k-i+j,10k-2i+j;8k+1+2i+j]_{d1} \mid i=0,1,\ldots,k-1, j=0,1,\ldots,12k\}$ $\cup\{[j,k-1-i+j,12k-3-2i+j;2k+2+i+j]_{d1} \mid i=0,1,\ldots,k-2, j=0,1,\ldots,12k\} \cup \{[j,k+j,12k-1+j;k+1+j]_{d1} \mid j=0,1,\ldots,12k\}$. Here and throughout we note that if any index ranges over an empty set of values then the corresponding blocks are omitted from the construction.

Case 2. Suppose $v \equiv 5 \pmod{12}$, say $v=12k+5$. Consider the set of blocks: $\{[j,6k+2-i+j,12k+4-2i+j;3k+1+i+j]_{d1},[j,5k+1-i+j,10k+2-2i+j;8k+5+2i+j]_{d1} \mid i=0,1,\ldots,k-1, j=0,1,\ldots,12k+4\}$ $\cup\{[j,k-1-i+j,12k+1-2i+j;2k+2+i+j]_{d1} \mid i=0,1,\ldots,k-2, j=0,1,\ldots,12k+4\} \cup\{[j,5k+2+j,10k+4+j;4k+1+j]_{d1} \mid j=0,1,\ldots,12k+4\}$ $\cup\{[j,k+j,12k+3+j;k+1+j]_{d1} \mid j=0,1,\ldots,12k+4,$ omit if $k=0\}$.

Case 3. Suppose $v \equiv 9 \pmod{12}$, say $v=12k+9$. Consider the set of blocks: $\{[j, 6k+4-i+j, 12k+8-2i+j; 3k+4+i+j]_{d1}, [j, 5k+3-i+j, 10k+6-2i+j; 8k+7+2i+j]_{d1}, [j, k-i+j, 12k+5-2i+j; 2k+4+i+j]_{d1} \mid i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, 12k+8\} \cup \{[j, 5k+4+j, 10k+8+j; 8k+6+j]_{d1}, [j, k+1+j, 12k+7+j; k+2+j]_{d1} \mid j = 0, 1, \ldots, 12k+8\}$.

In each of Cases 1–3, the given set of blocks forms a decomposition of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 4. Suppose $v \equiv 0 \pmod 4$, say $v = 4k$. Consider the set of blocks: $\{[j, 2+j, \infty; 1+j]_{d1} \mid j = 0, 1, \ldots, 4k-2\} \cup \{[j, k+1-i+j, k+2+i+j; 2k+1+2i+j]_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k-2\}$. In Case 4, the given set of blocks forms a decomposition of D_v where $V(D_v) = \{\infty, 0, 1, \ldots, v-2\}$ and numerical vertex labels in the blocks are reduced modulo $v-1$. ■

Corollary 2.3 *A d_3-decomposition of D_v and a d_4-decomposition of D_v each exist if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition follows as in Theorem 2.1. In the case $v \equiv 1 \pmod 4$, blocks for such a d_3-decomposition can be constructed from the d_1-decomposition of Theorem 2.2 by replacing every block of the form $[j, a+j, b+j; c+j]_{d1}$ with a block of the form $[a+j, b+j, j; a+c+j]_{d3}$. In the case $v \equiv 0 \pmod 4$, blocks for such a d_3-decomposition can be constructed from the d_1-decomposition of Theorem 2.2 by replacing every block of the form $[j, a+j, b+j; c+j]_{d1}$ with a block of the form $[a+j, b+j, j; a+c+j]_{d3}$ and by replacing every block of the form $[j, a+j, \infty; c+j]_{d1}$ with a block of the form $[a+j, \infty, j; a+c+j]_{d3}$.

Since d_4 is the converse of d_3, the construction of a d_4-decomposition of D_v will similarly follow. ■

Corollary 2.4 *A d_5-decomposition of D_v and a d_6-decomposition of D_v each exist if and only if $v \equiv 1$ (mod 4).*

Proof. As in Theorem 2.1, one necessary condition is that $v \equiv 0$ or 1 (mod 4). Notice that the vertices of d_5 are of in-degrees 0, 0, 2, and 2. Therefore another necessary condition for a d_5-decomposition on D_v (and similarly for a d_6-decomposition of D_v) is that each vertex of D_v is of even in-degree — that is, v must be odd. Therefore $v \equiv 1 \pmod 4$ is necessary.

Blocks for such a d_5-decomposition of D_v can be constructed from the d_1 system of Theorem 2.2 by replacing every block of the form $[j, a+j, b+j; c+j]_{d1}$ with a block of the form $[b+j, a+j, j; b+c+j]_{d5}$.

Since d_6 is the converse of d_5, the construction of a d_6-decomposition of D_v will similarly follow. ■

3 Packings

We now give necessary and sufficient conditions for the packing of D_v with each of the eight orientations of L.

