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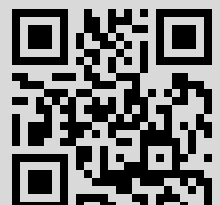
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**ON REGULARITY THEOREMS FOR LINEARLY INVARIANT FAMILIES OF HARMONIC FUNCTIONS**

**Abstract.** The classical theorem of growth regularity in the class  $S$  of analytic and univalent in the unit disc  $\Delta$  functions  $f$  describes the growth character of different functionals of  $f \in S$  and  $z \in \Delta$  as  $z$  tends to  $\partial\Delta$ . Earlier the authors proved the theorems of growth and decrease regularity for harmonic and sense-preserving in  $\Delta$  functions which generalized the classical result for the class  $S$ . In the presented paper we establish new properties of harmonic sense-preserving functions, connected with the regularity theorems. The effects both common for analytic and harmonic case and specific for harmonic functions are displayed.

**Key words:** *regularity theorem, linearly invariant family, harmonic function.*

**2010 Mathematical Subject Classification:** *30C55.*

**1. Introduction.** For a function  $u(z)$ , continuous in the disk  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ , we denote

$$M(r, u) = \max_{|z| \leq r} |u(z)| \quad \text{and} \quad m(r, u) = \min_{|z| \leq r} |u(z)|.$$

Let  $S$  be the class of all univalent analytic functions  $f(z) = z + \dots$  in  $\Delta$ . The theorem of growth regularity asserts that functions having the maximal growth in the given class, grows smoothly (regularly).

**Theorem A.** [1], [2], [3, pp. 104, 105], [4, pp. 8–9] *Let  $f \in S$ . Then there exist a  $\delta_0 \in [0, 1]$  with*

$$\lim_{r \rightarrow 1-} \left[ M(r, f) \frac{(1-r)^2}{r} \right] = \lim_{r \rightarrow 1-} \left[ M(r, f') \frac{(1-r)^3}{1+r} \right] = \delta^0,$$

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$\delta^0 = 1$  iff  $f(z) = z(1 - ze^{-i\theta})^{-2}$ . If  $\delta^0 \neq 1$ , then the functions under the sign of the limit increase on  $r$ .

If  $\delta^0 \neq 0$ , then there exists  $\varphi^0 \in [0; 2\pi)$  such that

$$\lim_{r \rightarrow 1-} \left[ |f(re^{i\varphi})| \frac{(1-r)^2}{r} \right] = \lim_{r \rightarrow 1-} \left[ |f'(re^{i\varphi})| \frac{(1-r)^3}{1+r} \right] = \begin{cases} \delta^0, & \varphi = \varphi^0 \\ 0, & \varphi \neq \varphi^0. \end{cases}$$

Here the functions under the sign of the limit are also increasing on  $r \in (0, 1)$ .

In [5], Ch. Pommerenke showed that many properties of functions from the class  $S$  can be extended to linearly invariant families (LIFs) of locally univalent analytic functions in  $\Delta$  of finite order. In [6] and [7], the theorem of growth regularity was obtained for such LIFs.

In [8], [9], the authors introduced the notion of LIF for complex-valued harmonic functions  $f$  in  $\Delta$ . Every such function can be presented, using analytic functions  $h$  and  $g$  in  $\Delta$  in the following way:

$$f(z) = h(z) + \overline{g(z)}, \quad (1)$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} \overline{a_{-n}(f)}z^n.$$

As in [5], L. E. Shaubroek considered locally univalent functions in  $\Delta$ . Moreover, these functions are sense-preserving in  $\Delta$ , i.e. the Jacobian  $J_f(z)$  satisfies

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \quad \forall z \in \Delta.$$

**Definition 1.** [8], [9] A set  $\mathfrak{M}_H$  of harmonic sense-preserving functions  $f$  in  $\Delta$  of form (1) is called the *linearly invariant family (LIF)* if for all  $f \in \mathfrak{M}_H$  and for any conformal automorphism  $\phi(z) = \frac{z+a}{1+\bar{a}z}$ ,  $a \in \Delta$ , the function  $e^{-i\theta} f_a(ze^{i\theta})$  belongs to  $\mathfrak{M}_H$ , where

$$f_a(z) = \frac{f(\phi(z)) - f(\phi(0))}{h'(\phi(0))\phi'(0)}. \quad (2)$$

It is assumed that the *order* of a family  $\mathfrak{M}_H$

$$\text{ord } \mathfrak{M}_H = \sup_{f \in \mathfrak{M}_H} |a_2(f)|$$

is finite.

In the analytic case (when  $g(z) \equiv 0$ ), the definitions of LIF and  $\text{ord } \mathfrak{M}_H$  coincide with the definitions of Pommerenke [5].

In [10], for LIFs of harmonic functions, the *strong order*

$$\overline{\text{ord}} \mathfrak{M}_H = \sup_{f \in \mathfrak{M}_H} \frac{|a_2(f) - a_{-1}(f)\overline{a_{-2}(f)}|}{1 - |a_{-1}(f)|^2}$$

was defined. The strong order proved to be convenient for investigation of LIFs, because it is not necessary to assume the affine invariance of a family. Moreover, for an affine LIF  $\mathfrak{M}_H$  the strong order does not exceed the old order:

$$\text{ord } \mathfrak{M}_H - \frac{1}{2} \leq \overline{\text{ord}} \mathfrak{M}_H \leq \text{ord } \mathfrak{M}_H.$$

This fact allows to describe properties of affine LIFs more precisely. For a LIF  $\mathfrak{M}$  of analytic functions,  $\text{ord } \mathfrak{M}_H = \overline{\text{ord}} \mathfrak{M}_H$ . Analogously to the analytic case in [10] the *universal* LIF  $\mathcal{U}_\alpha^H$  was introduced and studied. The family  $\mathcal{U}_\alpha^H$  is defined as the union of all LIFs  $\mathfrak{M}_H$  such that  $\overline{\text{ord}} \mathfrak{M}_H \leq \alpha$ . Equivalently,  $\mathcal{U}_\alpha^H$  is the set of all harmonic sense-preserving functions  $f$  in  $\Delta$  of the form (1) such that

$$\overline{\text{ord}} f \stackrel{\text{def}}{=} \overline{\text{ord}} \{e^{-i\theta} f_a(ze^{i\theta}) : a \in \Delta, \theta \in \mathbb{R}\} \leq \alpha.$$

It was shown in [10] that  $\overline{\text{ord}} \mathcal{U}_\alpha^H \geq 1$ .

