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GAMES IN PHILOSOPHICAL LOGIC*

Semantic games are an important evaluation method for a wide range of logical languages, and are frequently resorted to when traditional methods do not easily apply. A case in point is a family of independence-friendly (IF) logics, allowing regulation over information flow in formulas, amounting to the failure of perfect information in semantic games associated with IF formulas, and giving rise to informationally independent logical components. The mechanism of informational independence is studied in this paper. For example, we note that the imperfect information of players is often accompanied by the game-theoretic phenomenon of imperfect recall. We reply to a couple of misunderstandings that have occurred in the literature concerning the relation of IF first-order logic and game-theoretic semantics, intuitionism, constructivism, truth-definitions, negation, mathematical prose, and the status of set theory. By straightening out these misunderstandings, we also hope to show, at least partially, the importance semantic games and IF logics have in philosophical logic.

1.

It is an unmistakable truth that the method of games in logical studies has not yet been exploited in philosophical logic to the extent it deserves, although the game approach to meaning and truth has been a valuable resource to logicians for a long time, since Hintikka's introduction of game-theoretic semantics (with the roots in Peirce's interpretation of quantifiers, as shown in Hilpinen 1983). One of the original contributions of game-theoretic semantics was in delivering an

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alternative evaluation method to that of Tarski semantics for first-order logic, and by the same token providing insights into the meaning of logical concepts. From the technical perspective, a game toolkit has frequently been resorted to when traditional evaluation methods do not easily apply, as is the case of infinite languages or logics of imperfect information.

Nonetheless, games provide a polyergic set of tools for various areas of logic. In this paper, we furnish traditional game-theoretic assets in logical semantics by presenting applications relevant to philosophical investigations. In particular, we explain how game-theoretic semantics works for the family of *independence-friendly* (IF) logics of Hintikka (1996b) and Hintikka and Sandu (1997), which are logics evaluated by means of semantic games of imperfect information. We notice that the mechanism of imperfect information is fairly delicate, since it allows, among other things, the modelling of 'forgetting' (imperfect recall) in a logical manner, and also because it accounts for non-transitive dependencies between logical components. Also, we describe how the self-applied truth-predicate can be defined in IF logic, and address questions concerning intuitionism, constructivism, truth-definitions, negation, mathematical prose, as well as relations of IF logic to set theory. By straightening out some mistaken interpretations and distorted facts that have arisen in the literature concerning IF logics and game-theoretic semantics, we hope to demonstrate the importance games have in philosophical logic.

In the theory of semantic games, formulas of first-order logic are evaluated according to the rules prompted by logical ingredients, starting with the out-most component and proceeding outside in. Let $\mathcal{L}$ be language of first-order logic for a signature τ. We can define a game as a tuple $G = \langle P, \mathcal{R}, O \rangle$, where $P = \langle V, F \rangle$ is a tuple of players V (the verifier) and F (the falsifier), $\mathcal{R}$ is a set of well-defined rules, and O is a set of well-defined positions. A model for $\mathcal{L}$ is an ordered tuple $\langle \mathfrak{A}, g \rangle$, where $\mathfrak{A}$ is a τ-structure of the nonempty domain $|\mathfrak{A}|$ on which the game is being played, and g is an assignment $g \colon \mathcal{T} \to |\mathfrak{A}|$ from terms $\mathcal{T}$ to the domain of a structure. An assignment g in $\mathfrak{A}$ is restricted to the free variables of every formula of $\mathcal{L}$. If φ is a sentence, the assignment symbol can be omitted from the definition of a game.

The purpose of the players in a game is that F is trying to falsify the formula (i.e. to show that it is false in $\mathfrak{A}$) and V is trying to verify it (i.e. to show that it is true in $\mathfrak{A}$). For the sake of simplicity, it is assumed, without loss of generality, that $\mathcal{L}$ does not contain $\to$ (conditional) or $\leftrightarrow$ (biconditional), and that all formulas are in a negation normal form (i.e. all negation signs immediately precede the atomic formulas). The logical ingredients $\forall$ (universal quantifier) and $\wedge$ (conjunction) prompt a move by F, and $\exists$ (existential quantifier) and $\vee$ (disjunction) prompt a move by V. When the players come across $\sim$ (negation), they change roles, that is, V becomes F and F becomes V. Each application of a game rule reduces the complexity of a formula and hence an atomic formula is finally reached. The truth-value of an atomic formula (established by a given interpretation) determines which player wins the game.

Let $\mathcal{L}$ be a standard first-order language whose logical vocabulary has $\vee, \wedge, \sim, \exists$ and $\forall$, and let $\varphi[x/a]$ be a substitution of the occurrences of x in φ with a. A strictly competitive non-cooperative game $G(\varphi, g, \mathfrak{A})$ is defined by induction on the complexity of each formula $\varphi \in \mathcal{L}$ between two players, V and F by the following six rules in $\mathcal{R}$, where the antecedents are called the inputs I of each rule $r_i \in \mathcal{R}$, and the consequents are the outputs of $r_i \in \mathcal{R}$:

r_1. If $\varphi = \sim\psi$, V and F change roles, and the next choice is in $G(\psi, g, \mathfrak{A})$.

r_2. If $\varphi = (\varphi_1 \vee \varphi_2)$, V chooses $i \in \{1, 2\}$, and the next choice is in $G(\varphi_i, g, \mathfrak{A})$.

r_3. If $\varphi = (\varphi_1 \wedge \varphi_2)$, F chooses $i \in \{1, 2\}$, and the next choice is in $G(\varphi_i, g, \mathfrak{A})$.

r_4. If $\varphi = \exists x\psi x$, V chooses an individual of the domain of the structure $\mathfrak{A}$, e.g. a, and the next choice is in $G(\psi[x/a], g, \mathfrak{A})$.

r_5. If $\varphi = \forall x\psi x$, F chooses an individual of the domain of the structure $\mathfrak{A}$, e.g. a, and the next choice is in $G(\psi[x/a], g, \mathfrak{A})$.

r_6. If φ is atomic, the game ends, and V wins if φ is true, and F wins if φ is false.

Let $\mathrm{Sub}(\varphi)$ be an inductively defined set of sub-formulas of φ. A strategy for each player $P_i \in P$ in a game G for φ is a function $f\colon |\mathfrak{A}| \cap c(O) \to s(I)$, where $c\colon O \to \{V, F\}$ is a player function assigning to each $O = \mathrm{Sub}(\varphi)$ a player P_i, and $s\colon I \to |\mathfrak{A}| \cup \{1, 2\} \cup c(O)$ gives an output of an application of each rule $r_i \in \mathcal{R}$, given an input I of each rule, which is an element of the domain, a value in $\{1, 2\}$, or an instruction to change roles at O. A winning strategy is a strategy f by which a player can make operational choices such that every play results a win for him or her, no matter how the opponent chooses.

Let $\mathfrak{A} \models_{\mathrm{GTS}}$ denote truth under a game-theoretic evaluation. A formula φ is true in $\mathfrak{A}$, i.e. $\mathfrak{A} \models_{\mathrm{GTS}} \varphi^+$, iff there exists a winning strategy f for the initial V, and false in $\mathfrak{A}$, i.e. $\mathfrak{A} \models_{\text{GTS}} \varphi^-$, iff there exists a winning strategy f for the initial F.

These are the traditional game-theoretic definitions that give meanings to the formulas of $\mathcal{L}$, similar to Hintikka 1973 (quantifier clauses go back to Henkin 1961). Assuming the axiom of choice, game-theoretic semantics agrees with the classical Tarskian notion of truth (see Hodges 1983, p. 94). It follows that both players cannot simultaneously have winning strategies in a game for any $\varphi \in \mathcal{L}$.

There is a Skolem normal form theorem which says that every $\varphi \in \mathcal{L}$ is equisatisfiable (satisfiable in the same models) with the existential second-order formula Σ^1_1 of the form:

(1) $$\exists f_1 \ldots \exists f_m \forall x_1 \ldots \forall x_n \, \psi,$$

where $f_1 \ldots f_m, m \in \omega$ are new function symbols and ψ is a quantifier-free formula. Moreover, a Skolem normal form can effectively be found for every first-order sentence (which is put in a prenex normal form). For instance, one can convert the sentence $\forall x \exists y \, Sxy$ to a Skolem normal form $\exists f \forall x \, Sxf(x)$ which states the existence of a winning strategy for V. By the Skolem normal form theorem, it follows that $\exists f \forall x \, Sxf(x)$ iff $\forall x \exists y \, Sxy$.