Theorem 3.1 *A maximum m_1-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$,

(ii) $|A(L)| = 6$ *if* $v = 3$, *and* $|A(L)| = 2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$.

Maximum m_2-packings of D_v satisfy the same conditions.

Proof. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows. If $v \equiv 2$ or 3 (mod 4), then $|A(D_v)| \equiv 2$ (mod 4), and so a packing with leave L where $|A(L)| = 2$ would be maximum.
Case 1. Let $v \equiv 3$ (mod 4), say $v = 4k+3$ where $k \geq 4$. We note that:

$$D_{4k+3} = W_{4k+1}(\infty_1 : 2k-1) \cup W_{4k+1}(\infty_2 : 2k)$$

$$\cup C_{4k+1}(1, 2, \ldots, 2k-2) \cup \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}.$$

As shown in the proof of Theorem 2.1, there exists an m_1-decomposition of W_{4k+1} and W_{k-1} for $k \geq 4$. Since W_{k-1} is graceful, there exists a W_{k-1}-decomposition of $C_{4k+1}(1, 2, \ldots, 2k-2)$.

The result is trivial for $v = 3$.

For $v = 7$, we note that: $D_7 = W_5(\infty_1 : 1) \cup W_5(\infty_2 : 2) \cup \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}$.

For $v = 11$, the required packing is given by the set of blocks $\{[j, j+1, \infty_1; j+7]_{m1}, [j, j+5, \infty_2; j+3]_{m1}, [j, j+3, j+1; j+5]_{m1} \mid j = 0, 1, \ldots, 8\}$ where numerical vertex labels are reduced modulo 9.

For $v = 15$, we note that: $D_{15} = W_{13}(\infty_1 : 5) \cup W_{13}(\infty_2 : 6) \cup C_{13}(1, 2, 3, 4) \cup \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}$. As above, there exists an m_1-decomposition of W_{13}. An m_1-decomposition of $C_{13}(1, 2, 3, 4)$ is given by the set of blocks $\{[j, j+1, j+3; j+4]_{m1}, [j, j+3, j+1; j+9]_{m1} \mid j = 0, 1, \ldots, 12\}$ where vertex labels are reduced modulo 13.

In each case above, the leave of the packing is $\{(\infty_1, \infty_2), (\infty_2, \infty_1)\}$.
Case 2. Let $v \equiv 2$ (mod 4), say $v = 4k+2$ where $k \geq 8$. We note that:

$$D_{4k+2} = D_7 \cup C_{4k-5}(1, 2, \ldots, 2k-10) \cup_{i=1}^{7} W_{4k-5}(\infty_i : 2k-i-2).$$

As above, there exists an m_1-decomposition of W_{4k-5} and $C_{4k-5}(1, 2, \ldots, 2k-10)$ for $k \geq 8$. Further, there exists a maximum m_1-packing of D_7 with leave size two, as shown above.

For $v=6$, the required packing is given by the set of blocks $\{[0,1,5;2]_{m1}$, $[0,5,1;3]_{m1}, [4,0,2;1]_{m1}, [4,1,3;0]_{m1}, [5,3,2;4]_{m1}, [2,3,1;5]_{m1}, [3,5,4;0]_{m1}\}$. This packing has leave $\{(4,2), (2,1)\}$.

For $v=10$, the required packing is given by the set of blocks $\{[1+j,\infty_1,j;2+j]_{m_1} \mid j=0,1,\ldots,5\} \cup \{[2+3j,\infty_2,5+3j;6+3j]_{m1} \mid j=0,1,2,3,4\} \cup \{[2j,\infty_3,2j+2;2j+5]_{m_1} \mid i=0,1,\ldots,6\} \cup \{[6,0,3;\infty_1]_{m1}, [6,3,\infty_2;2]_{m_1}, [\infty_1,\infty_2,\infty_3;0]_{m_1}, [\infty_2,\infty_1,\infty_3;6]_{m_1}\}$, where all numerical vertex labels are reduced modulo 7. This packing has leave $\{(1,0),(\infty_2,2)\}$.

For $v=14$, the required packing is given by the set of blocks $\{[3j+3,\infty_1,3j;3j+6]_{m1} \mid j=0,1,\ldots,9\}\{[1+4j,\infty_2,5+4j;8+4j]_{m1} \mid j=0,1,\ldots,8\} \cup \{[2j,\infty_3,2j+2;2j+9]_{m1}, [j,j+1,j+5;j+10]_{m1} \mid j=0,1,\ldots,10\} \cup \{[8,0,4,\infty_1]_{m1}, [8,4,\infty_2;1]_{m1}, [\infty_1,\infty_2,\infty_3;0]_{m1}, [\infty_2,\infty_1,\infty_3;8]_{m1}\}$, where all numerical vertex labels are reduced modulo 11. The leave on this packing is $\{(3,0),(\infty_2,1)\}$.

