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An extremal problem for harmonic self-mappings of the unit disk

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Abstract

Let $\mathrm{Aut}_H(\mathbb{D})$ be the class of harmonic automorphisms of the unit disk $\mathbb{D}$. For $f(re^{i\varphi}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\varphi} \in \mathrm{Aut}_H(\mathbb{D})$ the upper bounds of $\mathrm{Re} \sum_{k=1}^{n} \{c_k + c_{-k}\}$ are given.

Keywords: Harmonic mapping; Extremal problem

1. Introduction

A function $f(z) = u(z) + iv(z)$ defined on a simply connected domain $G \subset \mathbb{C}$ is a harmonic mapping if and only if it is twice continuously differentiable and $\Delta f = 4f_{z\bar{z}} = 0$. The components u and v are real-valued harmonic functions which need not be conjugate. Because a composition $f \circ g$ with an analytic function g remains harmonic, the Riemann mapping theorem allows us to suppose that $G = \mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ (see e.g. [2] for details).

In the paper [3], Duren and Schober developed a variational method for solving extremal problems over families of sense-preserving univalent harmonic mappings which map the unit disk $\mathbb{D}$ onto given convex region. This method is most effective when specialised to harmonic mapping of the unit disk onto itself (see [3,4]).

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Let $\mathrm{Aut}_H(\mathbb{D})$ be the family of all orientation-preserving univalent harmonic mappings of the unit disk onto itself. Observe that $\mathrm{Aut}_H(\mathbb{D})$ is not compact with respect to the topology of locally uniform convergence. Furthermore, the closure $\overline{\mathrm{Aut}_H(\mathbb{D})}$ of this family consists of all integrals

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\} e^{i\theta(t)}\,dt, \tag{1}$$

where θ is a *circle mapping*, defined as left-continuous nondecreasing function on $[0, 2\pi)$ with $\theta(2\pi - 0) - \theta(0) \leqslant 2\pi$ (see [3] for details).

If $\mathcal{L}$ is a continuous linear functional on $\overline{\mathrm{Aut}_H(\mathbb{D})}$ then

$$\mathcal{L}(f) = \int_0^{2\pi} l(t)e^{i\theta(t)}\,dt,$$

where f is given by (1) and

$$l(t) = \mathcal{L}\left(\frac{1}{2\pi}\mathrm{Re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\}\right). \tag{2}$$

Duren and Schober proved in [3, Theorem 5.1] by a variational method the following result.

Theorem 1. [3] *Let $\mathcal{L}$ be a continuous linear functional, define l by* (2) *and let f maximize* $\mathrm{Re}\{\mathcal{L}\}$ *over $\overline{\mathrm{Aut}_H(\mathbb{D})}$. Then the circle mapping θ which represents f through* (1) *has the properties*:

(a) *θ is constant on any interval where* $\mathrm{Im}\{l(t)e^{i\theta(t)}\}$ *has constant sign,*
(b) $\mathrm{Im}\{l(t)e^{i\theta(t)}\}$ *has mean value zero on any interval whose endpoints are discontinuities of θ.*

They apply this theorem to get some sharp estimates for coefficients and other functionals. Note that the coefficient estimates for $f \in \mathrm{Aut}_H(\mathbb{D})$ given in [3,4], coincide with estimates obtained earlier by Kühnau (see [6]) for coefficients of Laurent series of certain functions F holomorphic and univalent in $\{z \in \mathbb{C}\colon 1 < |z| < R\}$, where R may depend on F. A trivial verification shows that these coefficient problems are equivalent.

2. Main results

Assume that $f \in \mathrm{Aut}_H(\mathbb{D})$ has a series expansion

$$f\left(re^{i\varphi}\right) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\varphi}, \quad 0 \leqslant r < 1. \tag{3}$$

For every given integer $n \geqslant 1$ and f with the Fourier expansion (3), set

$$\mathcal{I}_n(f) = \sum_{k=1}^{n}\{c_k + c_{-k}\}.$$

Consider now the extremal problem of finding the supremum of $\mathrm{Re}\{\mathcal{I}_n(f)\}$ as f ranges over the family $\mathrm{Aut}_H(\mathbb{D})$.

In [4, the proof of Theorem 3] it is shown that for a fixed integer $n \geqslant 1$,

$$\mathrm{Re}\{c_n + c_{-n}\} < \frac{4}{n\pi},$$

so it is obvious that

$$\operatorname{Re}\{\mathcal{I}_n(f)\} < \frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{k}, \qquad f \in \operatorname{Aut}_H(\mathbb{D}).$$

We have proved the following result:

Theorem 2. *Let f be a sense-preserving univalent harmonic mapping of the disk $\mathbb{D}$ onto itself, with the Fourier expansion* (3). *Then*

$$\operatorname{Re}\{\mathcal{I}_n(f)\} < \frac{4}{\pi}\mathcal{W}_n\left(\frac{\pi}{n+1}\right), \qquad n = 1, 2, \ldots, \tag{4}$$

where $\mathcal{W}_n(\alpha) = \sum_{k=1}^{n}\frac{\sin k\alpha}{k}$.