In [11] and [12], the following regularity theorems for harmonic functions were proved:

**Theorem B. (regularity of growth)** *Let  $f \in \mathcal{U}_\alpha^H$ . Set*

$$\Phi_1(r) = \int_0^r M(\rho, J_f) d\rho, \quad \Psi_1(r, \varphi) = \int_0^r J_f(\rho e^{i\varphi}) d\rho, \quad \text{and}$$

$$F_1(r) = \int_0^r \frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}} d\rho.$$

For each  $n \geq 2$  successively denote

$$\Phi_n(r) = \int_0^r \Phi_{n-1}(\rho) d\rho, \quad \Psi_n(r, \varphi) = \int_0^r \Psi_{n-1}(\rho, \varphi) d\rho, \quad \text{and}$$

$$F_n(r) = \int_0^r F_{n-1}(\rho) d\rho.$$

Then

a) for every  $\varphi \in [0; 2\pi)$  and  $n \in \mathbb{N}$ , the functions

$$J_f(re^{i\varphi}) \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}}, \quad M(r, J_f) \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}},$$

$$\frac{\Phi_n(r)}{F_n(r)}, \quad \frac{\Psi_n(r, \varphi)}{F_n(r)}, \quad \text{and} \quad \frac{\max_{\varphi} \Psi_n(r, \varphi)}{F_n(r)}$$

are non-increasing on  $r \in (0; 1)$ ;

b) there exist constants  $\delta^0 \in [0; 1]$  and  $\varphi^0 \in [0; 2\pi)$  such that for  $1 \leq n \leq 2\alpha + 2$ ,

$$\begin{aligned} \delta^0 &= \lim_{r \rightarrow 1^-} \left[ \frac{M(r, J_f) (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = \lim_{r \rightarrow 1^-} \left[ \frac{J_f(re^{i\varphi^0}) (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1^-} \left[ \frac{M(r, \frac{\partial}{\partial r} J_f) (1-r)^{2\alpha+3}}{J_f(0) 4(\alpha+1) (1+r)^{2\alpha-3}} \right] = \\ &= \overline{\lim}_{r \rightarrow 1^-} \left[ \frac{\left| \frac{\partial}{\partial r} J_f(re^{i\varphi^0}) \right| (1-r)^{2\alpha+3}}{J_f(0) 4(\alpha+1) (1+r)^{2\alpha-3}} \right] = \\ &= \lim_{r \rightarrow 1^-} \left[ \frac{\int_0^r M(\rho, \frac{\partial}{\partial \rho} J_f) d\rho (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1^-} \left[ \frac{\int_0^r \left| \frac{\partial}{\partial \rho} J_f(\rho e^{i\varphi^0}) \right| d\rho (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1^-} \frac{\Phi_n(r)}{J_f(0) F_n(r)} = \lim_{r \rightarrow 1^-} \frac{\Psi_n(r, \varphi^0)}{J_f(0) F_n(r)} = \lim_{r \rightarrow 1^-} \frac{\max_{\varphi} \Psi_n(r, \varphi)}{J_f(0) F_n(r)}; \end{aligned}$$

c)  $\delta^0 = 1$  for functions  $q_{\theta}(z) = e^{i\theta} k_{\alpha}(ze^{-i\theta}) + \sigma e^{i\theta} \overline{k_{\alpha}(ze^{-i\theta})}$ , where  $\sigma \in \Delta$ ,  $\theta \in \mathbb{R}$ , and

$$k_{\alpha}(z) = \frac{1}{2\alpha} \left[ \left( \frac{1+z}{1-z} \right)^{\alpha} - 1 \right]. \quad (3)$$

**Theorem C. (regularity of decrease)** Let  $f \in \mathcal{U}_\alpha^H$ . Set

$$Q_1(r) = \int_r^1 m(\rho, J_f) d\rho, \quad E_1(r) = \int_r^1 \frac{(1-\rho)^{2\alpha-2}}{(1+\rho)^{2\alpha+2}} d\rho.$$

For each  $n \geq 2$  successively denote

$$Q_n(r) = \int_r^1 Q_{n-1}(\rho) d\rho, \quad \text{and} \quad E_n(r) = \int_r^1 E_{n-1}(\rho) d\rho.$$

Then

a) for every  $\varphi \in [0; 2\pi)$  and  $n \in \mathbb{N}$  the functions

$$J_f(re^{i\varphi}) \frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}, \quad m(r, J_f) \frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}}, \quad \text{and} \quad \frac{Q_n(r)}{E_n(r)}$$

are non-decreasing on  $r \in (0; 1)$ ;

b) there exist constants  $\delta_0 \in [1; \infty]$  and  $\varphi_0 \in [0; 2\pi)$  such that

$$\begin{aligned} \delta_0 &= \lim_{r \rightarrow 1-} \left[ \frac{m(r, J_f) (1+r)^{2\alpha+2}}{J_f(0) (1-r)^{2\alpha-2}} \right] = \lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\varphi_0}) (1+r)^{2\alpha+2}}{J_f(0) (1-r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1-} \frac{Q_n(r)}{J_f(0) E_n(r)}; \end{aligned}$$

c) for  $\varphi \in [0; 2\pi)$  denote

$$R_1(r, \varphi) = \int_r^1 J_f(\rho e^{i\varphi}) d\rho,$$

and for  $n \geq 2$ , set

$$R_n(r, \varphi) = \int_r^1 R_{n-1}(\rho, \varphi) d\rho$$

(under the assumptions of Theorem C the integrals converge). If  $\delta_0 < \infty$  then for  $n \geq 1$  the function  $\frac{R_n(r, \varphi_0)}{E_n(r)}$  is non-decreasing on  $r \in (0; 1)$ . Moreover,

$$\delta_0 = \lim_{r \rightarrow 1-} \frac{R_n(r, \varphi_0)}{J_f(0) E_n(r)};$$

d) if  $J_f(z)$  is bounded in  $\Delta$ , then for every  $n \in \mathbb{N}$  and every  $\varphi \in [0; 2\pi)$ , the functions

$$\frac{R_n(r, \varphi)}{E_n(r)} \quad \text{and} \quad \frac{\min_{\varphi} R_n(r, \varphi)}{E_n(r)}$$

are non-decreasing on  $r \in (0; 1)$  and

$$\delta_0 = \lim_{r \rightarrow 1^-} \frac{\min_{\varphi} R_n(r, \varphi)}{J_f(0)E_n(r)};$$

e)  $\delta_0 = 1$  for functions  $q_{\theta}(z) = e^{i\theta}k_{\alpha}(ze^{-i\theta}) + \sigma e^{i\theta}\overline{k_{\alpha}(ze^{-i\theta})}$ , where  $\sigma \in \Delta$ ,  $\theta \in \mathbb{R}$ , and  $k_{\alpha}(z)$  is the function defined by (3).