A special form of the Skolem normal form theorem can be used in skolemising disjunctions, the only addition being that one can conjoin each disjunct with a Skolem function that has values in $\{1, 2\}$. For example, let $\varphi = \forall x \exists y \forall z (S_1xyz \vee S_2xy)$, where S_1xyz and S_2xy are atomic. Then it can be skolemised to $\exists f g_1 g_2 \forall x \forall z ((S_1xf(x) \wedge g_1(x, z) = 1) \vee (S_2xf(x) \wedge g_2(x, z) = 2))$.

If the language does not contain function symbols, the existence of winning strategies can still be stated using, instead of function symbols, special relation symbols taken to represent functions.

Now that one has a rigorous definition of a game, in a sense that it gives both the defining rules of a game as well as the strategies, the natural question to ask is what further properties do semantic games have? Which class of games corresponds to which logical language? Should the games be of perfect information?

The negative answer to the last question is a significant possibility in logic. As so often in the theory of games, which concerns human activities in situations involving decision-making and action, there is no pre-theoretic reason to assume that games should have perfect information. In real communities, no one has perfect information. Indeed, the failure of the perfectness assumption is one of the principle motivations for the development and study of IF logics.

Semantic games can be represented in an extensive form of games, which essentially is a tree structure with labelled edges as choices and nodes as current positions in a game. The non-terminal nodes (decision nodes) are partitioned into information sets $S_i^{P_j}$ (equivalence relations) for each player P_j, which represent information of players such that for any nodes at the same depth within one information set, a respective player cannot distinguish between these nodes, and hence does not know everything that has happened earlier in a game. Therefore, the non-singleton information sets provide a semantic method of representing imperfect information.

We define a game tree GT (together with the usual underlying components of a basic tree structure) for n-person game G on φ as a finite partially ordered set of well-defined moves as follows:

1. the distinguished node $v_0 \in \mathcal{V}$ is associated with the formula φ;
2. each node $v \in \mathcal{V} \backslash \{v_0\}$ is associated with a proper $\mathrm{Sub}(\varphi)$ (*i.e.* $\neq \varphi$);
3. all non-terminal decision nodes N of GT are partitioned into $|P|$ sets $S_1 \ldots S_{|P|}$, called the member sets;
4. for each member $P_j \in P$, S_i is sub-partitioned into subsets $S_i^{P_j}$, called information sets;
5. if the formulas $\psi = \mathrm{Sub}(\varphi)$ and $\eta = \mathrm{Sub}(\psi)$ at nodes $v, v' \in N$ lie on the same directed path, then $\psi \in S_i^{P_j}$ and $\eta \in S_l^{P_k}$ for any $S_i^{P_j} \neq S_l^{P_k}$.
6. for all $v, v' \in N$ in $S_i^{P_j}$, the number of predecessors of v is equal to the number of predecessors of v'.

A play of a game G is a maximal set of choices in a path, and the choices not in a play are alternatives to a choice at some $v \in N$.

We have defined games in a non-repetitive von Neumann–Morgenstern extensive form. Condition 5 ensures that the definition satisfies a *non-repetition hypothesis* (or ‘non-absentmindedness’): each node in the same information set may be visited at most once. This establishes that the information for players is persistent, that is, players have perfect memories in the sense that they do not forget information by confusing their location in GT, for otherwise such information sets would amount to ambiguities in interpreting games. Condition 6 states the ‘von Neumann–Morgenstern’ condition, namely that one information set includes nodes at the same depth only, which is a natural constraint in semantic games, since logical languages do not discern imperfect information modulo particular choices only, but rather work with whole moves (sets of choices). This is a more general stricture than non-repetition, in that also nodes in the different paths are not allowed to be contained in the same information set at varying depths in a game tree. Consequently, information sets always are assumed to be 'stratified,' i.e. they come in layers, each set including the nodes at a particular depth only. Games which are not in the form of this definition can sometimes be translated to it through a series of 'information-preserving' (and allegedly 'strategy-preserving') translations known as Thompson transformations (see Thompson 1952, Elmes and Reny 1994, cf. Osborne and Rubinstein 1994).

The non-singleton information sets provide us with a semantic method of representing imperfect information, which syntactically can be denoted in the family of IF logics by means of the Hintikka-Sandu slash operator "/". Its intuitive meaning is that the components occurring at the left-hand side of the slash are to be evaluated independently of the evaluation of the components at the right-hand side of the slash. For one thing, this notation allows us to express Henkin (branching) quantifiers in a linear one-row format.

Let the sign $\sim$ denote a game-negation, which is the negation to which the game rule r_1 as a role-swap applies. One rather general IF first-order logic is obtained as follows. Let $\varphi \in \mathcal{L}$ be a well-formed first-order formula in a negation normal form, and let $Qx\psi, Q \in \{\forall, \exists\}$ and $(\phi \diamond \psi), \diamond \in \{\wedge, \vee\}$ be Sub(φ) which are in the scope of $Q_1x_1 \ldots Q_nx_n$. Let $A = \{x_1 \ldots x_n\}$. Then the language $\mathcal{L}^*$ is formed as follows:

– *If $B \subseteq A$, then $(Qx/B)\psi$ and $(\phi(\diamond/B)\psi)$ are well-formed formulas of $\mathcal{L}^*$.*

We customarily write $\{x_1 \ldots x_n\}$ as $x_1 \ldots x_n$. For example, $\forall x(\exists y/x)$ Sxy, $\exists x\,(Sx(\vee/x)Sx)$ and $\sim\forall x_1 \ldots \forall x_n \sim(\exists y/x_1 \ldots x_n)\, Sx_1 \ldots x_n y$ are well-formed formulas of $\mathcal{L}^*$. To see an example of how game-theoretic semantics works, the evaluation of the formula $\forall x\exists y\forall z(\exists w/xy)\, Sxyzw$ would proceed such that first, F chooses an individual for x, say a, then V chooses an individual for y, say b, then F chooses an individual for z, say c, and when $\exists w$ finally is reached, the choice for w by V, say d, is done without her having the information of the previously made choices for x or y. The formula is true iff $\exists f\exists g\forall x\forall z\, Sxyf(x)g(y)$.

The formulas of these languages correspond to a Henkin quantifier introduced in Henkin 1961 as follows.

$$\forall x\exists y(\forall z/xy)(\exists w/xy)\, Sxyzw \quad \text{iff} \quad \begin{matrix}\forall x\exists y\\ \forall z\exists w\end{matrix} Sxyzw.$$

Let $\begin{matrix}\forall x\exists y\\ \forall z\exists w\end{matrix}$ be the Henkin prefix H. In Krynicki 1993, qualifying that of Blass and Gurevich (1986) and Walkoe (1970), it is shown that each complex Henkin quantifier prefix H can be defined by a quantifier prefix H^* of the following bifurcated form:

$$\begin{matrix} \forall x_1 \ldots x_n \, \exists y \\ \forall z_1 \ldots z_m \, \exists w \end{matrix} \quad \text{(for some } n, m \in \omega\text{).} \tag{2}$$

We call sentences whose prefixes are put into this form *Krynicki normal form* for Henkin quantifiers. The formulas in a Krynicki normal form can be defined in $\mathcal{L}^*$ as follows:

$$\forall x_1 \ldots x_n \exists y (\forall z_1 \ldots z_m / x_1 \ldots x_n y)(\exists w / x_1 \ldots x_n y) S x_1 \ldots x_n y z_1 \ldots z_m w$$

$$\text{iff}$$

$$H^* \, S x_1 \ldots x_n y z_1 \ldots z_m w.$$

All slashed connectives can also be defined using quantifier prefixes H^*, since the partially ordered connectives of Sandu and Väänänen (1992) can be defined using the normal form prefixes (2) (cf. Krynicki 1993). Therefore, partially ordered connectives do not generalise the language $\mathcal{L}^*$ over the language $\mathcal{L}_H$ with all Henkin quantifiers.

