Fuzzy Versions of Epistemic and Deontic Logic[1]

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1. INTRODUCTION

Epistemic and deontic logics are modal logics, respectively, of knowledge and of the normative concepts of *obligation, permission,* and *prohibition*. Epistemic logic is useful in formalizing systems of communicating processes [4] and knowledge and belief in AI. Deontic logic is useful in computer science wherever we must distinguish between actual and ideal behavior [9], as in fault tolerance and database integrity constraints. We here discuss fuzzy versions of these logics. In the crisp versions, various axioms correspond to various properties of the structures used in defining the semantics of the logics. Thus, any axiomatic theory will be characterized not only by its axioms but also by the set of properties holding of the corresponding semantic structures. Fuzzy logic does not proceed with axiomatic systems, but fuzzy versions of the semantic properties exist and can be shown to correspond to some of the axioms for the crisp systems in special ways that support dependency networks among assertions in a modal domain. This in turn allows one to implement truth maintenance systems. For the technical development of epistemic logic, see [4], and for that of deontic logic, see [1]. For previous work on fuzzy modal logic, see [10], [8], and [6]; we follow [7]. To our knowledge, we are the first to address fuzzy epistemic and fuzzy deontic logic explicitly and to consider the different systems and semantic properties available.

In section 2, we give the syntax and semantics of epistemic logic and discuss the correspondence between axioms of epistemic logic and properties of semantic structures. The same topics are covered for deontic logic in section 3. Section 4 presents fuzzy epistemic and fuzzy deontic logic and discusses the relationship between axioms and semantic properties for these logics. Section 5 discusses how our results can be exploited in truth maintenance systems.

2. SYNTAX AND SEMANTICS OF EPISTEMIC LOGIC

For the syntax of epistemic logic we start with primitive (or atomic) propositions in a set Φ and form more complicated formulas by closing under negation, conjunction, and the modal operators $K_1,\ldots,K_n$, where there are n agents named $1, 2,\ldots, n$. (Where φ is a formula, $K_i\varphi$ means that agent i knows φ. We use the standard symbols $\wedge$, $\vee$, $\neg$, and $\Rightarrow$ for conjunction, disjunction, negation, and implication, respectively.) Thus, if φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi\wedge\psi)$, and $K_i\varphi$, for $i = 1\ldots n$. Other propositional connectives are defined in the usual way.

The semantics of a language gives the conditions under which the sentences of the language are true or false. We use Kripke structures as formal models for this purpose. A Kripke structure M for n agents over a set Φ of primitive propositions is a tuple $(S, \pi, \kappa_1,\ldots, \kappa_n)$, where S is a set of states or possible worlds, π is an interpretation function that associates with each state in S a truth assignment to the primitive propositions of set Φ, and κ_i, $1\leq i\leq n$, is a binary relation on S, that is, a set of pairs of elements of S. We shall call each κ_i an *accessibility relation*. A binary relation κ is (a) *reflexive* if, for all $s\in S$, we have $(s,s)\in\kappa$, (b) *symmetric* if, for all $s,t\in S$, we have $(s,t)\in\kappa$ if and only if $(t,s)\in\kappa$, and (c) *transitive* if, for all $s,t,u\in S$, we have that, if $(s,t)\in\kappa$ and $(t,u)\in\kappa$, then $(s,u)\in\kappa$. A relation κ that enjoys all three of these properties is called an *equivalence relation*.

Where M is a Kripke structure, s is a possible world in M, and φ is any formula, we write $(M, s) \models \varphi$ to express that φ is true at s in M. We define such a notion of truth recursively as follows.

$$(M, s) \models p \text{ iff } \pi(s)(p) = \text{true, for } p\in\Phi$$
$$(M,s) \models \psi\wedge\varphi \text{ iff } (M, s) \models \psi \text{ and } (M, s) \models \varphi$$

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$(M, s) \models \neg\psi$ iff $(M, s) \not\models \psi$
$(M,s) \models K_i\varphi$ iff $(M, t) \models \varphi$ for all worlds t such that $(s, t) \in \kappa_i$.