**Definition 2.** We say that the constant  $\varphi^0$  from Theorem B is a *direction of maximal growth (d.m.g.)* of a function  $f(z)$ . The constant  $\varphi_0$  from Theorem C is a *direction of maximal decrease (d.m.d.)* of  $f(z)$ .

**Definition 3.** The numbers  $\delta^0$  from Theorem B and  $\delta_0$  from Theorem C are called the *Hayman numbers* of a function  $f(z)$ .

In the presented paper we establish new properties of  $\mathcal{U}_{\alpha}^H$ , connected with the regularity theorems.

**2. Main results.** For fixed  $c \in [0; 1)$  introduce the class  $\mathcal{U}_{\alpha, c}^H$ , consisting of all functions  $f = h + \bar{g} \in \mathcal{U}_{\alpha}^H$  such that  $|g'(0)| \leq c$ . That is,  $J_f(0) \geq 1 - c^2 > 0$  for all  $f \in \mathcal{U}_{\alpha, c}^H$ . The class  $\mathcal{U}_{\alpha, c}^H$  is not a LIF. Note that the family  $\mathcal{U}_{\alpha}^H$  is not compact in the topology induced by locally uniform convergence in  $\Delta$ , but for  $\mathcal{U}_{\alpha, c}^H$  the following theorem takes place.

**Theorem 1.** The family  $\mathcal{U}_{\alpha, c}^H$  is compact in the topology induced by locally uniform convergence in  $\Delta$ .

**Proof.** Let  $f_n \in \mathcal{U}_{\alpha, c}^H$ ,  $f_n = h_n + \bar{g}_n$ ,  $n \in \mathbb{N}$ ,  $h_n$  and  $g_n$  be analytic functions in  $\Delta$ . By  $A_{\alpha}$  denote the set of all analytic functions  $h$  in  $\Delta$  such that there exists an analytic function  $g$  in  $\Delta$  and  $f = h + \bar{g} \in \mathcal{U}_{\alpha}^H$ . In other words,  $A_{\alpha}$  is the set of analytic parts of functions  $f \in \mathcal{U}_{\alpha}^H$ . The linearly invariance of  $\mathcal{U}_{\alpha}^H$  implies that  $A_{\alpha}$  is a LIF of analytic functions. But for LIFs of analytic functions  $\overline{\text{ord } A_{\alpha}} = \text{ord } A_{\alpha}$ . Therefore for all  $h \in A_{\alpha}$

$$|h'(z)| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}, \quad |z| = r,$$

see [5]. Since  $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$  for all  $z \in \Delta$  and all  $f \in \mathcal{U}_\alpha^H$ , we have

$$|g'(z)| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}},$$

for all  $f = h + \bar{g} \in \mathcal{U}_\alpha^H$  and  $z \in \Delta$ ,  $|z| = r$ . Consequently,  $\mathcal{U}_{\alpha,c}^H \subset \mathcal{U}_\alpha^H$  is uniformly bounded on compact subsets of  $\Delta$ . According to the compactness principle, there exists a subsequence of  $f_n$  (let us save the notation) which converges locally uniformly in  $\Delta$  to a harmonic function  $f_0$ . Let us show that  $f_0 \in \mathcal{U}_{\alpha,c}^H$ .

For  $f \in \mathcal{U}_\alpha^H$  the following inequality holds (see [10])

$$\frac{(1-r)^{2\alpha-2}}{(1+r)^{2\alpha+2}} \leq \frac{J_f(z)}{J_f(0)} \leq \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}}, \quad |z| = r.$$

Therefore for  $f_n \in \mathcal{U}_{\alpha,c}^H$  we have

$$J_{f_n}(z) \geq \frac{(1-r)^{2\alpha-2}}{(1+r)^{2\alpha+2}}(1-c^2) > 0.$$

This implies  $J_{f_0}(z) > 0$  for all  $z \in \Delta$ . This means that the harmonic in  $\Delta$  function  $f_0$  is sense-preserving.

Next, we prove that  $\text{ord } f_0 \leq \alpha$ . Suppose not. Then, we may let  $\text{ord } f_0 = \beta > \alpha$ . Then, by the definition of the strong order, there exist a conformal automorphism  $\varphi(z) = \frac{z+a}{1+\bar{a}z}$  of  $\Delta$  and  $\theta \in \mathbb{R}$  such that for harmonic function

$$e^{-i\theta}(f_0)_a(ze^{i\theta}) = \frac{f_0(\varphi(ze^{i\theta})) - f_0(\varphi(0))}{h'_0(\varphi(0))\varphi'(0)e^{i\theta}} = \sum_{k=1}^{\infty} (A_k z^k + A_{-k} \bar{z}^k),$$

( $A_1 = 1$ ,  $f_0 = h_0 + \bar{g}_0$ ) the inequality

$$\frac{|A_2 - A_{-1}\overline{A_{-2}}|}{1 - |A_{-1}|^2} > \alpha + \frac{\beta - \alpha}{2} \quad (4)$$

is valid.

