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UNIVALENT HARMONIC MAPPINGS INTO TWO-SLIT DOMAINS

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Abstract

We study some classes of planar harmonic mappings produced with the *shear construction* devised by Clunie and Sheil-Small in 1984. The first section reviews the basic concepts and describes the shear construction. The main body of the paper deals with the geometry of the classes constructed.

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1. Introduction

A complex-valued function f on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ that is twice continuously differentiable and satisfies Laplace's equation $f_{z\overline{z}} = 0$ will be called harmonic. By a theorem of Lewy [3], the Jacobian $J_f = |f_z|^2 - |f_{\overline{z}}|^2$ of a locally univalent harmonic mapping never vanishes, so we may assume that $J_f > 0$ (that is, f is orientation-preserving), and consequently $|f_z| > 0$ everywhere in $\mathbb{D}$. It is easily verified that $f = h + \overline{g}$, where h and g are analytic on $\mathbb{D}$. Since $f_z = h'$ and $f_{\overline{z}} = \overline{g'}$, we see that $\omega = \overline{f_{\overline{z}}}/f_z = g'/h'$ is analytic and that $|\omega(z)| < 1$ on $\mathbb{D}$. By analogy with the complex dilation $\mu = f_{\overline{z}}/f_z$ the function ω will be called the analytic (or second complex) dilation of f.

Clunie and Sheil-Small introduced an effective tool for constructing univalent harmonic mappings with prescribed dilation. For completeness, we quote their theorem.

THEOREM 1.1 [1]. *Suppose that $f = h + \overline{g}$ is harmonic and locally univalent on the unit disk $\mathbb{D}$. Then f is univalent and its range is convex in the horizontal direction if and only if the analytic function $\varphi = h - g$ is a univalent mapping of $\mathbb{D}$ onto a domain that is convex in the horizontal direction.*

Henceforth, a domain $\Omega \subseteq \mathbb{C}$ is said to be *convex in the horizontal direction* if its intersection with each horizontal line is connected (or empty).

According to the theorem above, one begins with a conformal mapping φ of $\mathbb{D}$ onto a domain that is convex in the horizontal direction, such that $\varphi(0) = 0$, and an analytic function ω such that $|\omega(z)| < 1$ on $\mathbb{D}$ and $\omega(0) = 0$. The relations $\varphi = h - g$ and $\omega = g'/h'$ lead to a pair of linear equations for h' and g' that, together with the initial conditions $h(0) = g(0) = 0$, determine h and g. It follows immediately that

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re} \int_0^z \varphi'(\zeta) p(\zeta)\, d\zeta + i \operatorname{Im} \varphi(z) \quad \forall z \in \mathbb{D}, \tag{1.1}$$

where $p = (1 + \omega)/(1 - \omega)$; furthermore, p belongs to the class $\mathcal{P}$ of all analytic functions q with positive real part in $\mathbb{D}$ such that $q(0) = 1$.

For any $p \in \mathcal{P}$, the harmonic mapping f defined by (1.1) is orientation-preserving and univalent on $\mathbb{D}$. Moreover, Theorem 1.1 shows that the range of f is convex in the horizontal direction. On account of the remark above, it is natural to consider the family

$$\mathcal{F} = \{K(\cdot, p) \mid p \in \mathcal{P}\}$$

of univalent and orientation-preserving harmonic mappings, where

$$K(z, p) = \operatorname{Re} \int_0^z \varphi'(\zeta) p(\zeta)\, d\zeta + i \operatorname{Im} \varphi(z) \quad \forall z \in \mathbb{D}.$$

The Riesz–Herglotz representation theorem states that

$$p(z) = \int_{|\eta|=1} \frac{1 + \eta z}{1 - \eta z}\, d\mu(\eta) \quad \forall z \in \mathbb{D}, \tag{1.2}$$

where $\mu \in P_{\mathbb{T}}$, the family of all Borel probability measures on the boundary $\mathbb{T}$ of $\mathbb{D}$. Hence, if we set

$$k(z, \eta) = \int_0^z \varphi'(\zeta) \frac{1 + \eta\zeta}{1 - \eta\zeta}\, d\zeta,$$

then it may be concluded from (1.2) that, for each $f \in \mathcal{F}$,

$$f(z) = \operatorname{Re} \int_{|\eta|=1} k(z, \eta)\, d\mu(\eta) + i \operatorname{Im} \varphi(z) \quad \forall z \in \mathbb{D},$$

for a unique $\mu \in P_{\mathbb{T}}$. On the other hand, $P_{\mathbb{T}}$ is a weak-star compact and convex set, and all of its extreme points are unit point masses. Since

$$\mu \mapsto \operatorname{Re} \int_{|\eta|=1} k(\cdot, \eta)\, d\mu(\eta)$$

is a linear homeomorphism, it follows that $\mathcal{F}$ is convex and compact (with respect to the topology of locally uniform convergence), and finally that

$$\operatorname{Ext} \mathcal{F} = \{k_\eta(\cdot) = \operatorname{Re} k(\cdot, \eta) + i \operatorname{Im} \varphi(\cdot) : |\eta| = 1\}$$

where $\operatorname{Ext} \mathcal{F}$ denotes the set of extreme points of $\mathcal{F}$.