However, in a certain important sense $\mathcal{L}^*$ is more general than $\mathcal{L}_H$. One might not be able to define a formula $\forall x \exists y (\exists z / x) \, Sxyz$ in $\mathcal{L}_H$, for example. This is because the dependencies can be non-transitive. To understand what such quantifiers mean it is instructive to adopt a game-theoretic perspective and view the evaluation of the formulas as a flow of information from one logical component to another. For these non-transitive formulas can be evaluated using semantic games as before, the only difference being that information at $\forall x$ is not passed down to $(\exists z / x)$ through the mediating quantifier $\exists y$, although $(\exists z / x)$ can be dependent on $\exists y$. Consequently, this quantifier prefix is an example of a more general situation where non-transitive information flow is not captured by just resorting to finite partially ordered quantifiers or connectives.

We have arrived at a situation where one can logically capture the cases where information flows non-transitively. These dependencies can be expressed in extensive form games by lifting information sets for the corresponding player up in a game tree, for instance. There are other possibilities as well, but this dynamic lifting has a natural appeal as bringing out the real dynamics of information in games, as well as illustrating the general idea that, contrary to a common idea in game theory, the two notions of the player *having* an information set and the same player *moving* in that information set do not have to coincide (for example in game-theoretically interpreted modal logics this dynamic way of modelling players' information is almost inevitable). Now, to keep track of players and associated information sets, one can use indexing for players and respective information sets to prevent confusion, and our definition of games already allows this method. For instance, to express non-transitive information flow for $\forall x \exists y (\exists z/x)\, Sxyz$, the nodes $v \in N$ at the depth two in GT are included into the same information set $S_1^{V_2}$ for V_2.

In allowing completely unrestricted possibilities of representing information flow, IF logic becomes equipped with a way of capturing the phenomenon of forgetting, or imperfect recall, as game theorists would prefer to say. This phenomenon naturally follows from the definition of semantic games of imperfect information, and can game-theoretically be accounted for by occasionally viewing players as teams of players or 'multiple-selves' of a single player, where new members of a team or new selves become responsible for the decisions in the course of the game as the need arises. The team approach is by far the most common and natural way of explaining game-theoretic forgetting, and is spontaneously resorted to in a number of game- and decision-theoretic problems (cf. Rubinstein 1998).

The phenomenon of multiple players puts the games here broadly within Team Theory, which sees teams as groups of agents with identical interests but individual actions and individual information (see Kim and Roush 1987). Strategies are based on previous information in a game, but not on information the other members of the team might have. A connection is provided by a result which states that basic solution concepts of two-person zero-sum games hold also for games played by teams (Ho and Sun 1974). Also, there are links between this kind of imperfect information and NP-complete problems, shown by the Tsitsiklis–Athans theorem, stating that the problem of finding a team strategy that guarantees a certain minimum for finite 2×2 tables includes the Hamiltonian circuit problem and is NP-complete (Tsitsiklis and Athans 1984). And as it is known (Hella and Sandu 1995), the Hamiltonian circuit problem is expressible in IF first-order logic.

Since we have imposed a non-repetition hypothesis in the definition of games for IF logics, no ambiguities in the interpretation of game trees arise. Without the hypothesis, individual members inside one player can also have imperfect recall, but in this case we run the risk of confusing their location in a game tree altogether, for the information sets may contain more than one decision point in one path, and if so, there is no way of distinguishing which node a player has reached (on these interpretational ambiguities and the ensuing 'paradox of the absent-minded driver,' see Piccione and Rubinstein 1997). The team approach says that at the level of individual players the information is persistent and the players do not forget information, but the two principle players, observed as a whole, are seen to forget things. One can think of an implicit map from the 'information set' $\mathfrak{S} = \bigcup_i S_i^{P_j}$ containing all the information sets of the respective player to the information sets of members; in this way V and F can coordinate the individual players.

Further, the members are not allowed to communicate with each other, since this easily destroys the team's possibility, when regarded as one imperfect recall player, of forgetting information. On the one hand, when making a single choice a member does not have information about the choices other members have made earlier. On the other hand, there are certain minimal requirements as to what members are supposed to know. It is assumed that even under massive information hiding members know their own location in a game, for otherwise the interpretation of a game becomes ambiguous. In other words, members always know when it is their turn to move. Also, players have information about the existence of other members of the team and that new members can be recruited to take part in a game whenever called for. The latter piece of information is not essential for the general outcome of the game, it only makes the understanding of the positions of members easier.

From a slightly different perspective, we can think of players as playing the roles of all of the members, one at the time. When a subformula has a first component associated with a member of V or F, the player takes the role of a single member. As it happens, she or he is seen to forget information, since the players are not, during a particular turn, allowed to use the information available to the other members of the team.

In general, however, extensive forms are capable of representing some notions which the syntax of IF logic currently cannot handle. By including only some subset of the nodes at a certain depth into one information set, one is able to express situations where moves may become independent on some previous choices. Logical syntax, however, itself tells which moves depend on which other moves. It cannot directly describe all the situations possible in extensive forms. Perhaps it can be extended to cover these situations, describing 'intensional' information flow in the form of conditional statements such as "if F chose a for x, then V doesn't know what V chose for y, but for any other choice for x, she has perfect information" or "if F chose a for x and b for y, or b for x and a for y, then V does not know what F

has been chosen; otherwise perfect information." These kinds of game situations are conditioned on the individual choices previously made in the game, not on the total moves including all the alternatives.

We could also consider some further extensions of extensive forms and ask whether they are meaningful in logic. For example, some solution concepts have been found for extensive forms with additional cycles, i.e. cases where a player can move up in a game tree (Alpern 1991). This would correspond to the situation where one has more dependence than in classical first-order logic. One could say that the dependence or information flow can become bidirectional, whereas in received logics it can only be unidirectional. Interpretations for such dependencies on some logical constants residing deeper in a formula in a scope of the constant under evaluation are realistic, as illustrated by the fact that at least some special forms of pure strategies can be extracted from corresponding cycles in extensive forms. If dependencies in a cycle are non-transitive, one can even avoid infinite plays and evaluate formulas using a concurrent model of a game, where n games corresponding to n logical components in a cycle are played in a truly parallel fashion, the underlying game structure being a partial order of the positions. There is a series of potential developments largely unexploited in the informational approach to logic: from linear to partial to non-transitive to cyclic orders. One can even find a natural motivation for novel types of games in non-cooperative evolutional game theory, for instance, where so-called hypercycles are considered (Maynard Smith 1979).

2.

One interesting philosophical application of IF logic concerns its extensions to intensional notions. Using tools of epistemic logic with similar independence requirements between modal operators and quantifiers and connectives, for instance, one can see why the *de dicto* versus *de re* contrast has created unnecessary complications in philosophy (cf. Hintikka 1996a). Using intensional logic of imperfect information, one can capture the notion of intentional identity and resolve the problem of intentional identities without any need for resorting to extra-logical belief objects or discourse referents. Apart from providing representational frameworks for several non-linear concepts and natural language expressions, potential applications of IF intensional logics include information flow in communicating systems, knowledge representation in multi-agent architectures, and semantics for the fragments of natural language in general.

These non-linear IF intensional languages and their connections with natural language statements appear to give a new set of headaches for advocates of compositional approaches of language. Should they come up with an adequate compositional interpretation, they would need to delve into considerable tinkering and higher-order daedalion (such as using sets of subsets of possible worlds) in devising adequate compositional semantics for IF intensional languages.

The issue on compositionality indeed needs careful attention even in IF first-order logic. Hintikka (1996b, p. 112) says that IF languages cannot have a compositional truth-conditional semantics, even though no strict impossibility proof has been given to that effect. But in Tennant 1998, this statement has been criticised for not offering a "precise and general characterization of what it would mean to say that a semantics was compositional", and that a proof of Hintikka's claim "would require a strict definition of compositionality in advance, and an argument to the effect that it covered all possible forms of compositionality" (p. 97).