For $v=18$, the required packing is given by the set of blocks $\{[j+1,\infty_1,j;j+2]_{m1} \mid j=0,1,\ldots,13\} \cup \{[6+7j,\infty_2,13+7j;14+7j]_{m1} \mid j=0,1,\ldots,12\} \cup \{[2j,\infty_3,2j+2;2j+13]_{m1}, [j,j+4,j+9;j+3]_{m1}, [j,j+9,j+4;j+12]_{m1} \mid j=0,1,\ldots,14\} \cup \{[14,0,7;\infty_1]_{m1}, [14,7,\infty_2;6]_{m1}, [\infty_1,\infty_2,\infty_3;0]_{m1}, [\infty_2,\infty_1,\infty_3;14]_{m1}\}$, where all numerical vertex labels are reduced modulo 15. The leave is $\{(1,0),(\infty_2,6)\}$.

For $v=22$, we have $D_{22} = D_7 \cup_{i=1}^{7} W_{15}(\infty_i : i)$. This has a maximum packing with leave size two by the above comments.

For $v=26$, the required packing is given by the set of blocks $\{[j,j+18,\infty_1;j+4]_{m1}, [j,j+14,\infty_2;j+6]_{m_1}, [j,j+12,\infty_3;j+8]_{m1}, [j,j+10,\infty_4;j+11]_{m1}, [j,j+9,\infty_5;j+12]_{m1}, [j,j+6,\infty_6;j+14]_{m1}, [j,j+3,\infty_7;j+15]_{m1}, [j,j+1,j+3;j+2]_{m1} \mid j=0,1,\ldots,18\}$, where all numerical vertex labels are reduced modulo 19. The remaining arcs are isomorphic to D_7, which has a maximum packing with leave size two by the above comments.

For $v=30$, we have $D_{30} = D_7 \cup C_{23}(1,2,3,4) \cup_{i=1}^{7} W_{23}(\infty_i : 4+i)$. The required m_1-decomposition of $C_{23}(1,2,3,4)$ is given by the set of blocks $\{[j,j+1,j+3;j+4]_{m1}, [j,j+3,j+1;j+19]_{m1} \mid j=0,1,\ldots,22\}$, where are numerical labels on the vertices are reduced modulo 23. W_{23} has an m_1-decomposition by the above comments. D_7 has a maximum m_1-packing with leave size two.

Since m_2 is the converse of m_1, the construction of an m_2-packing of D_v will similarly follow. ■

Theorem 3.2 *A maximum d_1-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* $1 \ (mod\ 4)$,

(ii) $|A(L)| = 6$ *if* $v \in \{3,6\}$, *and* $|A(L)| = 2$ *if* $v \equiv 2$ *or* $3 \ (mod\ 4)$, $v \notin \{3,6\}$.

Maximum d_2-packings of D_v satisfy the same conditions.

Proof. The necessary conditions follow as in Theorem 3.1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.2 and the result follows.

Case 1. Suppose $v \equiv 2 \pmod 8$, say $v = 8k+2$ where $k \geq 1$. Consider the sets $A = \{[j, 5k-i+j, 5k+2+i+j; 2k+3-i+j]_{d1}, [j, 3k+2-i+j, 3k+3+i+j; 6+i+j]_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 8k-2\}$ and $B = \{[j, 1+j, \infty_1; 4+j]_{d1}, [j, 2+j, \infty_2; 3+j]_{d1}, [j, 5+j, \infty_3; 5k+1+j]_{d1} \mid j = 0, 1, \ldots, 8k-2\}$. Then $A \cup B \cup \{[\infty_2, \infty_1, \infty_3; 2]_{d1}, [\infty_1, \infty_2, \infty_2; 3]_{d1}, [0, 3, 2; \infty_2]_{d1}\} \setminus \{[2, 3, \infty_1; 6]_{d1}, [0, 2, \infty_2; 3]_{d1}\}$, where $V(D_v) = \{\infty_1, \infty_2, \infty_3, 0, 1, \ldots, v-4\}$ and numerical vertex labels are reduced modulo $8k-1$, is a maximum d_1-packing of D_v with leave L where $A(L) = \{(\infty_1, 2), (6, 2)\}$.

The result is trivial when $v = 2$.

Case 2. Suppose $v \equiv 3 \pmod 4$, say $v = 4k+3$ where $k \geq 1$. Consider the sets $A = \{[j, k+3-i+j, 4k-2i+j; 2k+3+2i+j]_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k\}$ and $B = \{[j, 1+2i+j, \infty_{i+1}; 2+2i+j]_{d1} \mid i = 0, 1, j = 0, 1, \ldots 4k\}$ where $V(D_v) = \{\infty_1, \infty_2, 0, 1, \ldots, v-3\}$ and numerical vertex labels are reduced modulo $4k+1$. Then $A \cup B$ is a maximum d_1-packing of D_v with leave L where $A(L) = \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}$.

