Acta Math. Hungar.
74 (1–2) (1997), 125–134.

SOME GENERALIZATIONS OF THE ENESTRÖM-KAKEYA THEOREM

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1. Introduction and statement of results

A classical result due to Eneström [3] and Kakeya [7] concerning the bounds for the moduli of the zeros of polynomials having positive real coefficients is often stated as (see Marden [9, p. 136]):

THEOREM A (Eneström–Kakeya). *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *whose coefficients satisfy* $0 \leqq a_0 \leqq a_1 \leqq \cdots \leqq a_n$. *Then* $p(z)$ *has all its zeros in the closed unit disk* $|z| \leqq 1$.

An equivalent, but perhaps more useful statement of the above theorem, due in fact to Eneström [3], is the following:

THEOREM B. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$, $n \geqq 1$, *be a polynomial of degree* n *with* $a_v > 0$ *for all* $0 \leqq v \leqq n$. *If*

$$\alpha = \alpha[p] := \min_{0 \leqq v < n} \{a_v / a_{v+1}\}, \quad \beta = \beta[p] := \max_{0 \leqq v < n} \{a_v / a_{v+1}\},$$

then all the zeros of $p(z)$ *are contained in the annulus* $\alpha \leqq |z| \leqq \beta$.

In the literature, there exists several generalizations of this result (see [1], [2], [4], [5], [6] and [8]).

In this paper we give some generalizations of this result for polynomials with complex coefficients when we have information only about the real or only the imaginary parts of the coefficients. As corollaries we obtain sharpened forms of several known results including those of Joyal, Labelle and Rahman [6], Kovačević and Milovanović [8], and of course the Eneström–Kakeya Theorem.

THEOREM 1. *Suppose* $p(z)=\sum_{v=0}^{n} a_v z^v$, Re $a_j=\alpha_j$ *and* Im $a_j=\beta_j$ *for* $j=0,1,\dots,n$, $a_n\neq 0$ *and for some* k,

$$\alpha_0 \leqq t\alpha_1 \leqq t^2\alpha_2 \leqq \cdots \leqq t^k\alpha_k \geqq t^{k+1}\alpha_{k+1} \geqq t^{k+2}\alpha_{k+2} \geqq \cdots \geqq t^n\alpha_n$$

for some positive t. *Then* $p(z)$ *has all its zeros in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = t|a_0| \Big/ \left(2t^k\alpha_k - \alpha_0 - t^n\alpha_n + t^n|a_n| + |\beta_0| + |\beta_n|t^n + 2\sum_{j=1}^{n-1}|\beta_j|t^j\right)$$

and

$$R_2 = \max\Bigg\{\Bigg(|a_0|t^{n+1} + (t^2+1)t^{n-k-1}\alpha_k - t^{n-1}\alpha_0 - t\alpha_n$$

$$+(t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j$$

$$+\sum_{j=1}^{n}\left(|\beta_{j-1}| + t|\beta_j|\right)t^{n-j}\Bigg) \Big/ |a_n|, \frac{1}{t}\Bigg\}.$$

We do not know if this result is best possible, however if we take $k=n$, $t=1$, $\beta_v=0$ for $0 \leqq v \leqq n$, and $a_0 \geqq 0$, we get that all the zeros of the polynomial $p(z)$ lie in the annulus $\dfrac{a_0}{2a_n - a_0} \leqq |z| \leqq 1$, which is best possible in the sense that the inner and outer radii of the annulus here cannot be improved (as is seen by considering the polynomial $p(z)=z^n+z^{n-1}+\cdots+z+1$).

If we take $k=n$ in Theorem 1, we get:

COROLLARY 1. *Suppose* $p(z)=\sum_{v=0}^{n} a_v z^v$, Re $a_j=\alpha_j$ *and* Im $a_j=\beta_j$ *for* $j=0,1,\dots,n$, $a_n\neq 0$ *and*

$$\alpha_0 \leqq t\alpha_1 \leqq t^2\alpha_2 \leqq \cdots \leqq t^n\alpha_n$$

for some positive t. *Then* $p(z)$ *has all its zeros in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = t|a_0| \Big/ \left(t^n\alpha_n - \alpha_0 + t^n|a_n| + |\beta_0| + |\beta_n|t^n + 2\sum_{j=1}^{n-1}|\beta_j|t^j\right)$$

and

$$R_2 = \max\left\{\left(|a_0|t^{n+1} + t^{-1}\alpha_n - t^{n-1}\alpha_0 + (t^2-1)\sum_{j=1}^{n-1} t^{n-j-1}\alpha_j \right.\right.$$
$$\left.\left. + \sum_{j=1}^{n}\left(|\beta_{j-1}| + t|\beta_j|\right)t^{n-j}\right)\Big/|a_n|, \frac{1}{t}\right\}.$$