2. Main results

Fix a number $\alpha \in (0, \frac{1}{2}\pi)$, and consider the function $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$ given by

$$\varphi_\alpha(z) = \frac{1}{2}\sin^2\alpha \log\left(\frac{1+z}{1-z}\right) + \cos^2\alpha\, \frac{z}{(1-z)^2},$$

where log denotes the principal branch of the logarithm. Note that

$$\mathrm{Re}\{(1-z)^2\varphi'_\alpha(z)\} > 0 \quad \forall z \in \mathbb{D},$$

so a theorem of Royster and Ziegler [5, Theorem 1] shows that for each α in $(0, \frac{1}{2}\pi)$, the function φ_α maps $\mathbb{D}$ univalently onto a domain that is convex in the horizontal direction. By direct calculation,

$$\varphi_\alpha(\mathbb{D}) = \mathbb{C}\backslash\{w \in \mathbb{C} \mid \mathrm{Re}\, w \le A(\alpha) \wedge |\mathrm{Im}\, w| = \tfrac{1}{4}\pi \sin^2\alpha\},$$

where

$$A(\alpha) = \mathrm{Re}\,\varphi_\alpha(-e^{-2i\alpha}) = \tfrac{1}{2}\sin^2\alpha \log(\tan\alpha) - \tfrac{1}{4}.$$

For a fixed $\alpha \in (0, \frac{1}{2}\pi)$, let $\mathcal{F}(\alpha)$ be the class of all mappings of the form

$$f(z) = \mathrm{Re}\int_0^z \varphi'_\alpha(\zeta)p(\zeta)\,d\zeta + i\,\mathrm{Im}\,\varphi_\alpha(z) \quad \forall z \in \mathbb{D},$$

where $p \in \mathcal{P}$. Theorem 1.1 and our preliminary considerations prove the following result.

LEMMA 2.1. *Suppose that $f \in \mathcal{F}(\alpha)$. Then f is harmonic, orientation-preserving and univalent on $\mathbb{D}$, and $f(\mathbb{D})$ is convex in the horizontal direction. Moreover, $\mathcal{F}(\alpha)$ is convex and compact (with respect to the topology of locally uniform convergence), and the set of its extreme points is $\{k_\eta : |\eta| = 1\}$, where*

$$k_\eta(z) = \mathrm{Re}\,k(z, \eta) + i\,\mathrm{Im}\,\varphi_\alpha(z) \quad \forall z \in \mathbb{D},$$

and

$$k(z, \eta) = \int_0^z \varphi'_\alpha(\zeta)\frac{1+\eta\zeta}{1-\eta\zeta}\,d\zeta \quad \forall z \in \mathbb{D}.$$

A simple calculation shows that for any mapping $f \in \mathcal{F}(\alpha)$,

$$f(0) = 0, \quad f_z(0) = 1, \quad f_{\bar{z}}(0) = 0, \tag{2.1}$$

and the following corollary is immediate.

COROLLARY 2.2. *Let S_H^0 denote the class of all harmonic, orientation-preserving and univalent mappings f that are normalized by (2.1). For any fixed $\alpha \in (0, \frac{1}{2}\pi)$, the inclusion $\mathcal{F}(\alpha) \subseteq S_H^0$ holds.*

Note also that, for each $f \in \mathcal{F}(\alpha)$, $f(z)$ is real if and only if z is real. Since $\operatorname{Re} p > 0$ in $\mathbb{D}$ and $\varphi_\alpha' > 0$ in $(-1, 1)$, the function f is increasing on $(-1, 1)$. Therefore the (possibly infinite) radial limits

$$\hat{f}(-1) = \lim_{r \to -1^+} f(r), \qquad \hat{f}(1) = \lim_{r \to 1^-} f(r)$$

exist, and $f((-1, 1)) = (\hat{f}(-1), \hat{f}(1))$. This leads to the following lemma.

LEMMA 2.3. *Fix a number $\alpha \in (0, \frac{1}{2}\pi)$ and let $f \in \mathcal{F}(\alpha)$. Then:*

(a) *f is a typically-real harmonic mapping;*
(b) *$k_{-1}(r) \le f(r) \le k_1(r)$ for all $r \in (-1, 1)$;*
(c) *$\hat{f}(-1) \in [\hat{k}_{-1}(-1), \hat{k}_1(-1)] = [-\infty, -\frac{1}{6}(1 + 2\sin^2\alpha)]$, $\hat{f}(1) = \infty$.*

PROOF. Part (a) of the lemma is evident. Assume that

$$f(r) = \operatorname{Re} \int_0^r \varphi_\alpha'(t) p(t)\, dt \quad \forall r \in (-1, 1),$$

for some function $p \in \mathcal{P}$. From the well-known inequality

$$\frac{1 - |z|}{1 + |z|} \le \operatorname{Re} p(z) \le \frac{1 + |z|}{1 - |z|} \quad \forall z \in \mathbb{D},$$

it follows that

$$k_{-1}(r) = \int_0^r \varphi_\alpha'(t) \frac{1 - t}{1 + t}\, dt \le f(r) \le \int_0^r \varphi_\alpha'(t) \frac{1 + t}{1 - t}\, dt = k_1(r) \quad \forall r \in (0, 1),$$

and

$$\begin{aligned} f(r) &= \operatorname{Re} \int_0^r \varphi_\alpha'(t) p(t)\, dt = -\operatorname{Re} \int_0^{-r} \varphi_\alpha'(-t) p(-t)\, dt \\ &\le -\int_0^{-r} \varphi_\alpha'(-t) \frac{1 - t}{1 + t}\, dt = k_1(r) \quad \forall r \in (-1, 0), \end{aligned}$$

justifying inequality (b). Finally, letting $r \to 1^-$ and $r \to -1^+$ in (b), we obtain (c). □

Lemma 2.1 is useful for describing the family $\mathcal{F}(\alpha)$. Roughly speaking, further properties of $f \in \mathcal{F}(\alpha)$ can be obtained by studying the ranges $k_\eta(\mathbb{D})$. We first observe that

$$\operatorname{Re} k_{\overline{\eta}}(z) = \operatorname{Re} k(z, \overline{\eta}) = \operatorname{Re} k(\overline{z}, \eta) = \operatorname{Re} k_\eta(\overline{z}) \quad \forall z \in \mathbb{D},\ \forall \eta \in \mathbb{T}. \tag{2.2}$$

Since $\operatorname{Im} \varphi_\alpha(z) = -\operatorname{Im} \varphi_\alpha(\overline{z})$ for any $\alpha \in (0, \frac{1}{2}\pi)$ and $z \in \mathbb{D}$, equality (2.2) shows that the sets $k_\eta(\mathbb{D})$ and $k_{\overline{\eta}}(\mathbb{D})$ are symmetric with respect to the real axis. We are now ready to describe some geometric properties of the extreme points.