The result is trivial when $v = 3$.

Case 3. Suppose $v \equiv 6 \pmod 8$, say $v = 8k+6$ where $k \geq 1$. Consider the sets $A = \{[j, 5k+3-i+j, 5k+5+i+j; 2k+4-i+j]_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 8k+2\} \cup [j, 3k+4-i+j, 3k+5+i+j; 6+i+j]_{d1} \mid i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, 8k+2\}$ and $B = \{[j, 1+j, \infty_1; 4+j]_{d1}, [j, 2+j, \infty_2; 3+j]_{d1}, [j, 5+j, \infty_3; 5k+4+j]_{d1} \mid j = 0, 1, \ldots, 8k+2\}$. Then $A \cup B \cup \{[\infty_2, \infty_1, \infty_3; 2]_{d1}, [\infty_1, \infty_2, \infty_3; 3]_{d1}, [0, 3, 2; \infty_2]_{d1}\} \setminus \{[2, 3, \infty_1; 6]_{d1}, [0, 2, \infty_2; 3]_{d1}\}$, where $V(D_v) = \{\infty_1, \infty_2, \infty_3, 0, 1, \ldots, v-4\}$ and numerical vertex labels are reduced modulo $8k+3$, is a maximum d_1-packing of D_v with leave L where $A(L) = \{(\infty_1, 2), (6, 2)\}$.

When $v = 6$, $|A(D_v)| = 30$ and a d_1-packing of D_6 could contain as many as seven copies of d_1. However, each vertex of D_6 is of in-degree 5 and d_1 contains a vertex of in-degree 3. Therefore the number of d_1s in a d_1-packing of D_6 cannot exceed the number of vertices in D_6—namely, six. So in a maximum d_1-packing of D_6 with leave L, we have $|A(L)| \geq 6$. A maximum packing is given by $\{[0, 2, 4; 3]_{d1}, [1, 2, 3; 0]_{d1}, [2, 4, 3; 1]_{d1}, [3, 5, 1; 0]_{d1}, [4, 5, 6; 3]_{d1}, [5, 1, 0; 4]_{d1}\}$ where $A(L) = \{(3, 5), (0, 2), (2, 5), (5, 2), (1, 4), (4, 1)\}$ and $|A(L)| = 6$.

Since d_2 is the converse of d_1, the construction of a d_2-packing of D_v will similarly follow. ∎

Corollary 3.3 *A maximum d_3-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod 4),*

(ii) $|A(L)| = 6$ *if* $v = 3$*, and* $|A(L)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod 4),* $v \neq 3$*.*

Maximum d_4*-packings of* D_v *satisfy the same conditions.*

Proof. The necessary conditions follow as in Theorem 3.1.

For $v \neq 6$, the blocks for such a d_3-packing of D_v can be constructed from the d_1-packing D_v of Theorem 3.2 by replacing every block of the form $[j, a+j, b+j; c+j]_{d1}$ with a block of the form $[a+j, b+j, j; a+c+j]_{d3}$, replacing every block of the form $[a, b, \infty_i; c]_{d1}$ with a block of the form $[-a, \infty_i, -b; c-2a]_{d3}$, and then (1) when $v \equiv 2 \pmod 8$ by replacing the two blocks $[2, \infty_2, 0; 5]_{d3}$ and $[5, \infty_3, 0; 5k+6]_{d3}$ with the three blocks $[\infty_2, \infty_1, \infty_3; 2]_{d3}$, $[\infty_3, \infty_1, \infty_2; 5]_{d3}$, and $[5, 2, 0; 5k+6]_{d3}$, and (2) when $v \equiv 6 \pmod 8$ by replacing the two blocks $[2, \infty_2, 0; 5]_{d3}$ and $[5, \infty_3, 0; 5k+9]_{d3}$ with the three blocks $[\infty_2, \infty_1, \infty_3; 2]_{d3}$, $[\infty_3, \infty_1, \infty_2; 5]_{d3}$, and $[5, 2, 0; 5k+9]_{d3}$. In the case $v \equiv 2 \pmod 4$, this is a d_3-packing of D_v, where $V(D_v) = \{\infty_1, \infty_2, \infty_3, 0, 1, \ldots, v-4\}$, with leave L where $A(L) = \{(\infty_2, 0), (\infty_3, 0)\}$. In the case $v \equiv 3 \pmod 4$, this is a d_3-packing of D_v, where $V(D_v) = \{\infty_1, \infty_2, 0, 1, \ldots, v-3\}$, with leave L where $A(L) = \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}$.