In any modal logic, we say that a formula ψ is *valid in* a Kripke structure *M* if ψ is true at all possible worlds in *M*; ψ is *valid for a family F* of Kripke structures if ψ is valid in all structures in *F*. Families of Kripke structures are defined by the properties their accessibility relations satisfy, such as the three above. (In a given epistemic Kripke structure, we assume that all the relations satisfy the same properties.)

Modal logics are generally presented as axiomatic systems. The set of theses of the system are all the formulas that follow from the axioms by zero or more applications of the rules of inference. The axioms and rules of inference for a modal logic system are generally chosen so that its set of theses is exactly the set of formulas valid for that family of structures. For epistemic logics (and analogously, most other modal logics), we have the following rules of inference, which preserve validity:

Modus Ponens: From φ and $\varphi \Rightarrow \psi$ infer ψ.
Modal Generalization: From φ infer $K_i j$

All axiomatic systems of epistemic logic include the following two axiom schemas – any instance of them is valid in any Kripke structure.

All tautologies of propositional calculus (**A1**)
$(K_i\varphi \wedge K_i(\varphi \Rightarrow \psi)) \Rightarrow K_i\psi$, i = 1, ..., n (**K**)

The following axioms, however, are valid for only certain families of Kripke structures, that is, only where certain accessibility-relation properties are satisfied.

$K_i\varphi \Rightarrow \varphi$ (**T**)
$K_i\varphi \Rightarrow K_iK_i\varphi$ (**4**)
$\neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$ (**5**)
$\varphi \Rightarrow K_i\neg K_i\neg\varphi$ (**S**)

T is valid in any reflexive structure, **4** is valid in any transitive structure, **S** is valid in any symmetric structure, and **5** is valid in any Euclidean structure. A binary relation κ on *S* is Euclidean if, for all $s,t,u \in S$, if $(s,t) \in \kappa$ and $(s,u) \in \kappa$, then $(t, u) \in \kappa$. Thus, the system called T, which has the usual rules of inference and axioms **A1**, **K**, and **T**, has as theses exactly those formulas valid for the family of reflexive structures. The theses of S4, which adds axiom **4**, are those formulas valid in reflexive and transitive structures. And the theses of S5, which further adds axiom **5**, are those formulas valid in reflexive, transitive, and Euclidean structures. But any binary relation that is reflexive and Euclidean is also symmetric; correspondingly, axiom **S** is derivable as a theses in S5. Thus the theses of S5 are the formulas valid in structures whose the accessibility relations are equivalence relations.

3. DEONTIC LOGIC

The syntax of deontic logic is just like that of epistemic except that

- K_i is replaced by *O*,
- no subscripts appear on modal operators (obligation, etc., apply equally to all), and
- we introduce additional modal operators *P* and *F*.

$O\varphi$ means that if φ is obligatory, $P\varphi$ means that φ is permitted, and $F\varphi$ means that φ is forbidden. In fact, one normally defines $P\varphi$ as $\neg O\neg\varphi$ and $F\varphi$ as $O\neg F$. For the semantics of deontic logic, we use Kripke structures each with only one accessibility relation,κ. The semantics for *O* is like that for K_i, and for P we have

$(M,s) \models P\varphi$ iff $(M,t) \models \varphi$ for some world t such that $(s,t) \in \kappa_i$

Again there are different axiomatic systems that we get by choosing different sets of axioms, and various properties of the accessibility relations correspond to various axioms. As before, the rules Modus Ponens and Modal Generalization with *O* in place of *K* are used in all systems, as are axioms **A1** and **K**. Henceforth, when referring to the deontic versions of these axioms, we shall precede the name with an "**O**". The following are some of the available axioms.