Proof. Let $n \geqslant 1$ be a fixed integer. The Poisson kernel has the expansion

$$\frac{1}{2\pi}\operatorname{Re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\} = \frac{1}{2\pi}\left(1 + \sum_{k=1}^{\infty} e^{-ikt}z^k + \sum_{k=1}^{\infty} e^{ikt}\overline{z^k}\right),$$

so the associated function l defined in (2) has the form

$$l(t) = \frac{1}{2\pi}\sum_{k=1}^{n}\left(e^{-ikt} + e^{ikt}\right) = \frac{1}{\pi}\sum_{k=1}^{n}\cos kt.$$

The basic idea of the proof is to maximize

$$\operatorname{Re}\{\mathcal{I}_n(f)\} = \frac{1}{\pi}\int_0^{2\pi}\left(\sum_{k=1}^{n}\cos kt\right)\cos\theta(t)\,dt \tag{5}$$

among all circle mappings θ. Note that

$$\operatorname{Im}\{l(t)e^{i\theta(t)}\} = \frac{1}{\pi}\left(\sum_{k=1}^{n}\cos kt\right)\sin\theta(t) = 0$$

on the vertical lines $t = \frac{(2s+1)\pi}{n+1}$, $t = \frac{2s\pi}{n}$, $s = 0, \pm 1, \pm 2, \ldots,$ and the horizontal lines $\theta(t) = s\pi$, $s = 0, \pm 1, \pm 2, \ldots.$

According to part (a) of Theorem 1 the extremal circle mapping θ is piecewise constant. In addition, property (b) says that if θ has consecutive discontinuities at points a and b, $0 \leqslant a < b < 2\pi$, then either $\theta(t) = s\pi$, $s = 0, \pm 1, \pm 2, \ldots,$ or $\theta(t)$ is constant for $a < t < b$ and

$$\int_a^b\left(\sum_{k=1}^{n}\cos kt\right)dt = 0.$$

On account of above remark the only intervals which make nonzero contributions to the integral (5) are those for which $\theta(t) = s\pi$.

We may assume without loss of generality that

$$\theta(t) = \begin{cases} 0, & 0 \leqslant t \leqslant a,\\ \text{piecewise constant}, & a < t \leqslant b,\\ \pi, & b < t \leqslant c,\\ \text{piecewise constant}, & c < t \leqslant d,\\ 2\pi, & d < t < 2\pi,\end{cases}$$

with possibility of degenerate cases (i.e., $a = 0$ or $a = b$, etc.). For a circle mapping of this type we have

$$\mathrm{Re}\{\mathcal{I}_n(f)\} = \frac{1}{\pi}\{\mathcal{W}_n(a) + \mathcal{W}_n(b) - \mathcal{W}_n(c) - \mathcal{W}_n(d)\} \leqslant \frac{4}{\pi}\mathcal{W}_n\left(\frac{\pi}{n+1}\right),$$

which is due to the fact that $\max_{\alpha\in(0,\pi)} \mathcal{W}_n(\alpha) = \mathcal{W}_n(\frac{\pi}{n+1})$ (see [8, Chapter VI, Examples 23 and 24]), $\mathcal{W}_n(\alpha) > 0$ for $\alpha \in (0, \pi)$ (see [5]), and $\mathcal{W}_n(2\pi - \alpha) = -\mathcal{W}_n(\alpha)$.

A trivial verification shows that the extremal value $\mathrm{Re}\{\mathcal{I}_n(f)\} = \frac{4}{\pi}\mathcal{W}_n(\frac{\pi}{n+1})$ is attained for the circle mapping

$$\theta(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \frac{\pi}{n+1}, \\ \pi, & \frac{\pi}{n+1} < t \leqslant 2\pi - \frac{\pi}{n+1}, \\ 2\pi, & 2\pi - \frac{\pi}{n+1} < t < 2\pi. \end{cases}$$

It follows immediately that the extremal mapping f is not from the family $\mathrm{Aut}_H(\mathbb{D})$. □

As we see, the proof of Theorem 2 is closely related to the properties of the trigonometric sums $\mathcal{W}_n(\alpha)$. Recently, it was shown in [1] that for $\alpha \in (0, \pi)$ we have

$$\mathcal{W}_n(\alpha) \leqslant \lambda(\pi - \alpha), \quad n = 1, 2, \ldots,$$

with the best possible constant factor

$$\lambda = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ 0.66395\ldots, & \text{if } n \text{ is even}. \end{cases}$$

Combining these inequalities with (4) we can assert that if $n \geqslant 1$ is an odd integer, then $\mathrm{Re}\{\mathcal{I}_n(f)\} < 4$ and if $n \geqslant 2$ is an even integer then $\mathrm{Re}\{\mathcal{I}_n(f)\} < 2.6558\ldots$. Those upper bounds can be improve by another way.

Lemma 1. *If $f \in \mathrm{Aut}_H(\mathbb{D})$ has the Fourier expansion* (3)*, then*

$$\mathrm{Re}\{\mathcal{I}_n(f)\} < \frac{4}{\pi}\int_0^\pi \frac{\sin\tau}{\tau}\,d\tau \approx 2.3579\ldots, \quad n = 1, 2, \ldots.$$

Proof. We first observe that the sequence $\{\mathcal{W}_n(\frac{\pi}{n+1})\colon n \in \mathbb{N}\}$ is increasing (see [8, Chapter VI, Example 25]) and moreover

$$\lim_{n\to\infty} \mathcal{W}_n\left(\frac{\pi}{n+1}\right) = \int_0^\pi \frac{\sin\tau}{\tau}\,d\tau$$

(see [7, Chapter II, Example 6]). By the above and Theorem 2 the proof is complete. □

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