For $v = 6$, consider the set of blocks $\{[4, 1, 3; 0]_{d3}, [4, 5, 2; 3]_{d3}, [5, 3, 0; 1]_{d3}, [3, 1, 2; 0]_{d3}, [0, 5, 1; 2]_{d3}, [1, 4, 0; 2]_{d3}, [2, 5, 4; 0]_{d3}\}$. This is a maximum d_3-packing of D_6 with leave L where $A(L) = \{(2, 3), (3, 5)\}$.

Since d_4 is the converse of d_3, the construction of a d_4-packing of D_v will similarly follow. ■

Theorem 3.4 *A maximum* d_5*-packing of* D_v *with leave* L *satisfies*

(i) $|A(L)| = v$ *if* $v \equiv 0$ *(mod 2),*

(ii) $|A(L)| = 0$ *if* $v \equiv 1$ *(mod 4), and*

(iii) $|A(L)| = 6$ *if* $v = 3$*, and* $|A(L)| = 2$ *if* $v \equiv 3$ *(mod 4),* $v \geq 7$*.*

Maximum d_6*-packings of* D_v *satisfy the same conditions.*

Proof. When $v \equiv 1 \pmod 4$, a decomposition exists by Corollary 2.4 and $|A(L)| = 0$ in this case. Notice that the vertices of d_5 are of in-degrees 0, 0, 2, and 2. So when v is even, a d_5-packing of D_v will have a leave L where the in-degree of each vertex of L is odd. So for v even, a d_5-packing of D_v with leave L where $|A(L)| = v$ would be maximum (and similarly for a d_6-packing of D_v). When $v \equiv 3 \pmod 4$, $|A(D_v)| \equiv 2 \pmod 4$ and in this case a d_5-packing (and similarly for a d_6-packing) of D_v with leave

L where $|A(L)| = 2$ would be maximum. In the following cases, we have $V(D_v) = \{0, 1, \ldots, v-1\}$.
Case 1. Suppose $v \equiv 0 \pmod 4$. Consider $A \cup B$ where $A = \{[2j, 4k-1+2j, 1+2j; 4k-2+2j]_{d5} \mid j = 0, 1, \ldots, 2k-1\}$ and $B = \{[j, 3k-3+j, 4k-2+j; 3k-2+j]_{d5}\} \cup \{[j, 2k-1+i+j, 2k+2+2i+j; 2k-3-2i+j]_{d5} \mid i = 0, 1, \ldots, k-3, j = 0, 1, \ldots 4k-1\}$ where vertex labels are reduced modulo $4k$. Then $A \cup B$ is a maximum d_5-packing of D_v with leave L where $A(L) = \{(j, j-1) \mid j = 0, 1, \ldots, 4k-1)\}$.
Case 2. Suppose $v \equiv 2 \pmod 4$, say $v = 4k+2$. Consider $\{[j, k+2+i+j, 1+2i+j; 2k+2+2i]_{d5} \mid i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, 4k+1\}$ where vertex labels are reduced modulo $4k+2$. This is a maximum d_5-packing of D_v with leave L where $A(L) = \{(j, j-1) \mid j = 0, 1, \ldots, 4k+1)\}$.
Case 3. Suppose $v \equiv 3 \pmod 4$. Consider $A \cup B$ where $A = \{[2i, 4k+2+2i, 1+2i; 4k+1+2i]_{d5} \mid i = 0, 1, \ldots, 2k\}$ $B = \{[j, 3k-1+j, 4k+2+j; 4k+j]_{d5} \mid j = 0, 1, \ldots, 4k+2\} \cup \{[j, 2k+i+j, 2k+4+2i+j; 2+2i+j]_{d5} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k+2\}$ where vertex labels are reduced modulo $4k+3$. Then $A \cup B$ is a maximum d_5-packing of D_v with leave L where $A(L) = \{(4k, 4k+2), (4k+1, 4k+2)\}$.

Since d_6 is the converse of d_5, the construction of a d_6-packing of D_v will similarly follow. ■

4 Covering

We now give necessary and sufficient conditions for the covering of D_v with each of the eight orientations of L.

Theorem 4.1 *A minimum m_1-covering of D_v, $v \geq 4$, with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* $1 \pmod 4$, *and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* $3 \pmod 4$.

Minimum m_2-coverings of D_v satisfy the same conditions.

