Fuzzy Propositional Logic

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1 Introduction and preliminaries

We introduce a multiple valued sentential logic, that is, a propositional logic which differs from classical two valued logic by allowing more truth values than only 'false' ($\mathbf{0}$) and 'true' ($\mathbf{1}$). The basic definitions of our many-valued logic are due to Pavelka [9] who in 1979 generalized the ideas of Lukasiewicz [5], and later Novàk [8] generalized these ideas to fuzzy predicate logic. We assumme all the time that the set L of values of truth forms a complete lattice. i.e. L is partially ordered and, for all subsets $\{y_i \mid i \in \Gamma\}$ of L, the greatest lower bound $\bigwedge_{\in\Gamma}\{y_i\}$ and the lowest upper bound $\bigvee_{\in\Gamma}\{y_i\}$ exist in L, in particular, for all pairs $x, y \in L$, $x \wedge y, x \vee y \in L$, and there is also in L the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$, corresponding to the absolute false and the absolute truth, respectively. Usually L is the real unit interval $[0,1]$ but also other alternatives are possible; we will see that any injective MV-algebra will offer sufficient conditions for the well behaviour of our multiple valued logic and necessary condition for the well behaviour of our logic is that L is a complete MV-algebra. To save space for examples, we omit some proofs. An interested reader will, however, find these proofs in [11], [12], [13].

Definition 1 *A* (complete) residuated lattice $L = \langle L, \leq, \wedge, \vee, \odot, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ *is a (complete) lattice* L *containing the least element* $\mathbf{0}$ *and the largest element* $\mathbf{1}$*, and endowed with two binary operations* $\odot$ *(called* product*) and* $\rightarrow$ *(called* residuum*) such that (i)* $\odot$ *is associative, commutative and isotone and, for all elements* $x \in L$, $x \odot \mathbf{1} = x$*, (ii) for all* $x, y, x \in L$, $x \odot y \leq z$ *iff* $x \leq y \rightarrow z$*.*

Residuated lattices are known also under other names, e.g. Höhle [4] calls them *integral, residuated, commutative l-monoids.* For the basic properties of residuated lattices, see [9], [13]. The following three structures are main examples of complete residuated lattices on the unit interval (cf.[6])

Example 1 *Gödel structure:*

$$x \odot y = \min\{x, y\},\ x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{elsewhere.} \end{cases}$$

We observe that, whatever the $\odot$, we always have $x \odot y \leq x \odot \mathbf{1} = x, x \odot y \leq \mathbf{1} \odot y = y$ so that $\min\{x, y\}$ is the greatest *t-norm* [3] on $[0, 1]$.

Example 2 *Product structure:*

$$x \odot y = xy,\ x \to y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{elsewhere.} \end{cases}$$

W.M. Faucett proved in [2] that any continuous t-norm with no idempotents other that $0, 1$ and no nilpotents (i.e. no-zero elements x such that $x^n = 0$ for some n) is equivalent to $\odot$.

Example 3 *Generalized Lukasiewicz structures:* for all natural numbers n, define

$$x \odot y = \sqrt[n]{\max\{0, x^n + y^n - 1\}},\ x \to y = \min\{1, \sqrt[n]{1 - x^n + y^n}.$$

In particular, for $n = 1$, we have *Lukasiewicz structure:*

$$x \odot y = \max\{0, x + y - 1\},\ x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{elsewhere.} \end{cases}$$

As proved in [7], any continuous t-norm on $[0, 1]$ with no idempotents other that $0, 1$ and at least one nilpotent is equivalent to $\odot$.

All these structures are also examples of *BL-algebras* [10], [3] on the real unit interval. Such algebras become *MV-algebras* [1] if we adjoin to the axioms the 'double negation law'; for all $x \in L$, $x = x^{**}$. More precisely, an MV-algebra is a structure $\langle L, \odot, \oplus, ^*, \mathbf{0}, \mathbf{1}\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle L, \oplus, \mathbf{0}\rangle$ is is an abelian monoid and for any $x, y \in L$, $x \oplus \mathbf{1} = \mathbf{1}$, $x = x^{**}$, $\mathbf{0}^* = \mathbf{1}$, $(x \oplus y)^* = x^* \odot y^*$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$. We do not list all the properties of MV-algebras, however, in the sequel the following

$$x \vee y = (x \to y) \to y,\ x \wedge y = (x^* \vee y^*)^*.$$

will be needed. Generalized Lukasiewicz structures are examples of MV-algebras, the other structures mentioned above are not. MV-algebras coincide with *Wajsberg algebras*;

Definition 2 *Let L be a non-void set, $\mathbf{1} \in L$ and $\to$, * be a binary and a unary operation, respectively, defined on L such that, for each $x, y, z \in L$,*

$$\mathbf{1} \to x = x, \tag{1}$$

$$(x \to y) \to [(y \to z) \to (x \to z)] = \mathbf{1}, \tag{2}$$

$$(x \to y) \to y = (y \to x) \to x, \tag{3}$$

$$(x^* \to y^*) \to (y \to x) = \mathbf{1}. \tag{4}$$

Then the system $\langle L, \to, ^*, \mathbf{1}\rangle$ is called a Wajsberg algebra.