For the automorphism  $\varphi$  and the number  $\theta$  denote

$$e^{-i\theta}(f_n)_a(ze^{i\theta}) = \sum_{k=1}^{\infty} (A_k^{(n)} z^k + A_{-k}^{(n)} \bar{z}^k), \quad (A_1^{(n)} = 1).$$



From locally uniform convergence of  $f_n$  to  $f_0$ , the Weierstrass theorem on series of analytic functions, and inequality (4) it follows that for sufficiently large  $n > N$

$$\frac{|A_2^{(n)} - A_{-1}^{(n)}\overline{A_{-2}^{(n)}}|}{1 - |A_{-1}^{(n)}|^2} > \alpha + \frac{\beta - \alpha}{2}.$$

Hence if  $n > N$  we have  $\overline{\text{ord}} f_n > \alpha + \frac{\beta - \alpha}{2}$  and  $f_n \notin \mathcal{U}_{\alpha,c}^H$ . This contradiction proves the theorem.  $\square$

In claim c) of Theorem B and claim e) of Theorem C some set of functions with the Hayman number  $\delta^0 = 1$  (or  $\delta_0 = 1$  for the theorem of decrease regularity) is described. These claims differ from the analytic case. In the analytic case  $\delta^0 = 1$  and  $\delta_0 = 1$  only for the functions  $e^{i\theta} k_\alpha(z e^{-i\theta})$ , where  $\theta \in \mathbb{R}$ ,  $k_\alpha(z)$  is the function defined by (3) [7], [13], [14]. The following example shows that in the harmonic case this set has more complicated structure. We construct the example of functions  $f$  of arbitrary strong order  $\beta \geq 3/2$  with  $\delta^0 = 1$ . These functions are not equal to the function  $q_\theta(z)$  from Theorem B. We use the Clunie and Sheil-Small shear construction [15] (see also [16, ch. 3.4]) to give our example. Let us note that our construction is not stable. As one can show, if we multiply the coanalytic part  $g$  of the function from our example by constant  $k \in (0, 1)$ , then the strong order of the function changes stepwise and  $\delta^0 \neq 1$  for this function.

**Example.** Put  $h'(z) = \frac{(1+z)^{\alpha-1}}{(1-z)^{\alpha+2}}$ ,  $g'(z) = zh'(z)$ ,  $z \in \Delta$ . Let  $\alpha \in [1, \infty)$  be fixed. If  $\varphi(z) = \frac{z+a}{1+\bar{a}z}$ ,  $a \in \Delta$ , is an automorphism of  $\Delta$ , then for  $f = h + \bar{g}$  we have

$$f_a(z) =: F(z) = H(z) + \overline{G(z)} = \frac{h(\varphi(z)) - h(\varphi(0))}{h'(\varphi(0))\varphi'(0)} + \overline{\left( \frac{g(\varphi(z)) - g(\varphi(0))}{h'(\varphi(0))\varphi'(0)} \right)},$$

where  $H$  and  $G$  are functions analytic in  $\Delta$ ,

$$H'(z) = \frac{h'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)} \quad \text{and}$$

$$G'(z) = \frac{g'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)} = \frac{\varphi(z)h'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)}.$$

Note that

$$J_F(z) = |H'(z)|^2 - |G'(z)|^2 = \frac{|h'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |\varphi(z)|^2)}{|h'(\varphi(0))|^2 |\varphi'(0)|^2},$$

and, in particular,

$$J_F(0) = 1 - |\varphi(0)|^2.$$

Therefore,

$$\begin{aligned} \frac{J_F(z)}{J_F(0)} &= \frac{\left|1 + \frac{z+a}{1+\bar{a}z}\right|^{2\alpha-2}}{\left|1+a\right|^{2\alpha-2}} \cdot \left|\frac{1-a}{1-\frac{z+a}{1+\bar{a}z}}\right|^{2\alpha+4} \cdot \frac{(1-|a|^2)^2}{|1+\bar{a}z|^4} \times \\ &\quad \times \left(1 - \left|\frac{z+a}{1+\bar{a}z}\right|^2\right) \frac{1}{(1-|a|^2)^3} = \\ &= \frac{\left|1 + z\frac{1+\bar{a}}{1+a}\right|^{2\alpha-2}}{\left|1 - z\frac{1-\bar{a}}{1-a}\right|^{2\alpha+4}} \cdot \frac{|1+\bar{a}z|^2 - |z+a|^2}{1-|a|^2} = \frac{\left|1 + z\frac{1+\bar{a}}{1+a}\right|^{2\alpha-2}}{\left|1 - z\frac{1-\bar{a}}{1-a}\right|^{2\alpha+4}} (1 - |z|^2), \end{aligned}$$

by generalized Schwarz's lemma. Consequently, for  $r \in (0, 1)$

$$\sup_{\substack{a \in \Delta, \\ |z|=r}} \frac{J_F(z)}{J_F(0)} = \frac{(1+r)^{2\alpha-1}}{(1-r)^{2\alpha+3}}.$$

Therefore for  $\beta = \alpha + \frac{1}{2}$ , all  $a \in \Delta$ , and  $|z| = r$  we get

$$\frac{J_F(z)}{J_F(0)} \leq \frac{(1+r)^{2\beta-2}}{(1-r)^{2\beta+2}}. \quad (5)$$

In [10] it was shown that for functions  $f$  harmonic and sense-preserving in  $\Delta$ ,

$$\overline{\text{ord}} f = \inf \left\{ \beta : \frac{J_F(z)}{J_F(0)} \leq \frac{(1+|z|)^{2\beta-2}}{(1-|z|)^{2\beta+2}}, \quad \forall F = f_a, \forall z \in \Delta \right\}. \quad (6)$$

From (5) and (6) we conclude that  $\overline{\text{ord}} f \leq \beta = \alpha + \frac{1}{2}$ . From Theorem B it follows that if for a function  $f$  harmonic and sense-preserving in  $\Delta$

$$\lim_{r \rightarrow 1-} \left[ \frac{J_f(z)}{J_f(0)} \frac{(1-r)^{2\beta+2}}{(1+r)^{2\beta-2}} \right] > 0, \quad (7)$$

then  $\overline{\text{ord } f} \geq \beta$ . For the considered function  $f$  the limit in (7) equals 1. Therefore,  $\overline{\text{ord } f} = \beta$  and

$$\delta^0 = \lim_{r \rightarrow 1^-} \left[ \frac{J_f(r) (1-r)^{2\beta+2}}{J_f(0) (1+r)^{2\beta-2}} \right] = 1.$$

It is interesting to find out if there exist functions with  $\delta^0 = 1$  which are not equal to the function from the example and the functions  $q_\theta(z)$ .