THEOREM 2.4. *Fix* $\alpha \in (0, \frac{1}{2}\pi)$. *Suppose that* $k_\eta \in \operatorname{Ext} \mathcal{F}(\alpha)$, *where* $\eta = e^{i\beta}$, *and define*

$$\lambda_1(c, \alpha, \beta) = \left(\frac{\pi}{4} \tan \frac{1}{2}\beta - \frac{\beta}{2 \sin \beta}\right)\sin^2\alpha + \left(\frac{\beta \sin \beta}{8 \sin^4 \frac{1}{2}\beta} - \frac{1}{2 \sin^2 \frac{1}{2}\beta} - \frac{(4c - \pi \sin^2 \alpha)\cot\frac{1}{2}\beta}{4 \cos^2 \alpha}\right)\cos^2\alpha,$$

$$\lambda_2(c, \alpha, \beta) = \left(\frac{c}{\sin^2 \alpha} \tan \frac{1}{2}\beta - \frac{\beta}{2 \sin \beta}\right)\sin^2\alpha + \left(\frac{\beta \sin \beta}{8 \sin^4 \frac{1}{2}\beta} - \frac{1}{2 \sin^2 \frac{1}{2}\beta}\right)\cos^2\alpha,$$

$$\lambda_3(c, \alpha, \beta) = \left(-\frac{\pi}{4} \tan \frac{1}{2}\beta - \frac{\beta - 2\pi}{2 \sin \beta}\right)\sin^2\alpha + \left(\frac{(\beta - 2\pi)\sin\beta}{8 \sin^4 \frac{1}{2}\beta} - \frac{1}{2 \sin^2 \frac{1}{2}\beta} - \frac{(4c + \pi \sin^2 \alpha)\cot\frac{1}{2}\beta}{4 \cos^2 \alpha}\right)\cos^2\alpha,$$

and

$$\mathcal{D}_1(\alpha, \beta) = \left\{(u, v) \in \mathbb{R}^2 \mid v < \lambda_1(u, \alpha, \beta) \wedge v \geq \tfrac{1}{4}\pi \sin^2 \alpha\right\},$$
$$\mathcal{D}_2(\alpha, \beta) = \left\{(u, v) \in \mathbb{R}^2 \mid v < \lambda_2(u, \alpha, \beta) \wedge |v| < \tfrac{1}{4}\pi \sin^2 \alpha\right\},$$
$$\mathcal{D}_3(\alpha, \beta) = \left\{(u, v) \in \mathbb{R}^2 \mid v < \lambda_3(u, \alpha, \beta) \wedge v < -\tfrac{1}{4}\pi \sin^2 \alpha\right\}.$$

Then:

(i) *for all* $\beta \in (0, \pi - 2\alpha)$, $k_\eta(\mathbb{D})$ *is equal to*

$$\mathcal{D}_1(\alpha, \beta) \cup \mathcal{D}_2(\alpha, \beta) \cup \mathcal{D}_3(\alpha, \beta) \cup \{u - i\tfrac{1}{4}\pi \sin^2 \alpha : u > \lambda_2(-\tfrac{1}{4}\pi \sin^2 \alpha, \alpha, \beta)\};$$

(ii) *for all* $\beta \in [\pi - 2\alpha, \pi)$, $k_\eta(\mathbb{D})$ *is equal to*

$$\mathcal{D}_1(\alpha, \beta) \cup \mathcal{D}_2(\alpha, \beta) \cup \mathcal{D}_3(\alpha, \beta) \cup \{u - i\tfrac{1}{4}\pi \sin^2 \alpha : u > \lambda_3(-\tfrac{1}{4}\pi \sin^2 \alpha, \alpha, \beta)\};$$

(iii) $k_1(\mathbb{D})$ *is equal to*

$$\mathbb{C}\backslash\{w \in \mathbb{C} : \operatorname{Re} w \leq -\tfrac{1}{6}(1 + 2 \sin^2 \alpha) \wedge |\operatorname{Im} w| \leq \tfrac{1}{4}\pi \sin^2 \alpha\};$$

(iv) $k_{-1}(\mathbb{D})$ *is equal to*

$$\{w \in \mathbb{C} : \operatorname{Re} w \leq -\tfrac{1}{2} \cos 2\alpha \wedge |\operatorname{Im} w| < \tfrac{1}{4}\pi \sin^2 \alpha\} \cup \{w \in \mathbb{C} : \operatorname{Re} w > -\tfrac{1}{2} \cos 2\alpha\}.$$

PROOF. We treat case (i) only. Fix $\beta \in (0, \pi)$ and let $\eta = e^{i\beta}$. Then, after integration,

$$\begin{aligned}\operatorname{Re} k_\eta(z) &\\ = \frac{\sin^2\alpha}{2}&\left[\cot\left(\frac{1}{2}\beta\right)\arg(1-z) + \tan\left(\frac{1}{2}\beta\right)\arg(1+z) - \frac{2}{\sin\beta}\arg(1-\eta z)\right]\\ &+ \cos^2\alpha\left[\frac{\sin\beta}{4\sin^4\frac{1}{2}\beta}\arg\left(\frac{1-\eta z}{1-z}\right) - \cot\left(\frac{1}{2}\beta\right)\operatorname{Im}\frac{1}{(1-z)^2}\right.\\ &\left.+\frac{1}{\sin^2\frac{1}{2}\beta}\operatorname{Re}\frac{z}{1-z} + \cot\left(\frac{1}{2}\beta\right)\operatorname{Im}\frac{z}{1-z}\right],\end{aligned} \tag{2.3}$$