In particular, taking $t=1$ and $\beta_v = 0$ for $0 \leqq v \leqq n$ in Corollary 1, we get that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial with real coefficients satisfying $a_0 \leqq a_1 \leqq \dots \leqq a_n$ then $p(z)$ has all its zeros in

$$\frac{|a_0|}{a_n - a_0 + |a_n|} \leqq |z| \leqq \frac{|a_0| + a_n - a_0}{|a_n|}. \tag{1.1}$$

This result sharpens a result due to Joyal, Labelle and Rahman [6]. The Eneström–Kakeya Theorem is implied by (1.1) when $a_0 \geqq 0$.

Similarly if we take $k = 0$ in Theorem 1, we get:

Corollary 2. *Suppose* $p(z) = \sum_{v=0}^{n} a_v z^v$, Re $a_j = \alpha_j$ *and* Im $a_j = \beta_j$ *for* $j = 0, 1, \dots, n$, $a_n \neq 0$ *and*

$$\alpha_0 \geqq t\alpha_1 \geqq t^2\alpha_2 \geqq \dots \geqq t^n\alpha_n$$

for some positive t. *Then* $p(z)$ *has all its zeros in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = t|a_0|\Big/\left(\alpha_0 - t^n\alpha_n + t^n|a_n| + |\beta_0| + |\beta_n|t^n + 2\sum_{j=1}^{n-1}|\beta_j|t^j\right)$$

and

$$R_2 = \max\left\{\left(|a_0|t^{n+1} + t^{n+1}\alpha_0 - t\alpha_n + (1-t^2)\sum_{j=1}^{n-1} t^{n-j-1}\alpha_j \right.\right.$$
$$\left.\left. + \sum_{j=1}^{n}\left(|\beta_{j-1}| + t|\beta_j|\right)t^{n-j}\right)\Big/|a_n|, \frac{1}{t}\right\}.$$

In particular, if $p(z)=\sum_{v=0}^{n} a_v z^v$ is with real coefficients satisfying $a_0 \geqq a_1 \geqq \cdots \geqq a_n$ then it has all its zeros in

$$\frac{|a_0|}{a_0-a_n+|a_n|} \leqq |z| \leqq \frac{|a_0|+a_0-a_n}{|a_n|}.$$

By applying Theorem 1 to the polynomial $z^n p(1/z)$ one easily gets:

THEOREM 2. *Suppose* $p(z)=\sum_{v=0}^{n} a_v z^v$, Re $a_j=\alpha_j$ *and* Im $a_j=\beta_j$ *for* $j=0,1,\ldots,n$, $a_n \neq 0$ *and for some* k,

$$t^n\alpha_0 \leqq t^{n-1}\alpha_1 \leqq t^{n-2}\alpha_2 \leqq \cdots \leqq t^k\alpha_{n-k} \geqq t^{k-1}\alpha_{n-k+1} \geqq \cdots \geqq t\alpha_{n-1} \geqq \alpha_n$$

for some positive t. *Then* $p(z)$ *has all its zeros in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1=\min\left\{|a_0| \Big/ \Big(|a_n|t^{n+1}+(t^2+1)t^{n-k-1}\alpha_{n-k}-t^{n-1}\alpha_n-t\alpha_0 + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_{n-j}+(1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_{n-j} + \sum_{j=1}^{n}\left(|\beta_{n-j+1}|+t|\beta_{n-j}|\right)t^{n-j}\Big), t\right\}$$

and

$$R_2=\left(2t^k\alpha_{n-k}-\alpha_n-t^n\alpha_0+t^n|a_0|+|\beta_0|t^n+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_{n-j}|t^j\right)\Big/\left(t|a_n|\right).$$

In particular, if we take $k=0$ and $\beta_v=0$ for $0 \leqq v \leqq n$, we get that if $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients satisfying

$$t^n a_0 \leqq t^{n-1}a_1 \leqq \cdots \leqq t a_{n-1} \leqq a_n$$

for some positive t, then all the zeros of $p(z)$ lie in

$$\min\left\{\frac{|a_0|}{|a_n|t^{n+1}+t^{n+1}a_n-ta_0+(1-t^2)\sum_{j=1}^{n-1} t^{n-j-1}a_{n-j}}, t\right\}$$

$$\leqq |z| \leqq \frac{a_n - t^n a_0 + |a_0| t^n}{t|a_n|}.$$

This result sharpens a recent result due to Kovačević and Milovanović [8]. For $t = 1$, this further reduces to (1.1) which, when $a_0 \geqq 0$, reduces to the Eneström–Kakeya Theroem.