**Definition 4.** A *direction of intensive growth (d.i.g.)* of a function  $f(z)$  is a constant  $\varphi \in [0; 2\pi)$  such that

$$\lim_{r \rightarrow 1^-} \left[ \frac{J_f(re^{i\varphi}) (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = \delta(f, \varphi) > 0.$$

A *direction of intensive decrease (d.i.d.)* of a function  $f(z)$  is a constant  $\varphi \in [0; 2\pi)$  such that

$$\lim_{r \rightarrow 1^-} \left[ \frac{J_f(re^{i\varphi}) (1+r)^{2\alpha+2}}{J_f(0) (1-r)^{2\alpha-2}} \right] = \delta'(f, \varphi) < \infty.$$

Since we study LIFs, it is important to know how d.i.g.-'s and d.i.d.-'s of a function  $f(z)$  are changed under the transformation  $e^{-i\theta} f_a(ze^{i\theta})$ . The case  $a = 0$  is trivial: a d.i.g. (d.i.d.)  $\varphi - \theta$  of the function  $e^{-i\theta} f(ze^{i\theta})$  corresponds to the d.i.g. (d.i.d.)  $\varphi$  of  $f(z)$ . In this situation  $\delta(f(z), \varphi) = \delta(f(ze^{i\theta}), \varphi - \theta)$  (and  $\delta'(f(z), \varphi) = \delta'(f(ze^{i\theta}), \varphi - \theta)$ ). It is also interesting to find out the relationship between the Hayman numbers of the functions  $f$  and  $f_a$  in general case. The following theorem concerns the non-obvious case  $a \neq 0$ .

**Theorem 2.** Let  $f \in \mathcal{U}_\alpha^H$ . Denote

$$R(r) = \left| \frac{re^{i\varphi} + a}{1 + \bar{a}re^{i\varphi}} \right|, \quad \gamma(r) = \arg \frac{re^{i\varphi} + a}{1 + \bar{a}re^{i\varphi}}, \quad a \in \Delta, \quad re^{i\varphi} \neq -a.$$

1)  $\varphi$  is a d.i.g. (d.i.d.) of the function  $f_a(z)$  iff  $\gamma$  is a d.i.g. (d.i.d.) of  $f(z)$  and

$$e^{i\varphi} = \frac{e^{i\gamma} - a}{1 - \bar{a}e^{i\gamma}}; \quad (8)$$

2) for all  $\gamma \in [0, 2\pi)$

$$\lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\gamma}) (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = \lim_{r \rightarrow 1-} \left[ \frac{J_f(R(r)e^{i\gamma(r)}) (1-R(r))^{2\alpha+2}}{J_f(0) (1+R(r))^{2\alpha-2}} \right],$$

and

$$\lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\gamma}) (1+r)^{2\alpha+2}}{J_f(0) (1-r)^{2\alpha-2}} \right] = \lim_{r \rightarrow 1-} \left[ \frac{J_f(R(r)e^{i\gamma(r)}) (1+R(r))^{2\alpha+2}}{J_f(0) (1-R(r))^{2\alpha-2}} \right].$$

Here  $\varphi$  and  $\gamma$  are connected by (8).

3) if  $\varphi$  is a d.i.g. of  $f_a(z)$ ,  $\gamma$  is a d.i.g. of  $f(z)$ , and  $\varphi$  is connected with  $\gamma$  by (8), then

$$\delta(f, \gamma) = \delta(f_a, \varphi) \frac{J_f(a) (1-|a|^2)^{2\alpha+2}}{J_f(0) |1 + \bar{a}e^{i\varphi}|^{4\alpha}};$$

if  $\varphi$  is a d.i.d. of  $f_a(z)$ ,  $\gamma$  is a d.i.d. of  $f(z)$ , and  $\varphi$  is connected with  $\gamma$  by (8), then

$$\delta'(f, \gamma) = \delta'(f_a, \varphi) \frac{J_f(a) |1 + \bar{a}e^{i\varphi}|^{4\alpha}}{J_f(0) (1-|a|^2)^{2\alpha-2}}.$$

**Proof.** 1) Let  $\varphi$  be a d.i.g. of  $f_a(z)$ . This means that there exists the limit

$$\delta(f_a, \varphi) = \lim_{r \rightarrow 1-} \left[ \frac{J_{f_a}(re^{i\varphi}) (1-r)^{2\alpha+2}}{J_{f_a}(0) (1+r)^{2\alpha-2}} \right] > 0.$$

Note that

$$J_{f_a}(z) = \frac{J_f\left(\frac{z+a}{1+\bar{a}z}\right)}{|h'(a)|^2 |1 + \bar{a}z|^4}, \quad (9)$$

and

$$J_{f_a}(0) = \frac{J_f(a)}{|h'(a)|^2}. \quad (10)$$

Let us calculate the following limit, using (9) and (10),

$$\begin{aligned} \delta &\stackrel{\text{def}}{=} \lim_{r \rightarrow 1-} \left[ \frac{J_f(R(r)e^{i\gamma(r)}) (1-R(r))^{2\alpha+2}}{J_f(0) (1+R(r))^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1-} \left[ \frac{J_{f_a}(re^{i\varphi})}{J_f(0)} |h'(a)|^2 |1 + \bar{a}re^{i\varphi}|^4 \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \left( \frac{1-R(r)}{1-r} \right)^{2\alpha+2} \right]. \end{aligned}$$

We have

$$\lim_{r \rightarrow 1^-} \frac{1 - R(r)}{1 - r} = \lim_{r \rightarrow 1^-} R'(r) = \frac{1 - |a|^2}{|1 + \bar{a}e^{i\varphi}|^2}. \quad (11)$$

Using (11), we obtain

$$\delta = \delta(f_a, \varphi) \frac{J_f(a)}{J_f(0)} |1 + \bar{a}e^{i\varphi}|^4 \left( \frac{1 - |a|^2}{|1 + \bar{a}e^{i\varphi}|^2} \right)^{2\alpha+2} > 0. \quad (12)$$