where we assume that $\arg(\cdot) \in (-\pi, \pi]$. Since any mapping from $\mathcal{F}(\alpha)$ is convex in the horizontal direction, we may assume that

$$\operatorname{Im} k_\eta(z) = \operatorname{Im} \varphi_\alpha(z) = c \tag{2.4}$$

for some $c \in \mathbb{R}$, and find the bounds on $\operatorname{Re} k_\eta(z)$. The main idea of the proof is to set $re^{i\theta} = (1+z)/(1-z)$, where $r > 0$ and $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and replace the variable z by the variables r and θ. This transforms (2.4) to the form $\operatorname{Im} \varphi_\alpha((re^{i\theta}-1)/(re^{i\theta}+1)) = c$, or equivalently,

$$2\theta\sin^2\alpha + r^2\cos^2\alpha\sin 2\theta = 4c. \tag{2.5}$$

If $c > \frac{1}{4}\pi\sin^2\alpha$, then (for given α and c) the positive solution

$$r = r_c(\theta) = \left(\frac{4c - 2\theta\sin^2\alpha}{\cos^2\alpha\sin 2\theta}\right)^{1/2}$$

of (2.5) is defined on $(0, \frac{1}{2}\pi)$. Substituting $r_c(\theta)$ into $\operatorname{Re} k_\eta((re^{i\theta}-1)/(re^{i\theta}+1))$ (see (2.3)) yields

$$g_c(\theta) = \operatorname{Re} k_\eta\left(\frac{r_c(\theta)e^{i\theta}-1}{r_c(\theta)e^{i\theta}+1}\right).$$

All mappings $k_\eta \in \operatorname{Ext}\mathcal{F}(\alpha)$ are open, and consequently the function $g_c(\theta)$ cannot assume boundary values inside the interval $(0, \frac{1}{2}\pi)$. Calculation shows that $\lim_{\theta\to 0^+} g_c(\theta) = +\infty$ and

$$\begin{aligned}\lim_{\theta\to\frac{1}{2}\pi^-} g_c(\theta) &= \left(\frac{1}{4}\pi\tan\frac{1}{2}\beta - \frac{\beta}{2\sin\beta}\right)\sin^2\alpha\\ &\quad + \left(\frac{\beta\sin\beta}{8\sin^4\frac{1}{2}\beta} - \frac{1}{2\sin^2\frac{1}{2}\beta} - \frac{(4c-\pi\sin^2\alpha)\cot\frac{1}{2}\beta}{4\cos^2\alpha}\right)\cos^2\alpha\\ &= \lambda_1(c, \alpha, \beta).\end{aligned}$$

Hence if $\operatorname{Im} k_\eta(z) = c$ and $c > \frac{1}{4}\pi\sin^2\alpha$, then $\operatorname{Re} k_\eta(z)$ varies over the interval $(\lambda_1(c, \alpha, \beta), +\infty)$, and finally

$$\begin{aligned}&k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid \operatorname{Im} w > \tfrac{1}{4}\pi\sin^2\alpha\}\\ &\quad = \{(u, v) \in \mathbb{R}^2 \mid v < \lambda_1(u, \alpha, \beta) \wedge v > \tfrac{1}{4}\pi\sin^2\alpha\}.\end{aligned}$$

Next, if we choose $c \in (0, \frac{1}{4}\pi \sin^2 \alpha)$, then the function r_c is defined on the interval $(0, \theta_1(c))$, where $\theta_1(c) = 2c \operatorname{cosec}^2 \alpha$. This, in turn, forces $\lim_{\theta\to 0^+} g_c(\theta) = +\infty$ and

$$\begin{aligned}\lim_{\theta\to\theta_1(c)^-} g_c(\theta) &= \left(\frac{c}{\sin^2 \alpha} \tan \frac{1}{2}\beta - \frac{\beta}{2 \sin \beta}\right)\sin^2\alpha \\ &\quad + \left(\frac{\beta \sin \beta}{8 \sin^4 \frac{1}{2}\beta} - \frac{1}{2 \sin^2 \frac{1}{2}\beta}\right)\cos^2\alpha \\ &= \lambda_2(c, \alpha, \beta),\end{aligned}$$

which gives

$$\begin{aligned}&k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid 0 < \operatorname{Im} w < \tfrac{1}{4}\pi \sin^2 \alpha\} \\ &\quad = \{(u, v) \in \mathbb{R}^2 \mid v < \lambda_2(u, \alpha, \beta) \wedge 0 < v < \tfrac{1}{4}\pi \sin^2 \alpha\}.\end{aligned}$$

In the case where $\operatorname{Im} k_\eta(z) = \frac{1}{4}\pi \sin^2 \alpha$, the function $r_{\frac{1}{4}\pi \sin^2 \alpha}$ is defined in $(0, \frac{1}{2}\pi)$. We see at once that

$$\lim_{\theta\to 0^+} g_{\frac{1}{4}\pi \sin^2 \alpha}(\theta) = +\infty$$

and

$$\lim_{\theta\to \frac{1}{2}\pi^-} g_{\frac{1}{4}\pi \sin^2 \alpha}(\theta) = \lambda_1(\tfrac{1}{4}\pi \sin^2 \alpha, \alpha, \beta) = \lambda_2(\tfrac{1}{4}\pi \sin^2 \alpha, \alpha, \beta) = a_\alpha(\beta), \quad (2.6)$$

say, which is due to the fact that

$$\lim_{\theta\to 0^+} r_{\frac{1}{4}\pi \sin^2 \alpha}(\theta) = 0, \qquad \lim_{\theta\to \frac{1}{2}\pi^-} r_{\frac{1}{4}\pi \sin^2 \alpha}(\theta) = \tan \alpha.$$