If we have information only about the imaginary parts of the coefficients, we have the following theorem which is of interest and follows by applying Theorem 1 to $-ip(z)$.

THEOREM 3. *Suppose* $p(z) = \sum_{v=0}^{n} a_v z^v$, Re $a_j = \alpha_j$ *and* Im $a_j = \beta_j$ *for* $j = 0, 1, \ldots, n$, $a_n \neq 0$ *and for some* k,

$$\beta_0 \leqq t\beta_1 \leqq t^2\beta_2 \leqq \cdots \leqq t^k\beta_k \geqq t^{k+1}\beta_{k+1} \geqq t^{k+2}\beta_{k+2} \geqq \cdots \geqq t^n\beta_n$$

for some positive t. *Then* $p(z)$ *has all its zeros in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = t|a_0| \Big/ \left(2t^k\beta_k - \beta_0 - t^n\beta_n + t^n|a_n| + |\alpha_0| + |\alpha_n|t^n + 2\sum_{j=1}^{n-1} |\alpha_j| t^j \right)$$

and

$$R_2 = \max\left\{ \left(|a_0| t^{n+1} + (t^2+1)t^{n-k-1}\beta_k - t^{n-1}\beta_0 - t\beta_n \right.\right.$$
$$+ (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\beta_j + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\beta_j$$
$$\left.\left. + \sum_{j=1}^{n} \left(|\alpha_{j-1}| + t|\alpha_j| \right) t^{n-j} \right) \Big/ |a_n|, \frac{1}{t} \right\}.$$

By making suitable choices of t and k in the above theorems, one can also obtain the following corollaries which appear to be interesting and useful. In each of these, $p(z) = \sum_{v=0}^{n} a_v z^v$, Re $a_j = \alpha_j$ and Im $a_j = \beta_j$ for $j = 0, 1, \ldots, n$, and $a_n \neq 0$.

COROLLARY 3. *If* $\alpha_0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_n$ *then all the zeros of* $p(z)$ *lie in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = |a_0| \Big/ \left(\alpha_n - \alpha_0 + |a_n| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1} |\beta_j| \right)$$

and

$$R_2 = \left(|a_0| - \alpha_0 + \alpha_n + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1} |\beta_j|\right) \Big/ |a_n|.$$

COROLLARY 4. *If* $\alpha_0 \geqq \alpha_1 \geqq \dots \geqq \alpha_n$ *then all the zeros of* $p(z)$ *lie in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = |a_0| \Big/ \left(\alpha_0 - \alpha_n + |a_n| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1} |\beta_j|\right)$$

and

$$R_2 = \left(|a_0| + \alpha_0 - \alpha_n + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1} |\beta_j|\right) \Big/ |a_n|.$$

COROLLARY 5. *If* $\beta_0 \leqq \beta_1 \leqq \dots \leqq \beta_n$ *then all the zeros of* $p(z)$ *lie in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = |a_0| \Big/ \left(\beta_n - \beta_0 + |a_n| + |\alpha_0| + |\alpha_n| + 2\sum_{j=1}^{n-1} |\alpha_j|\right)$$

and

$$R_2 = \left(\beta_n - \beta_0 + |a_0| + |\alpha_0| + |\alpha_n| + 2\sum_{j=1}^{n-1} |\alpha_j|\right) \Big/ |a_n|.$$

COROLLARY 6. *If* $\beta_0 \geqq \beta_1 \geqq \dots \geqq \beta_n$ *then all the zeros of* $p(z)$ *lie in* $R_1 \leqq |z| \leqq R_2$ *where*

$$R_1 = |a_0| \Big/ \left(\beta_0 - \beta_n + |a_n| + |\alpha_0| + |\alpha_n| + 2\sum_{j=1}^{n-1} |\alpha_j|\right)$$

and

$$R_2 = \left(\beta_0 - \beta_n + |a_0| + |\alpha_0| + |\alpha_n| + 2\sum_{j=1}^{n-1} |\alpha_j|\right) \Big/ |a_n|.$$

2. An example

In this section, we apply our results to a polynomial and compare the sizes of the zero containing region given by our theorems to those given by previously known results.