By (11),  $\lim_{r \rightarrow 1^-} R'(r) > 0$ , therefore the function  $R(r)$  increases on an interval  $(r_0, 1)$ . By Theorem B, for  $r_0 < r < r_1 < 1$

$$\frac{J_f(R(r_1))e^{i\gamma(r_1)}}{J_f(0)} \frac{(1 - R(r_1))^{2\alpha+2}}{(1 + R(r_1))^{2\alpha-2}} \leq \frac{J_f(R(r))e^{i\gamma(r_1)}}{J_f(0)} \frac{(1 - R(r))^{2\alpha+2}}{(1 + R(r))^{2\alpha-2}}.$$

Passing to the limit as  $r_1 \rightarrow 1^-$  and using (8), we get

$$\delta \leq \frac{J_f(R(r))e^{i\gamma}}{J_f(0)} \frac{(1 - R(r))^{2\alpha+2}}{(1 + R(r))^{2\alpha-2}}.$$

Thus,

$$\delta(f, \gamma) = \lim_{r \rightarrow 1^-} \left[ \frac{J_f(R(r))e^{i\gamma}}{J_f(0)} \frac{(1 - R(r))^{2\alpha+2}}{(1 + R(r))^{2\alpha-2}} \right] \geq \delta. \quad (13)$$

Taking into account (12), we conclude that  $\gamma$  is a d.i.g. of  $f(z)$ .

Now let us consider the sets

$$A = \{e^{i\gamma} : \gamma \text{ is a d.i.g. of } f(z)\},$$

$$B = \left\{ \frac{e^{i\varphi} + a}{1 + \bar{a}e^{i\varphi}} : \varphi \text{ is a d.i.g. of } f_a(z) \right\},$$

$$C = \{e^{i\eta} : \eta \text{ is a d.i.g. of } [f_a]_{(-a)}(z)\}.$$

Here  $[f_a]_{(-a)}(z)$  is the transformation (2) of the function  $f_a$  with the parameter  $-a$ . If  $\eta$  is a d.i.g. of  $[f_a]_{(-a)}(z)$ , then, as it was proved above,

$$e^{i\eta} = \frac{e^{i\varphi} + a}{1 + \bar{a}e^{i\varphi}},$$

where  $\varphi$  is a d.i.g. of  $f_a(z)$ . This implies that  $C \subset B$ . Let  $\varphi$  be a d.i.g. of  $f_a(z)$ . Then

$$e^{i\gamma} = \frac{e^{i\varphi} + a}{1 + \bar{a}e^{i\varphi}},$$

where  $\gamma$  is a d.i.g. of  $f(z)$ . Thus  $B \subset A$ . Since  $[f_a]_{(-a)}(z) = f(z)$ , we have  $A = C$  and, consequently,  $A = B$ . This completes the proof of the statement about d.i.g.-'s.

The statement about d.i.d.-'s is proved analogously.

2) Let us prove the first equality. If  $\gamma$  is not a d.i.g. of  $f(z)$ , then

$$\lim_{r \rightarrow 1^-} \left[ \frac{J_f(re^{i\gamma}) (1-r)^{2\alpha+2}}{J_f(0) (1+r)^{2\alpha-2}} \right] = 0.$$

Thus, by (13),

$$\delta \leq \lim_{r \rightarrow 1^-} \left[ \frac{J_f(R(r)e^{i\gamma}) (1-R(r))^{2\alpha+2}}{J_f(0) (1+R(r))^{2\alpha-2}} \right] = 0.$$

This implies  $\delta = 0$ .

Now let us consider the case when  $\gamma$  is a d.i.g. of  $f(z)$ . We have proved above that  $\delta(f, \gamma) \geq \delta$  (see (13)). It remains to show that  $\delta(f, \gamma) \leq \delta$ .

Denote

$$R_1(r) = \left| \frac{re^{i\gamma} - a}{1 - \bar{a}re^{i\gamma}} \right|.$$

Since  $[f_a]_{(-a)}(z) = f(z)$ ,  $\gamma$  is a d.i.g. of  $[f_a]_{(-a)}(z)$ , i. e.

$$\delta([f_a]_{(-a)}, \gamma) = \delta(f, \gamma) = \lim_{r \rightarrow 1^-} \left[ \frac{J_{[f_a]_{(-a)}}(re^{i\gamma}) (1-r)^{2\alpha+2}}{J_{[f_a]_{(-a)}}(0) (1+r)^{2\alpha-2}} \right] > 0.$$

Arguing as in the proof of claim 1), one can note that there exists

$$\delta^* \stackrel{\text{def}}{=} \lim_{r \rightarrow 1^-} \left[ \frac{J_{f_a} \left( \frac{re^{i\gamma} - a}{1 - \bar{a}re^{i\gamma}} \right) (1 - R_1(r))^{2\alpha+2}}{J_{f_a}(0) (1 + R_1(r))^{2\alpha-2}} \right].$$

Apply (13) to the function  $f_a(z)$ , using (9), (10), and (11):

$$\begin{aligned} \delta^* &\leq \lim_{r \rightarrow 1^-} \left[ \frac{J_{f_a}(re^{i\varphi}) (1-r)^{2\alpha+2}}{J_{f_a}(0) (1+r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1^-} \left[ \frac{J_f \left( \frac{re^{i\varphi} + a}{1 + \bar{a}re^{i\varphi}} \right) (1 - R(r))^{2\alpha+2}}{J_f(a) |1 + \bar{a}re^{i\varphi}|^4 (1 + R(r))^{2\alpha-2}} \right] \cdot \lim_{r \rightarrow 1^-} \left( \frac{1-r}{1-R(r)} \right)^{2\alpha+2} = \end{aligned}$$

$$= \frac{\delta J_f(0)}{J_f(a)|1 + \bar{a}e^{i\varphi}|^4} \left( \frac{|1 + \bar{a}e^{i\varphi}|^2}{1 - |a|^2} \right)^{2\alpha+2} = \frac{\delta J_f(0)}{J_f(a)} \frac{|1 + \bar{a}e^{i\varphi}|^{4\alpha}}{(1 - |a|^2)^{2\alpha+2}}. \quad (14)$$

On the other hand, by (9),

$$J_{f_a} \left( \frac{z - a}{1 - \bar{a}z} \right) = \frac{J_f(z)}{|h'(a)|^2 \left| 1 + \bar{a} \frac{z - a}{1 - \bar{a}z} \right|^4}.$$