Namely, given an MV-algebra $\langle L, \odot, \oplus, ^*, \mathbf{0}, \mathbf{1}\rangle$, we obtain a Wajsberg algebra $\langle L, \to, ^*, \mathbf{1}\rangle$ by setting $x \to y = x^* \oplus y$ and conversely, if a Wajsberg algebra is given, then we define $x \oplus y = x^* \to y$, $x \odot y = (x^* \oplus y^*)^*$, $\mathbf{0} = \mathbf{1}^*$ and obtain an MV-algebra. It is of importance to realize that if a complete residuated lattice L fulfils, for all elements $x \in L$ and all subsets $\{y_i \mid i \in \Gamma\} \subseteq L$,

$$\bigvee_{i\in\Gamma} (y_i \to x) = \bigwedge_{i\in\Gamma} y_i \to x, \tag{5}$$

$$\bigvee_{i\in\Gamma} (x \to y_i) = x \to \bigvee_{i\in\Gamma} y_i, \tag{6}$$

then this structure is an MV-algebra. From this fact it also follows that on $[0,1]$ the only residuated structures with continuous residuum on both variables (as a function $\to : [0,1]^2 \searrow [0,1]$) are complete MV-algebras.

Let L and K be two MV-algebras. A mapping $h : L \searrow K$ defined on the whole L is called a *MV-homomophisms* if, for all $x, y \in L$, it holds that $h(x \to y) = h(x) \to h(y)$ and $h(x^*) = h(x)^*$. It is easy to see that in such a case $h(x \oplus y) = h(x) \oplus h(y)$, $h(x \odot y) = h(x) \odot h(y)$, $h(x \vee y) = h(x) \vee h(y)$, $h(x \wedge y) = h(x) \wedge h(y)$ and if $x \leq y$ then $h(x) \leq h(y)$.

An MV-algebra K is called *injective* if, for all MV-algebras M, L, $M \subseteq L$, and for all MV-homomorphisms $h : M \searrow K$, h can be extended to all L. On the real unit interval, injective MV-algebras are structurally isomorphic with the Lukasiewicz structure.

2 Semantics of Fuzzy Propositional Logic

The formalized language of Fuzzy Propistional Logic is composed of four kind of building blocks:
(i) The set of *propositional variables* (*propositions*) is a set $L = \{p_i \mid i \in \mathcal{N}\}$. Propositions are sometimes denoted by p, q, r, s, t, w, too.
(ii) for any element $a \in L$ there is a *truth constant* $\mathbf{a}$ in the formalized language of Fuzzy Propositional Logic.
(iii) The logical connectives of Fuzzy Propositional Logic are `imp` (read 'implies') and `and` (read 'and' or 'conjunction').
(iv) There are auxiliary symbols $\},],), (, [, \{$ in the language of Fuzzy Propositional Logic.

Truth constants and propositional variables are *atomic formulas.*

Definition 3 *The set $\mathcal{F}$ of* well-formed formulas *is constructed in the following way:* (i) *atomic formulas are in $\mathcal{F}$,* (ii) *if α and β are in $\mathcal{F}$ then (α* `imp` *β) and (α* `and` *β) are in $\mathcal{F}$.*

Propositions correspond statements like *John is tall, John is very tall, It is raining, It is slightly raining, This power plant is performing well, This power* *plant is usually performing well,* etc. We do not study the mutual relation of propositions like *John is tall* and *John is very tall,* but take them as atoms. Instead, we tend to think that the fact *This power plant is usually performing well*, for example, can be expressed by associating a truth value close to 1 (but not 1!) with an expression *This power plant is performing well.* In general, we are more interested in the mathematical properties of the language of Fuzzy Propostional Logic as a whole.