EXAMPLE. Consider $p(z) = 1 + 10z + 20z^2 + 40z^3 + 80z^4 + 50z^5$. Notice that the result of Joyal, Labelle and Rahman [6] is not applicable. The result of Kovačević and Milovanović [8] says that the zeros of $p(z)$ lie in $|z| \leqq 1.602$. A theorem due to Cauchy (see p. 122 of [9]) gives that the zeros of $p(z)$ lie in $.083 \leqq |z| \leqq 2.2096$. Eneström's Theorem (Theorem B) gives that the zeros of $p(z)$ lie in $.100 \leqq |z| \leqq 1.600$. Applying Theorem 1, we find that $p(z)$ has all its zeros in $.499 \leqq |z| \leqq .840$. The inner radius is obtained by letting $t = .4995031$ and $k = 0$. The outer radius is obtained by letting $t = 1.190792$ and $k = 3$. In terms of area in the complex plane, Theorem 1 is an improvement over the Kovačević and Milovanović result by a factor of about 5.6, an improvement over Cauchy's Theorem by a factor of about 9.6, and an improvement over Eneström's Theorem by a factor of about 5.6.

3. Proof of Theorem 1

Consider the polynomial

$$P(z) = (t - z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv -a_n z^{n+1} + G_2(z).$$

We first note that

$$(3.1) \qquad |a_{j-1} - ta_j| = \left|\alpha_{j-1} - t\alpha_j + i(\beta_{j-1} - t\beta_j)\right| \leqq |\alpha_{j-1} - t\alpha_j| + |\beta_{j-1}| + t|\beta_j|.$$

Then

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| = \left|ta_0 z^n + \sum_{j=1}^{n}(ta_j - a_{j-1})z^{n-j}\right|$$

and on $|z| = t$, by (3.1),

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \leqq |ta_0|t^n + \sum_{j=1}^{n}|ta_j - a_{j-1}|t^{n-j}$$

$$\leqq |a_0|t^{n+1} + \sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^{n-j} + \sum_{j=1}^{n} \left(|\beta_{j-1}| + t|\beta_j|\right) t^{n-j}$$

$$= |a_0|t^{n+1} + \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1})t^{n-j} + \sum_{j=k+1}^{n} (\alpha_{j-1} - t\alpha_j)t^{n-j}$$

$$+ \sum_{j=1}^{n} \left(|\beta_{j-1}| + t|\beta_j|\right) t^{n-j}$$

$$= |a_0|t^{n+1} + (t^2+1)t^{n-k-1}\alpha_k - t^{n-1}\alpha_0 - t\alpha_n + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j$$

$$+(1-t^2) \sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{n} \left(|\beta_{j-1}| + t|\beta_j|\right) t^{n-j} \equiv M_2.$$

Hence, by the Maximum Modulus Principle,

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \leqq M_2 \quad \text{for} \quad |z| \leqq t$$

which implies

$$\left|G_2(z)\right| \leqq M_2|z|^n \quad \text{for} \quad |z| \geqq \frac{1}{t}.$$

From this follows

$$\left|P(z)\right| = \left| -a_n z^{n+1} + G_2(z)\right|$$

$$\geqq |a_n||z|^{n+1} - M_2|z|^n = |z|^n\left(|a_n||z| - M_2\right) \quad \text{for} \quad |z| \geqq \frac{1}{t}.$$

So if $|z| > \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\} \equiv R_2$, then $P(z) \neq 0$ and in turn $p(z) \neq 0$, thus establishing the outer radius for the theorem.

For the inner bound, consider

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv ta_0 + G_1(z).$$

Then for $|z| = t$, by (3.1)

$$|G_1(z)| \leqq \sum_{j=1}^{n} |a_{j-1} - ta_j| t^j + |a_n| t^{n+1}$$

$$\leqq \sum_{j=1}^{n} |\alpha_{j-1} - t\alpha_j| t^j + \sum_{j=1}^{n} \left(|\beta_{j-1}| + t|\beta_j|\right) t^j + |a_n| t^{n+1}$$

$$= -t\alpha_0 + 2t^{k+1}\alpha_k - t^{n+1}\alpha_n + |a_n|t^{n+1} + |\beta_0|t + |\beta_n|t^{n+1} + 2\sum_{j=1}^{n-1} |\beta_j| t^{j+1} \equiv M_1.$$

Applying Schwarz's Lemma (see, for example, p. 168 of Titchmarsh [10]) to $G_1(z)$, we get

$$|G_1(z)| \leqq \frac{M_1|z|}{t} \quad \text{for} \quad |z| \leqq t.$$

So

$$|P(z)| = |ta_0 + G_1(z)| \geqq t|a_0| - |G_1(z)| \geqq t|a_0| - \frac{M_1|z|}{t}.$$

Notice that $\frac{t^2|a_0|}{M_1} \leqq t$. So if $|z| < \frac{t^2|a_0|}{M_1} \equiv R_1$ then $P(z) \neq 0$ and in turn $p(z) \neq 0$.

Acknowledgment. The authors are grateful to the referee for his suggestions.

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(Received June 8, 1995; revised January 15, 1996)

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