Thus, using (8), (10), and (11), we can write  $\delta^*$  in the form

$$\begin{aligned} \delta^* &= \lim_{r \rightarrow 1^-} \left[ \frac{J_f(re^{i\gamma})}{J_f(a) \left| 1 + \bar{a} \frac{re^{i\gamma} - a}{1 - \bar{a}re^{i\gamma}} \right|^4} \frac{(1 - r)^{2\alpha+2}}{(1 + r)^{2\alpha-2}} \right] \times \\ &\quad \times \lim_{r \rightarrow 1^-} \left( \frac{1 - R_1(r)}{1 - r} \right)^{2\alpha+2} = \\ &= \delta(f, \gamma) \frac{J_f(0)}{J_f(a)|1 + \bar{a}e^{i\varphi}|^4} \left( \frac{1 - |a|^2}{|1 - \bar{a}e^{i\gamma}|^2} \right)^{2\alpha+2} = \\ &= \delta(f, \gamma) \frac{J_f(0)}{J_f(a)} \frac{|1 + \bar{a}e^{i\varphi}|^{4\alpha}}{(1 - |a|^2)^{2\alpha+2}}. \end{aligned}$$

Substituting

$$\delta^* = \delta(f, \gamma) \frac{J_f(0)}{J_f(a)} \frac{|1 + \bar{a}e^{i\varphi}|^{4\alpha}}{(1 - |a|^2)^{2\alpha+2}}$$

in (14), we get  $\delta(f, \gamma) \leq \delta$ . Therefore,  $\delta(f, \gamma) = \delta$ .

The second equality of claim 2) is proved analogously.

3) The formula, connected  $\delta(f, \gamma)$  and  $\delta(f_a, \varphi)$  is obtained from (12), using  $\delta = \delta(f, \gamma)$ .

The second equality is proved analogously.  $\square$

Theorem 2 implies the following

**Remark.** Let  $f \in \mathcal{U}_\alpha^H$ . For every  $\varphi \in [0; 2\pi)$  there exist  $\delta(f, \varphi) \in [0; 1]$  and  $\delta'(f, \varphi) \in [1; \infty]$  such that for any circle or straight line  $\Gamma \subset \Delta$ , orthogonal to  $\partial\Delta$  at the point  $e^{i\varphi}$ , we have

$$\lim_{\Gamma \ni z \rightarrow e^{i\varphi}} \left[ \frac{J_f(z)}{J_f(0)} \frac{(1 - |z|)^{2\alpha+2}}{(1 + |z|)^{2\alpha-2}} \right] = \delta(f, \varphi),$$

$$\lim_{\Gamma \ni z \rightarrow e^{i\varphi}} \left[ \frac{J_f(z) (1 + |z|)^{2\alpha+2}}{J_f(0) (1 - |z|)^{2\alpha-2}} \right] = \delta'(f, \varphi),$$

and the constants  $\delta(f, \varphi)$ ,  $\delta'(f, \varphi)$  do not depend on  $\Gamma$ .

By  $\mathcal{U}_\alpha^H(\delta^0)$  denote the set of all functions from  $\mathcal{U}_\alpha^H$  with the same Hayman number  $\delta^0$  from Theorem B.

Let  $\mathcal{U}_\alpha^H(\delta_0)$  be the set of all functions, having the Hayman number  $\delta_0$  from Theorem C.

**Theorem 3.** 1) If  $f \in \mathcal{U}_\alpha^H(\delta^0)$ ,  $\delta^0 \in (0; 1)$ , then for every  $\delta \in [\delta^0, 1)$  there exists  $a \in \Delta$  such that  $f_a(z) \in \mathcal{U}_\alpha^H(\delta)$ .

2) If  $f \in \mathcal{U}_\alpha^H(\delta_0)$ ,  $\delta_0 \in (1; \infty)$ , then for every  $\delta' \in (1, \delta_0]$  there exists  $a \in \Delta$  such that  $f_a(z) \in \mathcal{U}_\alpha^H(\delta')$ .

**Proof.** By Theorem B, for any  $\varphi \in [0; 2\pi)$  there exists

$$\lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\varphi}) (1 - r)^{2\alpha+2}}{J_f(0) (1 + r)^{2\alpha-2}} \right] = \delta(f, \varphi).$$

Let us fix  $a \in \Delta$   $\varphi \in [0; 2\pi)$ . Denote  $z = \frac{re^{i\varphi} - a}{1 - \bar{a}re^{i\varphi}}$ ,  $|z| = R(r)$  and consider the limit

$$\delta^*(\varphi) \stackrel{def}{=} \lim_{r \rightarrow 1-} \left[ \frac{J_{f_a}(z) (1 - R(r))^{2\alpha+2}}{J_{f_a}(0) (1 + R(r))^{2\alpha-2}} \right].$$

Let us calculate  $\delta^*(\varphi)$ , using (9) and (10)

$$\begin{aligned} \delta^*(\varphi) &= \lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\varphi}) (1 - R(r))^{2\alpha+2}}{J_f(a) \left| 1 + \bar{a} \frac{re^{i\varphi} - a}{1 - \bar{a}re^{i\varphi}} \right|^4 (1 + R(r))^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\varphi}) (1 - r)^{2\alpha+2}}{J_f(0) (1 + r)^{2\alpha-2}} \frac{J_f(0)}{J_f(a)} \left( \frac{1 - R(r)}{1 - r} \right)^{2\alpha+2} \right] \cdot \frac{1}{\left| 1 + \bar{a} \frac{re^{i\varphi} - a}{1 - \bar{a}re^{i\varphi}} \right|^4}. \end{aligned}$$

By (11),

$$\begin{aligned} \delta^*(\varphi) &= \delta(f, \varphi) \frac{J_f(0) (1 - |a|^2)^{2\alpha+2} |1 - \bar{a}e^{i\varphi}|^4}{J_f(a) |1 - \bar{a}e^{i\varphi}|^{4\alpha+4} (1 - |a|^2)^4} = \\ &= \delta(f, \varphi) \frac{J_f(0) (1 - |a|^2)^{2\alpha-2}}{J_f(a) |1 - \bar{a}e^{i\varphi}|^{4\alpha}} \leq \end{aligned}$$



$$\leq \lim_{R(r) \rightarrow 1^-} \left[ \frac{M(R(r), J_{f_a}) (1 - R(r))^{2\alpha+2}}{J_{f_a}(0) (1 + R(r))^{2\alpha-2}} \right] \stackrel{def}{=} \delta_a.$$