There may not exist one unanimous notion of a compositional interpretation of a language, but for instance the works of Hodges (1998), Partee (1984) and Pelletier (1994) aim at providing unambiguous clarifications of this notion. In the context of IF logic, compositionality could mean that independent quantifiers cannot be given a compositional interpretation in terms of the standard quantifiers $\forall x$ and $\exists x$ only. This is a natural requirement, given that the ingredients of which the independent quantifiers are built up consist of the two standard quantifiers. Under this assumption, Hintikka's claim has been given a strict impossibility proof in Cameron and Hodges 1999: it is shown that no compositional semantics is possible for IF languages on a first-order level, in a sense that the interpretation for any IF formula with one free variable would consist of a subset of the domain of a structure. To have compositional semantics, one has to introduce and apply the concept of a power set.

Various compositional semantics for IF logics have been designed (see Hodges 1997a, 1997c, 1999, Caicedo and Krynicki 1999, Väänänen 1999, Pietarinen 2000). There was no reason to expect otherwise, since the syntax of IF logic is finitely generated. The reason why it is still worth insisting on non-compositionality is that non-compositional semantics by means of games of imperfect information may turn out to be simpler and easier to apprehend than compositional ones; not to mention the philosophical virtues of game-theoretic semantics such as its being a representation of the human activities of seeking and finding connected with quantifiers, decision-making at connectives, and a responsibility-shift at game negation. In this sense, game-theoretic semantics codifies to a considerable extent the actual practices of what it means to verify and falsify sentences, practices which are also reflected in various ways in natural and mathematical languages.

In the realm of epistemic logic, the game-theoretic perspective is important also in the sense that it fills in the gap that is left in the research program of bringing epistemic logic and game theory closer to each other (Bacharach et al. 1997), for this project has almost unilaterally neglected the side of the convergence which begins with the game-theoretic interpretation of epistemic notions.

The usual framework for the logically oriented game-theorist is propositional epistemic logic. Nonetheless, imperfect information could make sense already at this level, so one does not need to look away from IF logics for having unnecessarily strong expressive resources for the purposes of game theory.

The simplest such propositional formula would perhaps be $K_1(K_2/K_1)\varphi$ for two agents, 1 and 2. Intuitively, this formula says that "1 knows that φ," and that "2 knows that φ," but there is no nesting, that is, 1 does not know that 2 knows that φ. Therefore, it seems that this formula is only a more compact version of the conjugated formula $K_1\varphi \wedge K_2\varphi$, and other formulas could be rewritten in a similar manner. One can ask, however, where the extra conjunctions come from, for the former represents only one proposition being in the focus of two agents. Are we dealing with an independent reading?

One could argue that the former formula is best read as "1 and 2 know that φ," but that it does not mean the same as the distributive reading "1 knows that φ and 2 knows that φ." These readings really are different, for the former can be seen to answer the question "who knows that φ?," whereas the latter is not as good an answer to the same question. For example, if one asks "who knows that it will be rainy tomorrow?" it can legitimately be answered that "1 and 2 know that it will be rainy tomorrow," but if an answer is "1 knows that it will be rainy tomorrow and 2 knows that it will be rainy tomorrow," the answerer, after asserting the first conjunct, is expected to assert something else than just the tautologous "1 knows it will be rainy tomorrow", thus giving, if not totally false, at least a slightly misleading impression about the intended meaning of the sentence.

Consider next a formula $K_1(\varphi \wedge (K_2/K_1)\psi)$. One can likewise try to rewrite this as $K_1(\varphi \wedge \psi) \wedge K_2\psi$. So what do we gain by asserting the former version? Again, a case can be made that the first formula is to be read as, say, "1 knows that it will be rainy tomorrow and 1 and 2 know that it will be cold tomorrow," and the latter as "1 knows that it will be rainy and cold tomorrow and 2 knows that it will be cold tomorrow." Now, if one is to ask "who knows that it will rain and be cold tomorrow?" the latter seems to be a better answer for this question than the former, concordant with the earlier example.

An extension of propositional epistemic logic to cover imperfect information in connectives is also feasible. For example, one can have sentences such as $K_1(\varphi(\vee/K_1)\psi)$. Now, $K_1(\varphi(\vee/K_1)\psi)$ obviously is equivalent to $K_1\varphi \vee K_1\psi$. These can both be read as "1 knows that it will be rainy tomorrow or 1 knows that it will be cold tomorrow." However, this means that "1 knows whether it will be rainy or cold tomorrow." Consequently, the mechanism of imperfect information seems to provide a formulation for *knowing whether* locutions.

More complex formulas, such as $K_1K_2(\varphi(\vee/K_1)\psi)$, do not generally reduce to linear propositional epistemic logic. One cannot rewrite $K_1K_2(\varphi(\vee/K_1)\psi)$ such that 2 knows $\varphi \vee \psi$, and 1 knows φ or 1 knows ψ such that the proper nesting K_1K_2 will be preserved. This formula would mean something like "1 knows whether it will be rainy or cold tomorrow, and that 2 knows that it will be rainy or cold tomorrow."

There may be fewer clear-cut examples of a type of knowledge with regard to conjunction, although the same distributiveness is seen to hold, e.g. $K_1(\varphi(\wedge/K_1)\psi)$ reduces to $K_1\varphi \wedge K_1\psi$. Without the slash, the latter sentence obviously can mean that "1 knows that one should not drink and drive," but it does not mean the same as "1 knows that one should not drink and 1 knows that one should not drive," for the agent may fail to conjoin the two propositions which, taken together, would mean something else.

These short remarks suggest that by means of IF logic one can make subtle distinctions in symbolising natural language statements. A logical example is provided by an observation that one can measure the expressive power of logic by means of translating the extensive form games into possible world structures, and then explore the possibilities of having standard translation of IF logic to some suitable background language.

One should also pay attention to possible restricted sub-fragments of fragments of $\mathcal{L}^*$, much in the same manner as one considers guarded fragments of first-order logic. In such guarded fragments of $\mathcal{L}^*$, it would make a difference whether a guarded formula $\exists\bar{y}(R\bar{x}\bar{y} \wedge \psi\bar{x}\bar{y})$ ($\bar{x}, \bar{y}$ are finite sequences of variables) exists in the scope of another guarded formula, such as $\forall\bar{z}(R\bar{u}\bar{z} \wedge \psi\bar{u}\bar{z})$. However, if variables $\bar{y}_1$ in $(\exists\bar{y}/\bar{y}_1)(R\bar{x}\bar{y} \wedge \psi\bar{x}\bar{y})$ do not occur in the matrix, the presence of imperfect information is perfectly admissible.

3.

Over and above its traditional counterpart, one of the advantages of IF first-order logic is that the notions of truth, falsity and logical equivalence become qualified, together with some interesting consequences. Especially the ordinary skolemisation process may not remain a sufficiently illuminating method for the general situation, for in game-theoretic terms skolemisation spells out the set of strategies for V, but it does not say anything about F's strategies, nor about which individuals are chosen. It only says that the original formula and its skolemised form are equisatisfiable. From the game-theoretic viewpoint, there is no reason to stop here. Also F's strategies should be incorporated into truth-conditions. To give an example of the resulting situation, consider $\forall x(\exists y/x)\, Sxy$ which would no longer be logically (strongly) equivalent with $\exists y \forall x\, Sxy$, but with $\exists y(\forall x/y)\, Sxy$, which expresses the fact that also F is ignorant of the choice made by V. Likewise, $\forall x(\exists y/x)\, Sxy$ is weakly equivalent with $\exists y \forall x\, Sxy$ (true in the same class of models), since in the latter there exists a strategy for F with one argument-place, which may be the winning one.

Therefore, IF logic provides different ways of *construction verborum* for different applications. One can ignore falsifier's strategies if one is primarily concerned with the usual notions of logical equivalence or consequence, for the Skolem normal form theorem says that in order to resolve the truth of the first-order sentence one only needs to know which existential quantifiers depend on which universal quantifiers, and this kind of dependence is brought out by the skolemisation process. Yet IF logic can account for more subtle equivalences.

By the same token, IF logic resembles the Strong-Kleene valuation schema in an interesting manner. It captures reasoning with different aspects of partial phenomena (the failure of perfect information), the difference to the partiality in the cases like Kleene being that partiality does not arise any longer at the level of non-logical constants (functions, predicates and relations) but at the level of logical ones (quantifiers and connectives). However, as shown in Sandu 1996, the resulting valuation schema for the connectives of IF logic agrees with the Strong-Kleene schema.