From this it may be concluded that

$$k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid \operatorname{Im} w = \tfrac{1}{4}\pi \sin^2 \alpha\} = \{u + i\tfrac{1}{4}\pi \sin^2 \alpha \mid u > a_\alpha(\beta)\}.$$

Application of Lemma 2.3 enables us to write

$$k_\eta((-1, 1)) = (\hat{k}_\eta(-1), \hat{k}_\eta(1)) = (\hat{k}_\eta(-1), +\infty),$$

where

$$\hat{k}_\eta(-1) = \lambda_2(0, \alpha, \beta) = -\frac{\beta \sin^2 \alpha}{2 \sin \beta} + \left(\frac{\beta \sin \beta}{8 \sin^4 \frac{1}{2}\beta} - \frac{1}{2 \sin^2 \frac{1}{2}\beta}\right)\cos^2\alpha.$$

Now we take $\operatorname{Im} k_\eta(z) = c$, where $c \in (-\frac{1}{4}\pi \sin^2 \alpha, 0)$. In this case, the function r_c is defined in $(\theta_1(c), 0)$, and it is easy to verify that

$$\lim_{\theta\to\theta_1(c)^+} r_c(\theta) = 0, \qquad \lim_{\theta\to 0^-} r_c(\theta) = +\infty.$$

Thus

$$\lim_{\theta\to\theta_1(c)^+} g_c(\theta) = \lambda_2(c, \alpha, \beta), \qquad \lim_{\theta\to 0^-} g_c(\theta) = +\infty$$

and therefore

$$\begin{aligned}&k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid -\tfrac{1}{4}\pi \sin^2\alpha < \operatorname{Im} w < 0\}\\ &\quad = \{(u, v) \in \mathbb{R}^2 \mid v < \lambda_2(u, \alpha, \beta) \wedge -\tfrac{1}{4}\pi \sin^2\alpha < v < 0\}.\end{aligned}$$

Let us now assume that $c < -\frac{1}{4}\pi \sin^2\alpha$. It is easy to check that r_c is defined on $(-\frac{1}{2}\pi, 0)$, and moreover, $\lim_{\theta\to 0^-} g_c(\theta) = +\infty$, while $\lim_{\theta\to -\frac{1}{2}\pi^+} g_c(\theta)$ is equal to

$$\begin{aligned}&\left(-\frac{1}{4}\pi \tan\frac{1}{2}\beta - \frac{\beta - 2\pi}{2\sin\beta}\right)\sin^2\alpha\\ &\qquad + \left(\frac{(\beta - 2\pi)\sin\beta}{8\sin^4\frac{1}{2}\beta} - \frac{1}{2\sin^2\frac{1}{2}\beta} - \frac{(4c + \pi\sin^2\alpha)\cot\frac{1}{2}\beta}{4\cos^2\alpha}\right)\cos^2\alpha\\ &\qquad = \lambda_3(c, \alpha, \beta).\end{aligned}$$

This clearly forces

$$\begin{aligned}&k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid \operatorname{Im} w < -\tfrac{1}{4}\pi \sin^2\alpha\}\\ &\quad = \{(u, v) \in \mathbb{R}^2 \mid v < \lambda_3(u, \alpha, \beta) \wedge v < -\tfrac{1}{4}\pi \sin^2\alpha\}.\end{aligned}$$

When $\operatorname{Im} k_\eta(z) = -\frac{1}{4}\pi \sin^2\alpha$, the function $r_{-\frac{1}{4}\pi\sin^2\alpha}(\theta)$ is defined on $(-\frac{1}{2}\pi, 0)$, and one can show that

$$\lim_{\theta\to 0^-} g_{-\frac{1}{4}\pi\sin^2\alpha}(\theta) = +\infty,$$

and

$$\lim_{\theta\to 0^-} g_{-\frac{1}{4}\pi\sin^2\alpha}(\theta) = \begin{cases}\lambda_2(-\frac{1}{4}\pi\sin^2\alpha, \alpha, \beta) = c_\alpha(\beta) & \text{if } \beta \in (0, \pi - 2\alpha)\\ \lambda_3(-\frac{1}{4}\pi\sin^2\alpha, \alpha, \beta) = d_\alpha(\beta) & \text{if } \beta \in (\pi - 2\alpha, \pi)\end{cases} \tag{2.7}$$

(observe that $c_\alpha(\beta) = d_\alpha(\beta)$ when $\beta = \pi - 2\alpha$). This completes the proof. □

REMARK 2.5. It is easy to check (see (2.6) and (2.7)) that

$$\begin{aligned}d_\alpha(\beta) - c_\alpha(\beta) &= -\frac{\pi\cos(\alpha - \frac{1}{2}\beta)\cos(\alpha + \frac{1}{2}\beta)}{2\sin^3\frac{1}{2}\beta\cos\frac{1}{2}\beta}\\ a_\alpha(\beta) - d_\alpha(\beta) &= \frac{\pi(\cot^2\alpha - \sin^2\frac{1}{2}\beta)\sin^2\alpha}{2\sin^2\frac{1}{2}\beta\tan\frac{1}{2}\beta}.\end{aligned}$$

This gives:

(i) for any fixed $\alpha \in (0, \frac{1}{2}\pi)$,

$$\begin{aligned}d_\alpha(\beta) &< c_\alpha(\beta) \quad \forall\beta \in (0, \pi - 2\alpha)\\ d_\alpha(\beta) &> c_\alpha(\beta) \quad \forall\beta \in (\pi - 2\alpha, \pi);\end{aligned}$$

(ii) for any fixed $\alpha \in (0, \frac{1}{4}\pi]$,

$$d_\alpha(\beta) < a_\alpha(\beta) \quad \forall \beta \in (0, \pi);$$

(iii) for any fixed $\alpha \in (\frac{1}{4}\pi, \frac{1}{2}\pi)$,

$$d_\alpha(\beta) < a_\alpha(\beta) \quad \forall \beta \in (0, \beta_0(\alpha))$$
$$d_\alpha(\beta) > a_\alpha(\beta) \quad \forall \beta \in (\beta_0(\alpha), \pi),$$

where $\beta_0(\alpha) = 2\arcsin(\cot\alpha)$.