$O(O\varphi \Rightarrow \varphi)$ (**OT**')

$$O\varphi \Rightarrow OO\varphi \qquad (\mathbf{O4})$$
$$PO\varphi \Rightarrow O\varphi \qquad (\mathbf{O5})$$
$$O(PO\varphi \Rightarrow \varphi) \qquad (\mathbf{OS'})$$

O4 and **O5** are the deontic versions of **4** and **5** and correspond, respectively, to the transitive and Euclidean properties of κ. **OT′** replaces **OT**, $O\varphi \Rightarrow \varphi$, because we do not require that, if some proposition is obligatory, then it must be true. Correspondingly, we eliminate reflexive accessibility relations. The property of κ corresponding to **OT′** is that of being *almost reflexive* (see [1]). **OS′** replaces **OS**, $PO\varphi \Rightarrow \varphi$, and we correspondingly eliminate the symmetric property – a relation that is transitive and symmetric must also be reflexive (which we do not want

The system OT′ includes the usual two inference rules and the axioms **OA1** and **OK**; it also includes axiom **OT′**. System OS4 includes in addition axiom **O4**, and system OS5 adds axiom **O5** to this. An additional axiom, $O\varphi \Rightarrow P\varphi$, states that what is obligatory is also permitted; the corresponding property of accessibility relations is *seriality* (see [1]). If we add this axiom to system OT′ (respectively, OS4 or OS5), we get a system called OT'^{+} (respectively, $OS4^{+}$ or $OS5^{+}$).

4. FUZZY EPISTEMIC AND DEONTIC LOGIC

Fuzzy modal logics give fuzzy versions of both π (the evaluation function) and κ_i (the accessibility relations). Now π, given a possible world and an atomic proposition, gives a value in the range [0,1]. Accessibility relations are now denoted by δ (or, when we have an indexed family, δ_i, $1 \le i \le n$) and δ is a function that assigns a value in [0,1] to a pair of worlds in S. Rather than using the |= notation , we write $V_s(\varphi)$ for the degree of truth of an arbitrary proposition φ at possible world s. We define V_s recursively. The base case and the cases for propositional connectives are as follows, where Δ is a triangular norm and ∇ is a triangular co-norm (see below).

$V_s(p) = \pi(s)(p)$, where p is an atomic proposition
$V_s(\neg\varphi) = 1 - V_s(\varphi)$
$V_s(\varphi \wedge \psi) = \Delta\, \{V_s(\varphi), V_s(\psi)\}$
$V_s(\varphi \vee \psi) = \nabla\, \{V_s(\varphi), V_s(\psi)\}$

For illustration, we shall consistently use the minimum function for the triangular norm and the maximum function for the triangular co-norm.

For a strong modality $\Box$ (e.g., O or K_i), we take $V_s(\Box\varphi)$ to be the minimum over all worlds $s' \in S$ of $max\{\neg\delta(s,s'), V_s(\varphi)\}$, where $\neg\delta(s,s')$ is defined as $1\text{-}\delta(s,s')$. The equation, then, is $V_s(\Box\varphi) = \Delta_{s' \in S}\nabla\{\neg\delta(s,s'), V_{s'}(\varphi)\}$. For a weak modality $\Diamond$ (e.g., P or $\neg K_i\neg$), we use $\Diamond\varphi \equiv \neg\Box\neg\varphi$ to get $V_s(\Diamond\varphi) = \nabla_{s' \in S}\Delta\{\delta(s,s'), V_{s'}(\varphi)\}$.

Because fuzzy logics work with numerical measures, axiomatic systems are not appropriate for presenting fuzzy logics. So, to capture properties of knowledge or properties of our normative concepts in fuzzy modal logics, we must in the first instance rely on the properties of the accessibility relations. Although other definitions are possible, we use the following for the three most common properties:

δ is reflexive if $\delta(s,s) = 1$ for all $s \in S$.
δ is symmetric if $\delta(s,t) = \delta(t,s)$ for all $s,t \in S$.
δ is transitive if $\delta(s,u) \ge min\{\delta(s,t), \delta(t,u)\}$ for all $s,t,u \in S$.