In addition to game negation $\sim$, one can introduce a weak, contradictory negation $\neg$ which cannot be captured by game rules. A simple example of the case that the failure of the law of excluded middle for a formula ψ of IF logic does not guarantee that its contradictory, classical negation $\neg\psi$ is not expressible in IF logic is given by considering $\forall x(\exists y/x)\,(x = y)$, which is not determined in a structure $\mathfrak{A}$ with two or more elements. Yet it is weakly equivalent to $\exists y \forall x\,(x = y)$, whose classical negation $\forall y \exists x \neg (x = y)$ is true in that structure. The general phenomenon here is that the presence of classical negation $\neg$ makes (*i*) the otherwise non-determined games determined, since from not ($\mathfrak{A} \models \psi^+$ and $\mathfrak{A} \models \psi^-$) it follows that not (not $\mathfrak{A} \models \neg\psi^+$ and not $\mathfrak{A} \models \neg\psi^-$), and (*ii*) the otherwise strictly competitive games non-strictly competitive games, since it can happen that both $\mathfrak{A} \models \psi^+$ and $\mathfrak{A} \models \psi^-$ (see Sandu and Pietarinen 2000).

In sum, IF logic constitutes, in a strong sense, the closing chapter in the logic of partiality which has been awaited for some time. For example, already Blamey (1986) and Langholm (1988) pointed out the need for a logic where partiality arises at the level of logical constants. In IF logic, we have got one.

Further evidence for the behaviour of negation can be gathered from natural language. Although negation in natural language can express classical negation, only having it is not enough in understanding the behaviour of negative constructions in general. Classical negation satisfies the usual laws of de Morgan (anti-addition), multiplication, and complementation. The classical negation can be found, for example, in sentences where the noun phrases are complete in a sense that they are closed with regard to complementation: if some representatives of a phrase do not belong to a range of quantification, their negations do so. As an example, consider (3) with a monotonically increasing determiner *some*:

(3) It is not the case that some player won.
$\not\rightarrow$
Some player didn't win.

Here one can think of a situation where one player lost and one won the game, making the consequent true and the antecedent false. Consequently, the negation in the consequent is not classical, but may rather satisfy weaker laws than the previous three and thus be stronger.

Examples of non-classical negations are *no N*, *none of the N* and *not a single*, which all are anti-additive, but not multiplicative or closed under complementation. To see this, consider (4), which violates multiplication:

(4) No student works hard and fails the examination.
$\not\rightarrow$
No student works hard and no student fails the examination.

These are just fairly cursory examples of possible relations between game negations and weaker-than-classical negations in natural language, and further work is needed to connect the properties of game-negation with properties of negative constructions existing in natural language.

4.

In Sandu 1996, an effective method is given which transforms any IF formula prefixed by negation into one in negation normal form. The method is analogous to the ordinary method used in first-order logic. To take an easy example, consider the following two IF formulas (5) and (6) (which in Sandu 1996 are denoted by the right-hand side expressions):

(5) $$\forall x \exists y \forall z (\exists w/x)\, \varphi = G^{1,1}_{1,1} xyzw\, \varphi$$

(6) $$\exists x \forall y \exists z (\forall w/x)\, \varphi = G^{*1,1}_{1,1} xyzw\, \varphi$$

In Definition 3 of Sandu 1996, p. 257, the truth-preserving mapping * and the falsity-preserving mapping $^\#$ on IF formulas are defined as follows (among other things):

$$\begin{aligned}
\varphi^* &= \varphi \\
\varphi^\# &= \sim\varphi, \quad \text{for } \varphi \text{ atomic} \\
(\sim\varphi)^* &= \varphi^\# \\
(\sim\varphi)^\# &= \varphi^* \\
(G^{1,1}_{1,1} xyzw\, \varphi)^* &= G^{1,1}_{1,1} xyzw\, \varphi^* \\
(G^{1,1}_{1,1} xyzw\, \varphi)^\# &= G^{*1,1}_{1,1} xyzw\, \varphi^\# \\
(G^{*1,1}_{1,1} xyzw\, \varphi)^* &= G^{*1,1}_{1,1} xyzw\, \varphi^* \\
(G^{*1,1}_{1,1} xyzw\, \varphi)^\# &= G^{*1,1}_{1,1} xyzw\, \varphi^\#.
\end{aligned}$$

We then have the following proposition:

Proposition 1. *For any IF formula φ, an assignment g, and a structure $\mathfrak{A}$:*

(i) *There exists a winning strategy for V in the game $G(\varphi, g, \mathfrak{A})$, iff there exists a winning strategy for V in the game $G(\varphi^*, g, \mathfrak{A})$.*

(ii) *There exists a winning strategy for F in the game $G(\varphi, g, \mathfrak{A})$, iff there exists a winning strategy for F in the game $G(\varphi^\#, g, \mathfrak{A})$.* □

Recalling that in game-theoretic semantics the existence of a winning strategy for V amounts to truth ($\mathfrak{A} \models_{\text{GTS}} \varphi^+$) and the existence of a winning strategy for F amounts to falsity ($\mathfrak{A} \models_{\text{GTS}} \varphi^-$), then by (i) of Proposition 1, by the definition of the mappings * and $^\#$, and by taking φ to be the atomic $Sxyzw$, we have the following equivalences:

$$\begin{array}{ll}
\mathfrak{A} \models_{\text{GTS}} \sim G^{1,1}_{1,1}xyzw\, Sxyzw^+ & \text{iff} \\
\mathfrak{A} \models_{\text{GTS}} (\sim G^{1,1}_{1,1}xyzw\, Sxyzw)^* & \text{iff} \\
\mathfrak{A} \models_{\text{GTS}} (G^{1,1}_{1,1}xyzw\, Sxyzw)^\# & \text{iff} \\
\mathfrak{A} \models_{\text{GTS}} G^{*1,1}_{\ 1,1}xyzw\, Sxyzw^\# & \text{iff} \\
\mathfrak{A} \models_{\text{GTS}} G^{*1,1}_{\ 1,1}xyzw \sim Sxyzw. &
\end{array}$$

Recalling further what $G^{1,1}_{1,1}xyzw$ and $G^{*1,1}_{\ 1,1}xyzw$ denote, we have just shown that

$$\begin{array}{ll}
\mathfrak{A} \models_{\text{GTS}} \sim \forall x \exists y \forall z (\exists w / \forall x) Sxyzw & \text{iff} \\
\mathfrak{A} \models_{\text{GTS}} \exists x \forall y \exists z (\forall w / \exists x) \sim Sxyzw. &
\end{array}$$

Now, an allegedly critical point made in Tennant 1998 is that no provision is made in the syntax of IF logic for the Henkin (branching) quantifier (7), where the rows begin with existential quantifiers:

$$(7) \qquad \begin{matrix} \exists x \forall y \\ \exists z \forall w \end{matrix} Sxyzw.$$

This branching quantifier would in Tennant's opinion be indispensable in order to understand the negation of $\forall x \exists y \forall z (\exists w / x)\, Sxyzw$. Recall that $\forall x \exists y \forall z (\exists w / x)\, Sxyzw$ in Hintikka 1996b corresponds to the branching quantifier notation

$$(8) \qquad \begin{matrix} \forall x \exists y \\ \forall z \exists w \end{matrix} Sxyzw.$$

Tennant is right that understanding (7) ipso facto implies understanding (8), for they both manifest a similar independence property. He writes, however, that "Sandu's quantifier prefix G^*xyzw [$G^{*1,1}_{\ 1,1}xyzw$], it should be noted, is *not* equivalent to the existential-universal prefix in the formula just displayed [(7)]. For the latter formula does *not* possess the truth conditions implicit in Sandu's clause (A.2.9) for formulae beginning with G^*xyzw" (Tennant 1998, p. 96). But Tennant simply is mistaken in claiming that no provision is made for (7). What corresponds to this formula is nothing other than $G^{*1,1}_{\ 1,1}xyzw\, Sxyzw$.