The following lemma will be extremely useful in proving our next results.

LEMMA 2.6. *Suppose that a_α, c_α, d_α are given by (2.6) and (2.7), and that $\beta_0(\alpha) =$ $2\arcsin(\cot\alpha)$. Then:*

(i) *for any fixed $\alpha \in (\frac{1}{4}\pi, \frac{1}{2}\pi)$, the function a_α is increasing on $(\beta_0(\alpha), \pi)$;*
(ii) *for any fixed $\alpha \in (0, \frac{1}{2}\pi)$, the function c_α is decreasing on $(0, \pi)$;*
(iii) *for any fixed $\alpha \in (0, \frac{1}{4}\pi]$, the function d_α is increasing on $(0, \pi)$;*
(iv) *for any fixed $\alpha \in (\frac{1}{4}\pi, \frac{1}{2}\pi)$, the function d_α is increasing on $(\pi - 2\alpha, \beta_0(\alpha))$.*

PROOF. We justify case (ii) only. Fix $\alpha \in (0, \frac{1}{2}\pi)$. By straightforward computation,

$$c'_\alpha(\beta) = \frac{\sin^2\alpha}{8\cos^2\frac{1}{2}\beta} f_1(\beta) + \frac{\cos^2\alpha}{8\sin^2\frac{1}{2}\beta} f_2(\beta),$$

where

$$f_1(\beta) = -\pi - 2\cot\tfrac{1}{2}\beta + \beta(\cot^2\tfrac{1}{2}\beta - 1),$$
$$f_2(\beta) = 6\cot\tfrac{1}{2}\beta - \beta(3\cot^2\tfrac{1}{2}\beta + 1).$$

It is evident that $f_1(\beta) < 0$ for $\beta \in (\frac{1}{2}\pi, \pi)$. Write $\beta = 2\,\mathrm{arccot}\, t$, where $\beta \in (0, \frac{1}{2}\pi)$; then

$$f_1(2\,\mathrm{arccot}\, t) = -\pi - 2t + 2(t^2 - 1)\,\mathrm{arccot}\, t \quad \forall t \in (1, +\infty).$$

The inequality

$$\mathrm{arccot}\, t \le \frac{1}{t} \quad \forall t \in (1, +\infty),$$

implies that $f_1(2\,\mathrm{arccot}\, t) \le -\pi - 2/t < 0$ for all $t > 1$. By the above, $f_1 < 0$ holds in $(0, \pi)$. Similarly, $f_2 < 0$ in the interval $(0, \pi)$, and finally $c'_\alpha < 0$ in $(0, \pi)$.

Parts (i), (iii) and (iv) follow in the same way, so we leave details to the reader. □

We illustrate our considerations concerning the sets $k_\eta(\mathbb{D})$ in Figure 1. Note that

$$A_\alpha(\beta) = a_\alpha(\beta) + i\tfrac{1}{4}\pi\sin^2\alpha, \quad C_\alpha(\beta) = c_\alpha(\beta) - i\tfrac{1}{4}\pi\sin^2\alpha$$

and

$$D_\alpha(\beta) = d_\alpha(\beta) - i\tfrac{1}{4}\pi\sin^2\alpha.$$

Making use of Theorem 2.4 and Lemma 2.6, we shall now prove the main theorem of this section.

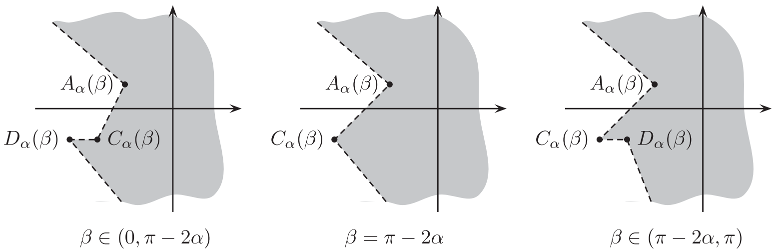

FIGURE 1. Domains $k_\eta(\mathbb{D})$, where $\arg \eta = \beta$.

THEOREM 2.7. *Fix* $\alpha \in (0, \frac{1}{2}\pi)$, *and suppose that* $\mathcal{K}(\alpha) = \bigcup_{k \in \operatorname{Ext} \mathcal{F}(\alpha)} k(\mathbb{D})$. *Then*

$$\mathcal{K}(\alpha) = \mathbb{C} \backslash \{w \in \mathbb{C} : \operatorname{Re} w \leq -\tfrac{1}{8}\pi \sin 2\alpha - \tfrac{1}{2} \wedge |\operatorname{Im} w| = \tfrac{1}{4}\pi \sin^2 \alpha\}. \tag{2.8}$$

PROOF. We first observe that

$$\mathbb{C} \backslash \{w \in \mathbb{C} : |\operatorname{Im} w| = \tfrac{1}{4}\pi \sin^2 \alpha\} \subseteq k_1(\mathbb{D}) \cup k_{-1}(\mathbb{D}),$$

for any fixed $\alpha \in (0, \frac{1}{2}\pi)$. Consequently, it is enough to find the set

$$\bigcup_{|\eta|=1} k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} : |\operatorname{Im} w| = \tfrac{1}{4}\pi \sin^2 \alpha\}.$$