Still, we would like to retain a relation between various properties of the accessibility relations and the corresponding formulas – the formulas relate more directly to our usual notions. We can no longer speak of formulas being valid for a family of structures. Instead, given a family of structures, we want some guarantee that the formulas in question are somehow distinguished. The two axioms that hold for any family of structures (**A1** and **K**) essentially fall back to non-modal fuzzy logic. The interesting cases are the formulas that correspond to the various accessibility-relation properties. Of the seven such formulas we have discussed, five (**T**, **4**, **5**, **S**, and **OO**) are conditionals, $\alpha \Rightarrow \beta$. The remaining two (**OT′** and **OS′**) are "strong conditionals", that is, of the form (using the general strong modal operator $\Box$ in place of O) $\Box(\alpha \Rightarrow \beta)$. For **T** and **4**, we can show that, as long as the corresponding property of the accessibility relation holds, at any world s we are guaranteed that $V_s(\alpha) \le Vs(\beta)$. The significance of these results comes out when we think of a schema $\alpha \Rightarrow \beta$ as a rule (admittedly dependent on Modus Ponens) that sanctions α as support for β. It runs out that, for **5**, **S**, and **OO**, even when the corresponding property of the accessibility relation holds, we cannot in general guarantee that $V_s(\alpha) \le Vs(\beta)$ at any world s. We have, however, found for each of these formulas conditions under which, when the property holds, so does the guarantee. The conditions tend to be quite general, so corresponding support rules would be available in most interesting cases. Similar results hold for the strong conditional forms.

Space restrictions allow us to prove the guarantee $V_s(\alpha) \le V_s(\beta)$ for only one, relatively simple case. Consider **T**, $\Box\varphi \Rightarrow \varphi$, and assume that δ is reflexive. We must show that, for any $s \in S$ and arbitrary S, π, and δ except that δ is reflexive, we have $V_s(\Box\varphi) \le V_s(\varphi)$, that is,

$$\Delta_{s' \in S} \nabla\{\neg\delta(s,s'), V_{s'}(\varphi)\} \le V_s(\varphi) \qquad (*)$$

For, since δ is reflexive, we have $\delta(s,s) = 1$ hence $\neg\delta(s,s) = 0$. Hence $\nabla\{\neg\delta(s,s), V_s(\varphi)\} = V_s(\varphi)$. Since the minimum operation ranges over s as well, the left-hand side of (*) cannot be greater than $V_s(\varphi)$ – Q.E.D.

5. Truth Maintenance Systems

Truth maintenance systems [5] are a general facility used in problem solving systems, in conjunction with inference engines, to manage logical relationships between statements. The inference engine is responsible for making inferences within the problem domain and the truth maintenance system is concerned with organizing the inferences which are consistent with each other. Every inference is communicated with the TMS by the inference engine. The consequent is the node of the inference engine datum that is infered. The antecedents are the nodes of the data used as antecedents to the inference rule. So a TMS maintains a dependency network with nodes representing *sentences* and *justifications*. A TMS is used for nonmonotonic reasoning where if we add new facts, we can still reason by removing an assumption which causes contradiction. This is called dependency driven backtracking.

There are four types of TMSs:

1. Justification-Based Truth Maintenance System (JTMS): This is a simple TMS where one can examine the consequences of the current set of assumptions. The meaning of sentences is not known.
2. Non-Monotonic Justification-based Truth Maintenance System (NMJTMS): This is much like a JTMS except that it supports non-monotonic inferences.
3. Assumption-based Truth Maintenance System (ATMS): This allows us to maintain and reason with a number of simultaneous, possibly incompatible current sets of assumption.
4. Logic-based Truth Maintenance System (LTMS): The LTMS incorporates negation explicitly and therefore can represent any propositional calculus formula.

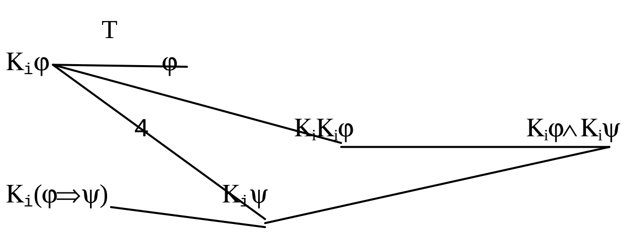
Truth maintenance systems were introduced for reasoning that uses crisp-logic. Recent research has introduced truth maintenance systems that work with multivalued logics [2]. We introduce truth maintenance systems that work with fuzzy epistemic and deontic logic since uncertainty is one of the main concerns of AI.