Tennant nevertheless goes further. According to him, it is not only ${G^*}^{1,1}_{1,1}xyzw\,Sxyzw$ which fails to have the right truth conditions, but all the informationally independent quantifiers (including $G^{1,1}_{1,1}xyzw\,Sxyzw$, which fails to represent (8)). The problem with the semantic interpretation of $G^{1,1}_{1,1}xyzw\,Sxyzw$ and ${G^*}^{1,1}_{1,1}xyzw\,Sxyzw$ is, according to Tennant, that no game rules are provided for the informationally independent moves.

Now, Sandu's description of the game associated with the former IF formula is that first, F chooses a, then V chooses b, then F chooses c, and finally V chooses d. (There are misprints in Hintikka 1996b, p. 256: the third move, i.e. the choice of a sequence $\bar{c}$, is made by F and not by V, and an index denoted by 1 should be replaced by l in two places; however, these are not the reasons why Tennant finds the game description inadequate.) The game associated with the latter IF formula is identical, except that the two players are reversed.

Tennant's objection is that no condition is stated to ensure the informational independence of the moves associated with w from the moves associated with x. Now, this indeed is the case, because the above descriptions pertain to the defining rules governing the choices of the two players in the game. There is nothing to distinguish these choices from any other choices made in the presence of perfect information. Thus the former choices describe also those associated with $\forall x \exists y \forall z \exists w\, Sxyzw$, and the latter choices describe those associated with $\exists x \forall y \exists z \forall w\, Sxyzw$. The difference, however, is in players' information, and this is codified in their strategies, as explained earlier in this paper and in Sandu 1996 and in Hintikka 1996b. In the game associated with $G^{1,1}_{1,1}xyzw\,Sxyzw$ (i.e. $\forall x \exists y \forall z (\exists w/x)\, Sxyzw$), the strategy of V consists of two functions f and g, as codified in the skolemised formula $\exists f \exists g \forall x \forall z\, Sxf(x)zg(z)$, while in the game of perfect information associated with $\forall x \exists y \forall z \exists w\, Sxyzw$, V's strategy consists likewise of two functions h, k, as codified in $\exists h \exists k \forall x \forall z\, Sxh(x)zk(x,z)$, where the function k is defined on longer sequence, having two argument places. The same holds for the sentences $\exists x \forall y \exists z (\forall w/x)\, Sxyzw$ and $\exists x \forall y \exists z \forall w\, Sxyzw$, except that now F's strategies are considered. All this is explicitly stated in the appendix of Sandu 1996 immediately after the game rules are given. In this light, one has difficulty in understanding Tennant's summary to the effect that: "Sandu does not provide for the existential-universal Henkin sentence as well-formed; and he does not provide correct interpretations even for the formulae that he *does* count as well-formed" (Tennant 1998, p. 97).

A further issue Tennant is dissatisfied with is that Hintikka (1996b) does not give any set of axioms or inference rules for defining the single turnstile of deductibility ($\vdash$) for IF first-order logic, especially because this sign is, as he correctly remarks, used in the proofs of some meta-theorems such as separation theorem. The reason why such axioms are not being attempted is the fact that the set of logical truths of IF first-order sentences is not recursively axiomatisable, and hence the deduction theorem does not hold in IF logic. So, Tennant is right that the notion of deductibility should not have been used, but this point should not be made more important than it is. The turnstile simply is intended to stand for the logical consequence sign, and in the proofs of the meta-theorems (such as separation theorem and Beth's definability theorem), this is how it should be interpreted. (As to the status of downwards Löwenheim-Skolem Theorem, Tennant is right that Hintikka's proof of it is given only for a finite set of sentences, but the general case for countable theories should be obvious from that proof.)

In Tennant 1998, also the question of the status of conditional is being raised. We note that Hintikka assumes it to be defined using disjunction and game-negation: $\varphi \supset \psi$ is defined as $(\sim\varphi) \vee \psi$. Tennant is right in saying that "it is an important lacuna for the intuitionist" (p. 98) how the conditional is understood, but in the context of IF logic the understanding is trifling. Attempts to implement the idea of conditional as a warrant for a passage from the truth of the antecedent to the truth of the consequent are discussed in Hintikka and Sandu 1997, pp. 391–393, for example. One possibility is to extend IF logic truth-functionally, which would amount to the adding of a weak (classical) negation $\neg$ into the language. One could then express conditional $\varphi \supset_{\mathrm{T}} \psi$ as $(\neg\varphi) \vee \psi$, which is weaker than just $\varphi \supset \psi$, and hence better for the inferential practices.

Tennant also claims that an important form of completeness is overlooked in Hintikka 1996b, namely expressive completeness, since one is not able, due to the lack of classical negation, to express in the language of IF logic the correct denial of an arbitrary sentence. What is achieved in IF logic, namely the descriptive completeness (categoricity) of some mathematical concepts, is obtained, according to Tennant, by sacrificing expressive completeness. Tennant says that the second-order logician fares better on this point, since he achieves both descriptive completeness and expressive completeness.

There is some truth in these remarks, but not the whole truth. What Hintikka in his book tries to achieve is a reasonable tradeoff between the various forms of completeness and some desirable model-theoretic properties of a logical system. It is reasonable to strive for a logic which is manageable model-theoretically, and which scores well on both descriptive and expressive completeness. The result is a logic which is compact, has the downwards Löwenheim-Skolem and separation properties, defines its own truth-predicate (in certain models), and achieves in descriptive completeness a lot more than ordinary first-order logic does. So praising second-order logic for achieving both expressive and descriptive completeness and criticising IF logic for sacrificing the former does not tell the whole story: the second-order logician sacrifices all the previously mentioned model-theoretic properties and more.

The alleged immolation of expressive completeness is one of the main objections Tennant raises against IF logic, and he repeatedly returns to this criticism. First, here is a fact of IF logic. The game-negation admits only of the elimination rule (*i*) but not the introduction rule (*ii*):

$$\frac{\phi \quad \sim\phi}{\bot}\ (i) \qquad\qquad \frac{\begin{matrix}\phi \\ \vdots \\ \bot\end{matrix}}{\sim\phi}\ (ii)$$

Now, Tennant (pp. 109–110) concludes that "the game-theoretic notion of 'falsity' (*i.e.*, existence of a winning strategy for player F) does not have the right relationship to the notion of mathematical refutation for any associated 'negation' operator to function the way *ordinary mathematical reasoning—be it classical or constructive—requires it to.* Not even the essentials of intuitionistic reasoning are conserved in the IF account!"

This conclusion is too hasty, however. It depends on what "ordinary mathematical reasoning" amounts to. If it means reasoning under the influence of perfect information, then probably either the classical or intuitionistic negation should be enough for some sense of "ordinary mathematical reasoning." Certainly, this does not exhaust the whole of mathematical reasoning, and partial recursive functions provide a case in point. For the undefined arguments, it is appropriate to say that such a function is neither true nor false, and hence the notion of falsity is not the classical one. Indeed, considerations as these originally prompted Kleene to devise the Strong-Kleene valuation schema. So, one could say that this schema is intended to capture mathematical reasoning with partial objects and partial structures.

In the light of these observations, to blame IF logic for being neither classical nor intuitionistic only expresses a useless *epizeuxis* of the previous misunderstandings of the behaviour of games in IF logic. IF logic is classical when formalising mathematical reasoning with total objects (since determinacy holds), but neither classical nor intuitionistic when dealing with partial ones. One should not assume that non-determinacy could or should be assimilated or reconciled with some standard views on intuitionism. There are subtle distinctions to be made here. For one thing, game-negation behaves classically for the standard quantifiers but not for the slashed informationally independent ones. Yet in the latter case, game-negation obeys the Strong-Kleene valuation schema. Partiality of IF logic is a result of classical premises.

The conclusions in Tennant 1998 are unwarranted in other respects as well. There is no need to restate most of them, but a couple of things need to be corrected in order to prevent further misunderstandings. One thing concerns the definition of truth-predicate in IF language itself. In Tennant (1998, p. 106) we find that "The reader would be justified in refusing to accept Sandu's proof of Proposition 1 [of Sandu 1996, p. 261]."