Let  $\varphi$  be equal to d.m.g.  $\varphi^0$  of  $f(z)$  and  $a = \rho e^{i\varphi^0}$ . Then  $\delta(f, \varphi) = \delta^0$  and

$$\delta^0 \frac{J_f(0)}{J_f(\rho e^{i\varphi^0})} \frac{(1 - \rho^2)^{2\alpha-2}}{(1 - \rho)^{4\alpha}} = \delta^0 \frac{J_f(0)}{J_f(\rho e^{i\varphi^0})} \frac{(1 + \rho)^{2\alpha-2}}{(1 - \rho)^{2\alpha-2}} \leq \delta_a. \quad (15)$$

By Theorem B, there exists a d.m.g.  $\varphi_1 \in [0; 2\pi)$  of  $f_a(z)$  such that

$$\begin{aligned} \delta_a &= \lim_{r \rightarrow 1^-} \left[ \frac{J_{f_a}(r e^{i\varphi_1}) (1 - r)^{2\alpha+2}}{J_{f_a}(0) (1 + r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1^-} \left[ \frac{J_f \left( \frac{r e^{i\varphi_1} + a}{1 + \bar{a} r e^{i\varphi_1}} \right) (1 - r)^{2\alpha+2}}{J_f(a) |1 + \bar{a} r e^{i\varphi_1}|^4 (1 + r)^{2\alpha-2}} \right]. \end{aligned}$$

Denote  $R_1(r) e^{i\gamma_1(r)} = \frac{r e^{i\varphi_1} + a}{1 + \bar{a} r e^{i\varphi_1}}$ , where  $\gamma_1(r)$  is a real-valued function. Then, using (11) for  $R(r) = R_1(r)$ , we obtain

$$\begin{aligned} \delta_a &\leq \lim_{r \rightarrow 1^-} \left[ \frac{M(R_1(r), J_f) (1 - r)^{2\alpha+2}}{J_f(a) |1 + \bar{a} r e^{i\varphi_1}|^4 (1 + r)^{2\alpha-2}} \right] = \\ &= \lim_{r \rightarrow 1^-} \left[ \frac{M(R_1(r), J_f) (1 - R_1(r))^{2\alpha+2}}{J_f(0) (1 + R_1(r))^{2\alpha-2}} \right] \times \\ &\times \frac{J_f(0)}{J_f(a) |1 + \bar{a} e^{i\varphi_1}|^4} \cdot \lim_{r \rightarrow 1^-} \left( \frac{1 - r}{1 - R_1(r)} \right)^{2\alpha+2} = \\ &= \delta^0 \frac{J_f(0)}{J_f(a) |1 + \bar{a} e^{i\varphi_1}|^4} \left( \frac{|1 + \bar{a} e^{i\varphi_1}|^2}{1 - |a|^2} \right)^{2\alpha+2} = \delta^0 \frac{J_f(0)}{J_f(a)} \frac{|1 + \bar{a} e^{i\varphi_1}|^{4\alpha}}{(1 - |a|^2)^{2\alpha+2}} \leq \\ &\leq \delta^0 \frac{J_f(0)}{J_f(a)} \frac{(1 + \rho)^{4\alpha}}{(1 - \rho^2)^{2\alpha+2}} = \delta^0 \frac{J_f(0)}{J_f(a)} \frac{(1 + \rho)^{2\alpha-2}}{(1 - \rho)^{2\alpha+2}}. \end{aligned}$$

Taking into account inequality (15), we get

$$\delta^0 \frac{J_f(0)}{J_f(\rho e^{i\varphi^0})} \frac{(1 + \rho)^{2\alpha-2}}{(1 - \rho)^{2\alpha+2}} = \delta_a.$$

Since the continuous function  $\frac{J_f(0)}{J_f(\rho e^{i\varphi^0})} \frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}}$  decreases on  $\rho$ , equals 1 as  $\rho = 0$ , and tends to zero as  $\rho \rightarrow 1-$ , then we can find  $\rho \in [0; 1)$  such that  $\delta_a$  takes preassigned value from  $[\delta^0; 1)$ .

Claim 2 of the theorem is proved analogously.  $\square$

In [7] (see also [17], [14]) it was proved that the set of all d.i.g.-'s and d.i.d.-'s of a given analytic function is at most countable. The following theorem shows that this statement is true for set of d.i.g.-'s of harmonic function too. But we don't know whether this fact is true for set of d.i.d.-'s.

**Theorem 4.** *Let  $f \in \mathcal{U}_\alpha^H$ . Then the set of all d.i.g.-'s of  $f$  is at most countable.*

**Proof.** If  $f = h + \bar{g} \in \mathcal{U}_\alpha^H$ , then  $\overline{\text{ord}} h \leq \alpha$ . Since

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 \leq |h'(z)|^2$$

for all  $z \in \Delta$ , then for  $\varphi \in [0, 2\pi)$  and  $r \in [0, 1)$

$$\frac{J_f(re^{i\varphi})}{J_f(0)} \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \leq \left[ |h'(re^{i\varphi})| \frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}} \right]^2 \frac{1}{J_f(0)}. \quad (16)$$

By Theorem B and theorem of growth regularity from [7], there exist the limits

$$\delta(f, \varphi) = \lim_{r \rightarrow 1-} \left[ \frac{J_f(re^{i\varphi})}{J_f(0)} \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \right],$$

and

$$\tilde{\delta}(h, \varphi) = \lim_{r \rightarrow 1-} \left[ |h'(re^{i\varphi})| \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \right].$$

From (16) we get  $\delta(f, \varphi) \leq \frac{\tilde{\delta}^2(h, \varphi)}{J_f(0)}$ . If  $\varphi$  is a d.i.g. of  $f$ , then  $\delta(f, \varphi) > 0$ .

Consequently,  $\tilde{\delta}(h, \varphi) > 0$  and  $\varphi$  is a d.i.g. of  $h$ . Therefore the set  $V$  of all d.i.g.-'s of  $f$  is contained in the set  $W$  of all d.i.g.-'s of  $h$ . As it was proved in [7],  $W$  is at most countable. Hence  $V$  is at most countable too.  $\square$

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