Proof. If $v \equiv 0$ or $1 \pmod 4$, then there is a decomposition by Theorem 2.1 and the result follows. If $v \equiv 2$ or $3 \pmod 4$, then $|A(D_v)| \equiv 2 \pmod 4$, and so a covering with padding P where $|A(P)| = 2$ would be minimum.
Case 1. Let $v \equiv 2 \pmod 4$, say $v = 4k+2$ where $k \geq 5$. We note that:

$$D_{4k+2} = D_3 \cup C_{4k-1}(1, 2, \ldots, 2k-4) \cup_{i=1}^{3} W_{4k-1}(\infty_i : 2k-i).$$

As above, there exists an m_1-decomposition of W_{4k-1} and $C_{4k-1}(1, 2, \ldots, 2k-4)$ for $k \geq 5$. The remaining arcs are covered by the set $\{[\infty_1, \infty_2, \infty_3; 0]_{m1}, [\infty_3, \infty_2, \infty_1; 1]_{m1}\}$. This covering has padding $\{(0, \infty_1), (1, \infty_3)\}$.

For $v = 6$, the required covering is obtained from the packing in Theorem 3.1 along with the set $\{[2, 1, 4; 3]_{m1}\}$. This covering has padding $\{(1, 4), (3, 2)\}$.

For $v = 10$, the required covering is obtained from the packing in Theorem 3.1 along with the set $\{[2, 1, 0; \infty_2]_{m1}\}$. This covering has padding $\{(2, 1), (0, 2)\}$.

For $v = 14$, the required covering is obtained from the packing in Theorem 3.1 along with the set $\{[1, 3, 0; \infty_2]_{m1}\}$. This covering has padding $\{(1, 3), (0, 1)\}$.

For $v = 18$, the required covering is obtained from the packing in Theorem 3.1 along with the set $\{[6, 1, 0; \infty_2]_{m1}\}$. This covering has padding $\{(6, 1), (0, 6)\}$.

Case 2. Let $v \equiv 3 \pmod 4$, say $v = 4k + 3$ where $k \geq 7$. We note that:

$$D_{4k+3} = \cup_{i=1}^{6} W_{4k-3}(\infty_i : 2k - 1 - i) \cup C_{4k-3}(1, 2, \ldots, 2k - 8) \cup D_6.$$

As shown in the proof of Theorem 2.1, there exists an m_1-decomposition of W_{4k-3} and W_{k-5} for $k \geq 7$. Since W_{k-4} is graceful, there exists a W_{k-4}-decomposition of $C_{4k-3}(1, 2, \ldots, 2k - 8)$. Further, there exists a minimum covering of D_6 as given above. Thus there exists a minimum covering of D_{4k+3} for $k \geq 8$ with padding $\{(\infty_1, \infty_4), (\infty_3, \infty_2)\}$.

For $v = 7$, the covering is given by the set of blocks $\{[0, 6, 1; 4]_{m1}, [0, 1, 6; 2]_{m1}, [5, 1, 3; 0]_{m1}, [5, 0, 2; 1]_{m1}, [4, 3, 0; 6]_{m1}, [3, 4, 6; 0]_{m1}, [3, 6, 2; 5]_{m1}, [5, 2, 4; 6]_{m1}, [4, 2, 1; 5]_{m1}, [6, 3, 2; 5]_{m1}, [1, 5, 2; 4]_{m1}\}$. The padding is $\{(6, 3), (5, 3)\}$.

For $v = 11$, the covering is given by the set of blocks $\{[1 + 3i, \infty_1, 7 + 3i; 3 + 3i]_{m1}, [2 + 3i, \infty_1, 8 + 3i; 4 + 3i]_{m1} \mid i = 0, 1, 2\} \cup \{[4 + 4i, \infty_2, 4i; 5 + 4i]_{m1} \mid i = 0, 1, \ldots, 7\} \cup \{[i, i + 1, i + 3; i + 4]_{m1} \mid i = 0, 1, \ldots, 8\} \cup \{[\infty_1, 6, 0; \infty_2]_{m1}, [\infty_2, 5, 0; \infty_1]_{m1}, [3, \infty_1, 0; 5]_{m1}, [6, \infty_1, 3; 8]_{m1}, [0, 3, 1; 2]_{m1}\}$, where all numerical vertex labels are reduced modulo 9. The padding is $\{(0, 3), (3, 1)\}$.

For $v = 15$, the covering is given by the set of blocks $\{[5i, \infty_1, 5i+5; 5i+8]_{m1}, [6i, \infty_2, 6i+6; 6i+7]_{m1} \mid i = 0, 1, \ldots, 11\} \cup \{[i, i+1, i+3; i+4]_{m1}, [i, i+3, i+1; i+9]_{m1} \mid i = 0, 1, \ldots, 12\} \cup \{[\infty_1, 0, 8; \infty_2]_{m1}, [\infty_2, 0, 7; \infty_1]_{m1}, [7, 3, 8; 1]_{m1}\}$, where all numerical vertex labels are reduced modulo 13. The padding is $\{(7, 3), (8, 7)\}$.