The definability result says that truth-predicate for a suitable IF first-order logic can be defined in the very same language. The result stems from the fact that the expressive capacity of IF logic is suitably strong to create self-applied truth-predicates, much in the same spirit as the set of Gödel numbers of true Σ_1^1-sentences can be defined using Σ_1^1-sentences only. The expressive strength stems from the well-known results that every Henkin quantifier can be defined as a Σ_1^1-sentence (Henkin 1961), and that every Σ_1^1-sentence can be defined in a language of all Henkin quantifiers (Walkoe 1970).

The usual way of formulating a truth-definition for the language of first-order Peano Arithmetic is to take the truth-defining formula $\mathrm{Tr}(x)$ to have the form $\exists X(L(X) \wedge X(x))$, with X a second-order variable, and $L(X)$ a formula which runs through all the recursive clauses of the truth-definition. In $L(X)$, there exists thus a clause

$$X(\ulcorner \forall x\varphi \urcorner) \leftrightarrow \forall x X\, Sub(\ulcorner \varphi \urcorner, \ulcorner x \urcorner, x), \tag{9}$$

where *Sub* is the substitution function. All the other clauses are similar, and the formula $L(X)$ will be the conjunction of all such clauses. In case of the underlying logic is IF first-order logic, additional conjuncts are prepared for the independent quantifiers. One observation of truth-definition in Sandu 1996 is that one can get rid of the biconditional in favour of the simple conditional, since the second-order variables are interpreted in the standard way. Therefore, the above clause could be replaced with

$$X(\ulcorner \forall x\varphi \urcorner) \rightarrow \forall x X\, Sub(\ulcorner \varphi \urcorner, \ulcorner x \urcorner, x). \tag{10}$$

A similar procedure applies for all the other conjuncts, except for the clause for the atomic formulas where the biconditional is retained. Now

Proposition 1 in Sandu (1996, pp. 261–264) states that, for every IF formula φ and for the standard model of Peano Arithmetic $\mathcal{N}$:

$$(11) \qquad \mathcal{N} \models \exists X(L(X) \wedge X(\ulcorner\varphi\urcorner)) \quad \text{iff} \quad \mathcal{N} \models_{\text{GTS}} \varphi.$$

The method of replacing the biconditional by a simple conditional is also adopted in Hintikka 1996b. In Tennant 1998 this is claimed to be a perpetuated mistake, and his recommendation is that the proof should be rejected.

However, the proof is plainly valid. From right-to-left it goes by induction on the complexity of φ (see also Hintikka 1998, Sandu 1998). The set A which will satisfy the right side is defined as $A = \{\ulcorner\varphi\urcorner : \mathcal{N} \models_{\text{GTS}} \varphi\}$. Thence the induction shows that

$$(12) \qquad \text{if} \quad \mathcal{N} \models_{\text{GTS}} \varphi \quad \text{then} \quad (\mathcal{N}, A) \models L(X) \wedge X(\ulcorner\varphi\urcorner).$$

The clause for the universal quantifier goes as follows: Assume $\mathcal{N} \models_{\text{GTS}} \forall x\psi$, from which it follows that $\ulcorner\forall x\psi\urcorner \in A$. Therefore, the second conjunct of (12) has been taken care of. In order to show that $(\mathcal{N}, A) \models L(X)$, it is enough to consider only the clause in $L(X)$ for the universal quantifier. So assume $(\mathcal{N}, A) \models X(\ulcorner\forall x\psi\urcorner)$. Hence $(Sub(\ulcorner\varphi\urcorner, \ulcorner x\urcorner, x)) \in A$, and then it is immediate that $(\mathcal{N}, A) \models \forall x X\, Sub(\ulcorner\varphi\urcorner, \ulcorner x\urcorner, x)$. Contrary to what is claimed by Tennant, we do not need the stronger conclusion $(\mathcal{N}, A) \models X(\ulcorner\forall x\psi\urcorner)$.

One might question whether the defining truth-predicate is in some sense formally correct and materially adequate. Certainly the biconditional (11) similar to the Tarski truth-predicate is satisfied. In the proof of this biconditional given in Sandu 1996 only two clauses in the induction are considered: the clause for the universal quantifier and the clause for the independent quantifiers, that is, for the formulas such as $\forall x\exists y\forall z(\exists w/x)\,\varphi$. Given the fact that the detailed treatment of the latter clause is proffered, one starts to doubt writer's good will when one reads the following: "Worse, Sandu omits the reasoning for the most difficult case in the inductive step, namely the one which would have dealt with the innovative, information-independent quantifiers! Given the grave defects in his treatment of these quantifiers already pointed out above, this is an inexcusable omission" (Tennant 1998, p. 107).

The fact that IF logic defines truth-in-$\mathcal{N}$ should not be seen as an extraordinary thing, as similar results are known in the literature. Historically, Myhill (1950) has shown that an existentially quantified fragment of ordinary first-order logic defines its own truth-predicate. Smorynski (1977) systematises similar results for the existential fragments of the arithmetical hierarchy, and related cases have been considered on a number of other occasions as well. The definability result states the same thing for the first existential level of the hyper-arithmetical hierarchy. The point is to restate this result using game-theoretic terminology and to relate it to Kripke's work on the definability of truth (Kripke 1975). However, the game-theoretic approach certainly introduces a novel perspective to the earlier results. Whereas some truth definitions are bound to the partial interpretation of the truth predicate in the sense that the truth-value gaps arise at the level of the atomic sentences, in a game-theoretic interpretation the truth-value gaps arise at the level of complex sentences involving at least two logical constants, because of the failure of the principle of the determinacy in games of imperfect information. In the presence of imperfect information, it indeed is possible to find a formula which defines truth-in-$\mathcal{N}$.

Consequently, one sees that in languages where perfect information fails, remarkable things can happen. Therefore, this definability result opens up intriguing questions of the precise conditions under which the failure of perfect information gives rise to self-applied truth-predicate. It appears that it is not only the expressive capacity and the strength which allows this to happen, for the Π^1_1-fragment fails to include a truth-predicate, although some fragments of first-order logic may well have this property.

5.

The mechanism of imperfect information is used in mathematical languages in various connections, some of them pointed out in Hintikka 1996b. One of the cases where IF logic contributes to mathematical reasoning is the notion of uniform differentiability. One can replace a standard existential quantifier with an independent one, because in the formulation of uniform differentiability a point of the derivation has to be assumed independent of another one. This can be expressed in IF logic with the aid of the quantifier string $\forall x \exists y \forall \epsilon (\exists \delta / x) \forall z$. However, Tennant criticises this example and claims that the same result can be achieved in the language of ordinary first-order set theory, defining δ as a function $\exists \delta \colon R \to R$, followed by the string of quantifiers $\forall x \exists y \forall \epsilon \forall z$.

Tennant's *paragnosia* here stems from the inability to perceive the general situation, however. For sometimes IF quantifiers can be in the guise of choice functions. In the example given in Tennant 1998, the function δ depends only on the tolerance parameter ϵ; it does not need x as its argument. Therefore, this example as it is described can be expressed in IF logic, but not in the linear first-order logic, vindicating the original argument in Hintikka 1996b.

Further, Tennant argues that "the move to IF logic deprives one of comprehensive deductive grasp of the inferential commitments one makes with one's mathematical assertions. If we wish to be able still to *prove* interesting theorems about uniform differentiability, we would be well advised to eschew IF logic and do our mathematics as usual" (1998, p. 113).

Yet again, things are not as simple as described. The idea is that there are fragments of mathematical prose corresponding to informationally independent quantifiers. In addition to the example with uniform differentiability and others in Hintikka 1996b, one can easily find more. For instance, consider Lang (1966, pp. 85–86):

Lemma. Let $y_1 \dots y_m$ be [...] formal power series linearly independent over $K(z)$, and satisfying the linear differential equation $Y' = QY$, where Q is a matrix of rational functions. Let $P_1 \dots P_m$ be polynomials in $K[z]$, and let

$$F_1 = P_1 y_1 + \cdots + P_m y_m.$$

Let T be a polynomial denominator for Q, and define inductively

$$F_k = TDF_{k-1} = P_{k_1} y_1 + \cdots + P_{k_m} y_m.$$

Let r be the rank of the matrix $(P_{k_j})(k, j = 1 \dots m)$ and suppose $r < m$. Then

$$\operatorname{ord} F_1 \leq r(\max \deg P_j) + c_0,$$

where c_0 is a positive number depending only on $y_1 \dots y_m, Q$ and not on the P_j.