For example, if proposition Xp is derived from propositions X_2 and X_1, then Xp depends on X_2 and X_1. X_1 and X_2 are supporters of Xp. It is said that Xp is justified by X_1 and X_2. [X_1 X_2 Xp] is called a *justification*. If a proposition is supported by more than one justification then its justification can be expressed as a *disjunction*. Many propositions derived from problem solver create a network of propositions and justifications called *dependency network*.

Axioms of epistemic and deontic logics are used as rules for reasoning. A justification is actually an instance of axiom within the TMS. Thus, a justification can be thought of having a truth value in the interval [0, 1] based on the truth values of the assumptions. Fuzzy logic methods are used to evaluate *conjunction*, *disjunction* and *implication*. Certain Axioms of epistemic and deontic logics are proved to be valid using fuzzy evaluation techniques. In particular, the truth value of the consequent is shown to be greater than or equal to the truth value of the antecedent in the conditional axioms we use for justifications.

Fuzzy epistemic LTMS

The following represents dependencies that would exist in a fuzzy epistemic LTMS. For example, given that $K_i\varphi$ has a high truth value, formula **T** would ensure that φ also has a high truth value (in fact, at least as high as $K_i\varphi$). The other relations show similar support. Note that **4** requires that both $K_i\varphi$ and $K_i(\varphi \Rightarrow \psi)$ have high truth values to ensure that $K_i\psi$ has a high truth value.



Such an epistemic LTMS can be used for distributed systems where we can have many agents with common knowledge about what each one knows [4]. If we come up with a contradiction of some statement, we can backtrack and remove the assumption causing the contradiction. We can fix a cutoff value for our derived fact and, if the value is less than the cutoff value, we can retract the fact. Our fuzzy epistemic LTMS can be used for communicaion protocols.

Fuzzy deontic LTMS

Deontic logic is used where there is a conflict between ideal and actual behavior of a system [9]. Applications of deontic logic include policies of an organization. We can put constraints on salaries, ages, deadlines, etc. Our obligation rules (constraints) can be used for specifying those constraints. By using our fuzzy deontic LTMS, we can allow some amount of violation without any penalty. For example, a student can be allowed to return a book to the library within three days after the due date. So this is not a crisp date. These ranges of violation can be given in noncrisp values. If someone violates a rule by more than a certain extent, he or she can be penalized to that extent.
To consider some xamples, let

φ = Person *p* receives a driver's license.
ψ = Person *p* is 18 or older.
ρ = Person *p* is an employee of company *c*.
σ = Person *p* is over 80 years old.
T = Person *p* is under 20 years old.
μ = Company *c* gives its employees a bonus.
ν = The employees of company *c* arrive at work not more than ten minutes late.

Then the following are interesting deontic statements:

$O(\varphi \Rightarrow \psi)$ which implies $O\varphi \Rightarrow O\psi$
$\rho \Rightarrow O(\neg\sigma \wedge \neg T)$
$\mu \Rightarrow O\nu$

These obligations need not be crisp, but can be in a permissible range which is given in the fuzzy truth value in the interval [0, 1]. One obligation may lead to another obligation. So we can derive other obligations from the antecedents which can be nodes of TMS and implications are used as rules for deriving consequents which are also denoted by nodes. Thus, TMS keeps all the facts regarding our reasoning system and removes some obligations if contradiction a occurs.

6. CONCLUSION

We have reviewed various systems of epistemic and deontic logic and shown how to characterize fuzzy versions of these logics. We have also reported our results that show that a special relationship often exists even in the fuzzy versions between certain axioms and certain properties of the accessibility relations. For conditional axioms $\alpha \Rightarrow \beta$ (the most important group), this is that, when the relevant property holds, the degree of truth of the consequent is at least as great as the degree of truth of the antecedent. This allows us to consider an instance of α as supporting a corresponding instance of β. This is the foundation of a family of truth maintenance systems we are building to support inference engines in modal domains. This work complements earlier work on (non-modal) fuzzy spatial logic [3]. Other logics that could be handled in similar fashion are logics of time (usually modal) and logics of motion. The general drift is to produce fuzzy versions of logics that have rich structures and express well-defined features of domains important to computer science and AI.

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