For $v = 19$, we note that: $D_{19} = \cup_{i=1}^{6} W_{13}(\infty_i : i) \cup D_6$. There exists an m_1-decomposition of W_{13} by the above comments. Further, there exists a minimum m_1-covering of D_6 by above.

For $v = 23$, the covering is given by the set of blocks $\{[10i, \infty_1, 10i + 10; 10i+11]_{m1}, [9i, \infty_2, 9i+9; 9i+12]_{m1} \mid i = 0, 1, \ldots, 19\} \cup \{[\infty_1, 0, 11; \infty_2]_{m1}, [\infty_2, 0, 12; \infty_1]_{m1}, [11, 3, 12; 1]_{m1}\}$, where all numerical vertex labels are reduced modulo 21. The remaining arcs are isomorphic to $C_{21}(1, 2, \ldots, 8)$, which has an m_1-decomposition by the above comments. The padding is $\{(11, 3), (12, 11)\}$.

For $v = 27$, the covering is given by the set of blocks $\{[12i, \infty_1, 12i + 12; 12i+13]_{m1}, [11i, \infty_2, 11i+11; 11i+14]_{m1} \mid i = 0, 1, \ldots, 23\} \cup \{[\infty_1, 0, 13; \infty_2]_{m1}, [\infty_2, 0, 14; \infty_1]_{m1}, [13, 3, 14; 1]_{m1}\}$, where all numerical vertex labels are reduced modulo 25. The remaining arcs are isomorphic to $C_{25}(1, 2, \ldots, 10)$, which has an m_1-decomposition by the above comments. The padding is $\{(13, 3), (14, 13)\}$.

Since m_2 is the converse of m_1, the construction of an m_2-covering of D_v will similarly follow. ■

Theorem 4.2 *A minimum d_1-covering of D_v where $v \geq 4$ with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod 4), and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod 4).*

Minimum d_2-coverings of D_v satisfy the same conditions.

Proof. The necessary conditions follow as in Theorem 4.1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.2 and the result follows. In the following cases, we have $V(D_v) = \{\infty_1, \infty_2, 0, 1, \ldots, v - 3\}$.
Case 1. Suppose $v \equiv 2$ (mod 4), say $v = 4k + 2$ where $k \geq 2$. Take the d_1-packing of D_v given in Theorem 3.2 and replace the block $[0, 3, 2; \infty_2]_{d1}$ with the two blocks $[0, 2, \infty_2; 3]_{d1}$ and $[2, 3, \infty_1; 6]_{d1}$. This is a minimum covering of D_v with padding P where $A(P) = \{(2, \infty_2), (3, \infty_1)\}$.

For $v = 6$, consider the set of blocks $\{[5, 0, 1; 4]_{d1}, [1, 5, 4; 2]_{d1}, [3, 1, 0; 5]_{d1}, [2, 4, 3; 1]_{d1}, [4, 3, 1; 0]_{d1}, [0, 2, 4; 3]_{d1}, [5, 2, 3; 4]_{d1}, [2, 5, 0; 3]_{d1}\}$. This is a minimum d_1-covering of D_6 with padding P where $A(P) = \{(3, 2), (4, 5)\}$.
Case 2. Suppose $v \equiv 3$ (mod 4), say $v = 4k + 3$. Consider the blocks in $A \bigcup B \setminus \{[0, 3, \infty_2; 4]_{d1}\} \cup \{[0, \infty_2, \infty_1; 4]_{d1}, [\infty_2, 3, 0; \infty_1]_{d1}\}$ where sets A and B are defined in Theorem 3.2 Case 2. This is a minimum covering of D_v with padding P where $A(P) = \{(0, \infty_2), (\infty_1, 0)\}$.

Since d_2 is the converse of d_1, the construction of a d_2-covering of D_v will similarly follow. ■

Theorem 4.3 *A minimum d_3-covering of D_v where $v \geq 4$ with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod 4), and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod 4).*

Minimum d_4-coverings of D_v satisfy the same conditions.

Proof. The necessary conditions follow as in Theorem 4.2. When $v \equiv 0$ or 1 (mod 4), a decomposition exists by Corollary 2.3 and $|A(P)| = 0$ in this case.
Case 1. Suppose $v \equiv 2 \pmod 8$, $v \neq 6$. Take the d_3-packing of D_v given in Corollary 3.3 and replace the block $[5, 2, 0; 5k+6]_{d3}$ with the two blocks $[2, \infty_2, 0; 5]_{d3}$ and $[5, \infty_3, 0; 5k + 6]_{d3}$. This is a minimum covering of D_v with padding P where $A(P) = \{(2, \infty_2), (5, \infty_3)\}$.