We note that similar structures of dependencies and independencies can be found on pages 4, 5, 18, 33, 73, 94, 96 in Lang 1966. To see more, consider an application of the Ramsey theorem (Hodges 1997b, p. 155):

Corollary. Let G be a group, and let $H_0 \dots H_{n-1}$ be subgroups of G and $a_0 \dots a_{n-1}$ elements of G. Suppose that G is the union of the set of cosets $X = \{H_0 a_0 \dots H_{n-1} a_{n-1}\}$, but not the union of any proper subset of X. Then for each $i < n$, $(G\colon H_i) < l$ for some finite l depending only on n; in particular H_i has finite index in G.

Yet another example from Nešetřil (1995, p. 1378), which describes an iterated version of the Ramsey theorem:

For every t and n there exists N such that for every coloring of the power set $\mathcal{P}([N])$ by t colors there exists a subset X of $[N]$ of size n such that the color of a subset of X depends only on its size.

Finally, consider repeated applications of Ramsey functions, described in Lifsches and Shelah 1999, p. 288 (and similar results on pp. 298, 304, 305):

LEMMA. For every $n, d < \omega$ there is $K^* = K^*(n, d) < \omega$ such that if $\mathcal{I}$, of dimension d and depth n, is an interpretation of an infinite K-random graph $\mathcal{G}$ on C and $K > K^*$, then there is $m < \omega$, that depends only on K, n and d, such that if (L, R) is a Dedekind cut of C:

L (or R) has an infinite bouquet size,

R (or L) has bouquet size that is at most m.

Informational independence is also ubiquitous in concurrent game semantics for fragments of linear logic (see e.g. Abramsky and Melliès 1999). All these examples show that mathematical constructions involving independent quantifiers are ingredients of mathematical language, and the number of examples can easily be multiplied beyond necessity.

Some pieces of mathematical English cannot be represented using just the standard linear quantifiers. One can of course go to set theory, but the same phenomenon will emerge again: reasoning in the jargon of set theory with tacit independent quantifiers will not be adequately represented in the standard set theory, which would involve linearly ordered quantifiers only.

Therefore, the claim that mathematics should be done "as usual," because one is then able to "*prove* interesting theorems" misses the target. IF logic does not have to be a particularly good or practical framework for doing mathematics, albeit a surprisingly large fragment of conventional mathematics can be captured or expressed or formulated in it, at least in principle. But if such resources were not available, there could not be true progress in logic, since one aspect of mathematical discoveries involves concepts going well beyond the detection of stronger and stronger proof-theoretic methods or stronger and stronger set-theoretic axioms.

In Tennant 1998, further criticism is given of Hintikka's account of the relations between set theory and IF logic. In the criticism, Tennant appeals to Friedman's results in the foundations of mathematics and in set theory. In particular, a claim is put forward that Friedman's result (that a body of finite combinatorial problems is provable only by using large cardinals such as $\mathfrak{k}$-subtle cardinals) "completely undermines" Hintikka's combinatorial approach to the foundations of mathematics, since "such combinatorial statements themselves [...] can be just as imponderable as some very-large-cardinal existence claims" (p. 105).

However, the matter is not quite as straightforward as Tennant's remarks would have us to believe. First, Friedman's results pertain to the field of finite combinatorics, and his particular results of interest here concern finite combinatorial statements such as inserting vertices into finite labelled trees (cf. Friedman 1998). However, the questions of the validity of mathematical statements expressible in IF logic are not necessarily finite, and IF logic, being non-axiomatisable, is itself not only finite in character, but embodies problems of infinitary combinatorics. It is therefore premature to put forward a strong claim that "combinatorial statements are just as imponderable as some very-large-cardinal existence claims," unless one does not distinguish the cases where finite and (un)countable combinatorics are involved in Friedman's independence proofs and in Hintikka's IF logic. Without attempting any further characterisation of the precise sense in which IF logic is combinatorial, we note that combinatorial questions may arise at least in the following areas: (*i*) Questions of the reduction of an IF formula to the ordinary linear first-order formula, (*ii*) questions of measuring the expressive power of IF logic (interpretations do not require the idea of a power set), (*iii*) questions of validity both in finite and countable structures. Also, the formal results which aim to establish a logical link between the consistency statements of a particular set theory and finite combinatorial statements still leave the door open for many other no less important issues, such as the philosophical justifications of very-large-cardinal existence claims.

One egregious misunderstanding of game-theoretic semantics made in Tennant 1998 concerns the relation between constructivity and semantic games. For there a theorem is questioned: the equivalence of the game-theoretic truth definition and Tarski's truth definition, *assuming the axiom of choice*. According to Tennant (pp. 113–114), the equivalence does not need the axiom of choice, "for the notion 'player T (or player F) possesses a winning strategy in such-and-such state of play' can be rigorously defined without any explicit quantification over functions". To underpin this claim, it is referred to Tennant 1990 where the argument for the equivalence without the axiom of choice is allegedly given (on pp. 35–37).

The proof of the equivalence can be found, for example, in Hodges 1983, where it is emphasised that in the proof of the clause for the universal quantifier the axiom of choice is needed. The argument goes as follows. Let g be an assignment to the free variables of $\forall x\psi$. From right to left, we have to prove the claim

$$(13) \qquad (\mathfrak{A}, g) \models_{\text{GTS}} \forall x\psi \quad \text{iff} \quad (\mathfrak{A}, g) \models_{\text{Tarski}} \forall x\psi.$$

We have that $(\mathfrak{A}, g) \models_{\text{Tarski}} \psi[x/a]$ for all individuals a in the domain of the structure of $\mathfrak{A}$. By the inductive hypothesis, V has a winning strategy in every game played on $\psi[x/a]$. From every set of winning strategies of V in the corresponding game, we *choose* one. We then go on forming the winning strategy for V in the game played on $\forall x\psi$, but the rest of the details will not be interesting here. The crucial point is that the axiom of choice is needed in order to compress many choices into a single strategy function.

Being eager to learn how to avoid the use of the axiom of choice, we looked at the corresponding proof in Tennant 1990. On page 37 the notion of V (resp. F) being the possessor of a winning strategy in a play of the game is inductively defined. Therefore, the possessor of a winning strategy in a play of the game associated with e.g. $(\varphi \vee \psi)$ is V, if and only if either she is the possessor of a winning strategy in a play of the game associated with φ, or she is the possessor of a winning strategy in a play of the game associated with ψ. Similarly, F possesses a winning strategy in a play of $\forall x\,\psi$, if and only if there is an individual a such that F possesses a winning strategy in the game associated with $\psi[x/a]$ (and likewise for the rest of the formulas). Then Tennant asserts that for arbitrary formulas the possessor of a winning strategy is V in the corresponding game if and only if that formula is satisfiable in the sense of the Tarski's truth definition.

It is not difficult to see why the axiom of choice is not needed: the concept of strategy does not have authentic game-theoretic content. The necessary and sufficient conditions for a player to 'possess' a strategy are defined in Tarski's spirit. Some game-related terminology is mentioned, such as winning strategies, but never genuinely used. For one thing, this definition does not say how a winning strategy in a complex game is obtained from the winning strategies in simpler games, crucial in understanding game-theoretic truth-definitions. Game-theoretic semantics in Tennant 1990 is Tarski's semantics, and in the proof of the equivalence between the two the axiom of choice is not needed, because there is nothing to be proved. This monoideism recurs in the alleged 'game-theoretic' interpretation of intuitionistic logic presented in Tennant 1979.

Since Tennant comes unstuck with his 'game-theoretic semantics' without any real game-theoretic content, it is obvious why he is in an impossible situation even to attempt reliably to criticise the proper game-theoretic approach to the meaning of sentences involving informationally independent components gone places by Henkin, Hintikka, Hodges, Sandu, and others. His discussion on game-theoretic semantics is *ignis fatuus*, based on the fallacious idea that the meanings of logical constants or truth-definitions for formulas could somehow be characterised in terms of rules of inference or 'game' rules of his own invention. For example, to mention a concept of a strategy without really using it to construct further strategies is like saying that one has found an effective method of playing games without actually using those strategies in real situations, which is next to indulging in word sorcery.

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