Due to the symmetry of the domains $k_\eta(\mathbb{D})$ and $k_{\overline{\eta}}(\mathbb{D})$, we need only consider the case where $\arg \eta = \beta \in [0, \pi]$. By Theorem 2.4, $k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid \operatorname{Im} w = \frac{1}{4}\pi \sin^2 \alpha\}$ is equal to

$$\begin{cases} \{(u, \frac{1}{4}\pi \sin^2 \alpha) \mid u > -\frac{1}{6}(1 + 2\sin^2 \alpha)\} & \text{if } \beta = 0, \\ \{(u, \frac{1}{4}\pi \sin^2 \alpha) \mid u > a_\alpha(\beta)\} & \text{if } \beta \in (0, \pi), \\ \{(u, \frac{1}{4}\pi \sin^2 \alpha) \mid u > -\frac{1}{2}\cos 2\alpha\} & \text{if } \beta = \pi, \end{cases}$$

and $k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} \mid \operatorname{Im} w = -\frac{1}{4}\pi \sin^2 \alpha\}$ is equal to

$$\begin{cases} \{(u, -\frac{1}{4}\pi \sin^2 \alpha) \mid u > -\frac{1}{6}(1 + 2\sin^2 \alpha)\} & \text{if } \beta = 0 \\ \{(u, -\frac{1}{4}\pi \sin^2 \alpha) \mid u > c_\alpha(\beta)\} & \text{if } \beta \in (0, \pi - 2\alpha] \\ \{(u, -\frac{1}{4}\pi \sin^2 \alpha) \mid u > d_\alpha(\beta)\} & \text{if } \beta \in (\pi - 2\alpha, \pi) \\ \{(u, -\frac{1}{4}\pi \sin^2 \alpha) \mid u > -\frac{1}{2}\cos 2\alpha\} & \text{if } \beta = \pi, \end{cases}$$

where $c_\alpha(\pi - 2\alpha) = d_\alpha(\pi - 2\alpha) = -\frac{1}{8}\pi \sin 2\alpha - \frac{1}{2}$. When $\beta = \arg \eta$, let $\mathcal{T}_\alpha(\beta)$ denote the projection of the set

$$k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} : |\operatorname{Im} w| = \tfrac{1}{4}\pi \sin^2 \alpha\}$$

onto the real axis. Note that $a_\alpha(\beta) - c_\alpha(\beta) > 0$ for any $\alpha \in (0, \frac{1}{2}\pi)$ and $\beta \in (0, \pi)$. Therefore $\mathcal{T}_\alpha(\beta) = (c_\alpha(\beta), \infty)$ for all $\beta \in (0, \pi - 2\alpha)$, by Remark 2.5. Lemma 2.6 now implies that

$$\bigcup_{\beta\in(0,\pi-2\alpha)} \mathcal{T}_\alpha(\beta) = (c_\alpha(\pi - 2\alpha), \infty).$$

The case where $\beta \in [\pi - 2\alpha, \pi)$ depends on α. If $\alpha \in (0, \frac{1}{4}\pi]$, then

$$\bigcup_{\beta\in[\pi-2\alpha,\pi)} \mathcal{T}_\alpha(\beta) = (c_\alpha(\pi - 2\alpha), \infty).$$

If $\alpha \in (\frac{1}{4}\pi, \frac{1}{2}\pi)$, then Remark 2.5 and Lemma 2.6 show that $\mathcal{T}_\alpha(\beta) = (d_\alpha(\beta), \infty)$, for any $\beta \in [\pi - 2\alpha, \beta_0(\alpha))$, and

$$\bigcup_{\beta\in[\pi-2\alpha,\beta_0(\alpha))} \mathcal{T}_\alpha(\beta) = (d_\alpha(\pi - 2\alpha), \infty).$$

Similarly,

$$\bigcup_{\beta\in[\beta_0(\alpha),\pi)} \mathcal{T}_\alpha(\beta) = \bigcup_{\beta\in[\beta_0(\alpha),\pi)} (a_\alpha(\beta), \infty) = (a_\alpha(\beta_0(\alpha)), \infty).$$

Since

$$a_\alpha(\beta_0(\alpha)) = d_\alpha(\beta_0(\alpha)) \geq d_\alpha(\pi - 2\alpha) = c_\alpha(\pi - 2\alpha),$$

we finally have

$$\bigcup_{\beta\in(0,\pi)} \mathcal{T}_\alpha(\beta) = (d_\alpha(\pi - 2\alpha), \infty), \tag{2.9}$$

for $\alpha \in (0, \frac{1}{2}\pi)$. Moreover, Theorem 2.4 gives

$$\mathcal{T}_\alpha(0) = (-\tfrac{1}{6}(1 + 2\sin^2\alpha), \infty), \quad \mathcal{T}_\alpha(\pi) = (-\tfrac{1}{2}\cos 2\alpha, \infty). \tag{2.10}$$

Combining (2.9) with (2.10), we conclude that

$$\bigcup_{\beta\in[0,\pi]} \mathcal{T}_\alpha(\beta) = \mathcal{T}(\alpha) = (d_\alpha(\pi - 2\alpha), \infty). \tag{2.11}$$

Consequently,

$$\bigcup_{|\eta|=1} k_\eta(\mathbb{D}) \cap \{w \in \mathbb{C} : |\mathrm{Im}\, w| = \tfrac{1}{4}\pi \sin^2\alpha\}$$
$$= \{w \in \mathbb{C} : \mathrm{Re}\, w \in \mathcal{T}(\alpha) \wedge |\mathrm{Im}\, w| = \tfrac{1}{4}\pi \sin^2\alpha\},$$

which completes the proof. □

We can now formulate our main result.