For $v = 6$, take the d_3-packing of D_6 given in Corollary 3.3, along with the block $[2, 3, 5; 1]_{d3}$. This yields a minimum covering of D_6 with padding P where $A(P) = \{(1, 2), (2, 5)\}$.
Case 2. Suppose $v \equiv 3 \pmod 4$. Take the d_3-packing of D_v given in Corollary 3.3 and replace the block $[0, \infty_1, 4k; 2]_{d3}$ with the two blocks $[\infty_1, 0, 4k; \infty_2]_{d3}$ and $[0, \infty_1, \infty_2; 2]_{d3}$. This is a minimum covering of D_v with padding P where $A(P) = \{(\infty_1, 0), (0, \infty_2)\}$.
Case 3. Suppose $v \equiv 6 \pmod 8$. Take the d_3-packing of D_v given in Corollary 3.3 and replace the block $[5, 2, 0; 5k + 9]_{d3}$ with the two blocks $[2, \infty_2, 0; 5]_{d3}$ and $[5, \infty_3, 0; 5k + 9]_{d3}$. This is a minimum covering of D_v with padding P where $A(P) = \{(2, \infty_2), (5, \infty_3)\}$.

Since d_4 is the converse of d_3, the construction of a d_4-covering of D_v will similarly follow. ■

Theorem 4.4 *A minimum d_5-covering of D_v where $v \geq 4$ with padding P satisfies*

(i) $|A(L)| = v$ *if* $v \equiv 0 \ (mod\ 2)$,

(ii) $|A(L)| = 0$ *if* $v \equiv 1 \ (mod\ 4)$, *and*

(iii) $|A(L)| = 2$ *if* $v \equiv 3 \ (mod\ 4)$.

Minimum d_6-coverings of D_v satisfy the same conditions.

Proof. When $v \equiv 1 \pmod 4$, a decomposition exists by Corollary 2.4 and the result follows. Notice that the vertices of d_5 are of in-degrees 0, 0, 2, and 2. So when v is even, a d_5-covering of D_v will have a padding P where the in-degree of each vertex of P is odd. So for v even, a d_5-covering of D_v with padding P where $|A(P)| = v$ would be minimum (and similarly for a d_6-covering of D_v). When $v \equiv 3 \pmod 4$, $|A(D_v)| \equiv 2 \pmod 4$ and in this case a d_5-covering (and similarly for a d_6-covering) of D_v with padding P where $|A(P)| = 2$ would be minimum. In the following cases, we have $V(D_v) = \{0, 1, \ldots, v - 1\}$.
Case 1. Suppose $v \equiv 0 \pmod 4$, say $v = 4k$. Consider the blocks in $A \cup B$ where $A = \{[j, 2k + j, 2k - 1 + j; 4k - 1 + j]_{d5} \mid j = 0, 1, \ldots, 4k - 1\}$ and $B = \{[j, k + 1 + i + j, 1 + 2i + j; 4k - 2 - 2i + j]_{d5} \mid i = 0, 1, \ldots, k - 2, j =$

$0, 1, \ldots, 4k-1\}$ where vertex labels are reduced modulo $4k$. Then $A \cup B$ is a minimum d_5-covering of D_v with padding P where $A(P) = \{(j, j+1) \mid j = 0, 1, \ldots, 4k-1\}$.

Case 2. Suppose $v \equiv 2 \pmod 4$, say $v = 4k+2$. Consider the blocks in $A \cup B$ where $A = \{[2j, 4k+1+2j, 1+2j; 4k+2j]_{d5} \mid j = 0, 1, \ldots, 2k\}$ and $B = \{[j, k+1+j, 1+j; 2k+1+j]_{d5} \mid j = 0, 1, \ldots, 4k+1\} \cup \{[j, k+2+i+j, 3+2i+j; 4k-2-2i+j]_{d5} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k+1\}$ where vertex labels are reduced modulo $4k+2$. Then $A \cup B$ is a minimum d_5-covering of D_v with padding P where $A(P) = \{(j, j+1) \mid j = 0, 1, \ldots, 4k+1\}$.

Case 3. Suppose $v \equiv 3 \pmod 4$, say $v = 4k+3$. Consider the blocks in $A \cup B$ where $A = \{[2j, 4k+2+2j, 1+2j; 4k+1+2j]_{d5} \mid j = 0, 1, \ldots, 2k+1\}$ and $B = \{[j, 3k-1+j, 4k+2+j; 4k+j]_{d5} \mid j = 0, 1, \ldots, 4k+2\} \cup \{[j, 2k+i+j, 2k+4+2i+j; 2+2i+j]_{d5} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k-1\}$ where vertex labels are reduced modulo $4k+3$. Then $A \cup B$ is a minimum d_5-covering of D_v with padding P where $A(P) = \{(4k+1, 0), (4k+2, 0)\}$.

Since d_6 is the converse of d_5, the construction of a d_6-covering of D_v will similarly follow. ■

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