THEOREM 2.8. *Fix α, $\alpha \in (0, \frac{1}{2}\pi)$, and suppose that $\mathcal{K}(\alpha)$ is given by (2.8). Then*

$$\bigcup_{f\in\mathcal{F}(\alpha)} f(\mathbb{D}) = \mathcal{K}(\alpha).$$

PROOF. We first recall that for any fixed $\alpha \in (0, \frac{1}{2}\pi)$, the family $\mathcal{F}(\alpha)$ is convex and compact. By the Krein–Milman theorem, the closed convex hull $\overline{\text{conv}}(\text{Ext}\,\mathcal{F}(\alpha))$ is all of $\mathcal{F}(\alpha)$. Hence, the convex hull $\text{conv}(\text{Ext}\,\mathcal{F}(\alpha))$ is dense in $\mathcal{F}(\alpha)$ in the topology of locally uniform convergence (which makes $\mathcal{F}(\alpha)$ compact). This implies that each function $f \in \mathcal{F}(\alpha)$ can be locally uniformly approximated by functions f_n of the form

$$f_n = \sum_{j=1}^{n} \mu_s k_{\eta_s}, \tag{2.12}$$

where $\mu_s > 0$, $s = 1, 2, \ldots, n$, $\sum_{s=1}^{n} \mu_s = 1$ and $k_{\eta_s} \in \text{Ext}\,\mathcal{F}(\alpha)$. Taking any mapping $k_\eta \in \text{Ext}\,\mathcal{F}(\alpha)$, we see that $\text{Im}\, k_\eta(z) = \text{Im}\,\varphi_\alpha(z)$ for all $z \in \mathbb{D}$, so for f_n defined by (2.12),

$$\text{Im}\, f_n(z) = \text{Im}\,\varphi_\alpha(z), \quad \text{Re}\, f_n(z) = \sum_{s=1}^{n} \mu_s\,\text{Re}\, k_{\eta_s}(z) \quad \forall z \in \mathbb{D}.$$

Observe that if we restrict ourselves to the set $\{z \in \mathbb{D} \mid \text{Im}\,\varphi_\alpha(z) = \frac{1}{4}\pi \sin^2\alpha\}$, then $\text{Im}\, f_n(z) = \frac{1}{4}\pi \sin^2\alpha$ and $\text{Re}\, f_n(z) \in \mathcal{T}(\alpha)$, and this follows from Theorem 2.7.

The same reasoning applies to the case $\{z \in \mathbb{D} \mid \text{Im}\,\varphi_\alpha(z) = -\frac{1}{4}\pi \sin^2\alpha\}$. □

Our knowledge of extreme points is very useful for solving extremal problems on $\mathcal{F}(\alpha)$. In particular, if Λ is a real continuous convex functional on $\mathcal{F}(\alpha)$, it is sufficient (by the Krein–Milman theorem) to find the maximum of Λ over the set of extreme points $\text{Ext}\,\mathcal{F}(\alpha)$. Repeating the arguments in the proof of Theorem 2.7, we can prove the following result.

LEMMA 2.9. *Fix a number $\alpha \in (0, \frac{1}{2}\pi)$, and suppose that $f \in \mathcal{F}(\alpha)$. Then*

$$|\text{Re}\, f(-e^{-2i\alpha})| \le |\text{Re}\, k_{-e^{2i\alpha}}(-e^{-2i\alpha})| = |c_\alpha(\pi - 2\alpha)| = \tfrac{1}{8}\pi \sin 2\alpha + \tfrac{1}{2}.$$

From this lemma we deduce that

$$|\text{Re}\,\varphi_\alpha(-e^{-2i\alpha})| < \tfrac{1}{8}\pi \sin 2\alpha + \tfrac{1}{2} \quad \forall \alpha \in (0, \tfrac{1}{2}\pi),$$

and hence establish the following corollary.

COROLLARY 2.10. *Fix $\alpha \in (0, \frac{1}{2}\pi)$ and let φ_α be the generating function for the class $\mathcal{F}(\alpha)$. Then*

$$\varphi_\alpha(\mathbb{D}) \subset \mathcal{K}(\alpha),$$

where $\mathcal{K}(\alpha)$ is given by (2.8).

Note that when $\alpha \to \frac{1}{2}\pi^-$, conformal slits vanish and we obtain the class $\mathcal{F}(\frac{1}{2}\pi)$ of harmonic univalent functions related to the strip $\Omega = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{1}{4}\pi\} = \varphi_{\frac{1}{2}\pi}(\mathbb{D})$. In fact, Hengartner and Schober [2] showed that $\mathcal{F}(\frac{1}{2}\pi)$ is the closure of the family of harmonic orientation-preserving univalent mappings from $\mathbb{D}$ onto Ω, normalized by $f(0) = f_{\bar{z}}(0) = 0$ and $f_z(0) > 0$. On the other hand, φ_0 is the Koebe function and

$$\bigcup_{f \in \mathcal{F}(0)} f(\mathbb{D}) = \mathbb{C} \backslash \big(-\infty, -\tfrac{1}{2}\big],$$

so the family $\mathcal{F}(0)$ is related to the whole plane $\mathbb{C}$ slit along an infinite ray $(-\infty, a]$ where $a < 0$ (see [4]).

References

[1] J. Clunie and T. Sheil-Small, 'Harmonic univalent functions', *Ann. Acad. Sci. Fenn. Ser. A. I Math.* **9** (1984), 3–25.

[2] W. Hengartner and G. Schober, 'Univalent harmonic functions', *Trans. Amer. Math. Soc.* **299**(1) (1987), 1–31.

[3] H. Lewy, 'On the non-vanishing of the Jacobian in certain one-to-one mappings', *Bull. Amer. Math. Soc.* **42** (1936), 689–692.

[4] A. E. Livingston, 'Univalent harmonic mappings', *Ann. Polon. Math.* **LVII**(1) (1992), 57–70.

[5] W. C. Royster and M. Ziegler, 'Univalent functions convex in one direction', *Publ. Math. Debrecen* **23**(3–4) (1976), 339–345.

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