At this stage it should be pointed out that the language of Fuzzy Propositional Logic is much simplier that natural language. For example, there is no causal link between formulas α and β in the formula $(\alpha \texttt{ imp } \beta)$, nor is there no temporal order among them in the formula $(\alpha \texttt{ and } \beta)$. The statement *If I swich of the light, then there will be dark in this room* is quite correct in natural language and contains an idea of causal link between the statements *I swich of the light* and *There is dark in this room* while the statement *If the Moon is a big cheese then A is equal to A* sounds odd and meaningless in natural language; whatever the Moon is A is equal to itself anyway. In the formalized language of Fuzzy Propositional Logic both statements, however, are well-formed formulas of the form $(\alpha \texttt{ imp } \beta)$. Similarly, the expression *During the lunch hour Giuseppe went home and made love* has different meaning in natural language than the expression *During the lunch hour Giuseppe made love and went home.* In the formalized language of Fuzzy Propositional Logic this kind of temporal difference between formulas $(\alpha \texttt{ and } \beta)$ and $(\beta \texttt{ and } \alpha)$ does not occur. To avoid unnecessary paranthesis we may write $(\alpha \texttt{ and } \beta \texttt{ and } \gamma)$ instad of $[(\alpha \texttt{ and } \beta) \texttt{ and } \gamma]$ or $[\alpha \texttt{ and } (\beta \texttt{ and } \gamma)]$. An expression $(\alpha \texttt{ imp } \beta \texttt{ imp } \gamma)$ is not, however, a well-formed formula; one should write either $[(\alpha \texttt{ imp } \beta) \texttt{ imp } \gamma]$ or $[\alpha \texttt{ imp } (\beta \texttt{ imp } \gamma)]$ which are two different formulas.

The truth constants $\mathbf{a}$, $a \in L$, generalize the falsum sign $\perp$ of classical logic. We follow the idea of intuitionistic logic and abbreviate $(\alpha \texttt{ imp } \mathbf{0})$ by $(\texttt{non}-\alpha)$. This new logical connective non (read 'non') is called *negation.*

Giving *semantic interpretation* to a formula $\alpha \in \mathcal{F}$ means we associate a value of truth $v(\alpha) \in L$ to α, in other words, we define *truth value function* $v : \mathcal{F} \searrow L$. Before introducing an exact mathematical definition of truth value function v, let us discuss the properties this function should obey. A very natural assumption is (V1):

$v(\alpha)$ *is defined for each well-formed formula* $\alpha \in \mathcal{F}$.

Since we are generalizing Classical Propositional Logic and since for all classical truth value functions v holds $v(\perp) = 0$, we set (V2):

For all truth constants $\mathbf{a}$, $v(\mathbf{a}) = a$.

We adopt *Principle of Truth Functionality* into Fuzzy Propositional Logic (V3):

For all $\alpha, \beta \in \mathcal{F}$, $v(\alpha \texttt{ imp } \beta)$, $v(\alpha \texttt{ and } \beta)$ *depend only on* $v(\alpha)$, $v(\beta)$.

A crucial question in defining truth value for a formula containing the logical connective imp is *What should the value* $v(\alpha \texttt{ imp } \beta)$ *tell us?* As there is not necessarily any causal link between formulas α and β in a formula $(\alpha \texttt{ imp } \beta)$ we accept *Principle for Implication* (PI):

The degree of truth of a formula $(\alpha \texttt{ imp } \beta)$ *quantifies the degree by which* β *is at least as true as* α.

In other words, (PI) states that the degree $v(\alpha \texttt{ imp } \beta)$ quantifies the degree by which $v(\beta)$ is at least as large as $v(\alpha)$. A direct consequence of (PI) is *Boundary Condition* (BC):

$$\text{For all } \alpha, \beta \in \mathcal{F},\ v(\alpha \texttt{ imp } \beta) = \mathbf{1} \text{ iff } v(\alpha) \leq v(\beta).$$

Remark 1 *One of the most important properties of the logical connective* imp *is that* imp *defines a quasi-order on the set of well-formed formulas. Accepting* (PI)*, this property holds in Fuzzy Propositional Logic, too.*

Indeed, define, for all $\alpha, \beta \in \mathcal{F}$, $\alpha \preceq \beta$ iff $v(\alpha \texttt{ imp } \beta) = \mathbf{1}$. Then, as $v(\alpha) \leq v(\alpha)$, we have $v(\alpha \texttt{ imp } \alpha) = \mathbf{1}$, thus $\alpha \preceq \alpha$ for any $\alpha \in F$. Moreover, if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $v(\alpha \texttt{ imp } \beta) = \mathbf{1}$, $v(\beta \texttt{ imp } \gamma) = \mathbf{1}$, thus $v(\alpha) \leq v(\beta) \leq v(\gamma)$ and so $v(\alpha \texttt{ imp } \gamma) = \mathbf{1}$, hence $\alpha \preceq \gamma$. We assume *Monotonicity of Implication* (MI):

$$\text{For all } \alpha, \beta, \gamma \in \mathcal{F},\ \text{if } v(\alpha) \leq v(\beta),\ \text{then } v(\gamma \texttt{ imp } \alpha) \leq v(\gamma \texttt{ imp } \beta),$$
$$v(\beta \texttt{ imp } \gamma) \leq v(\alpha \texttt{ imp } \gamma).$$

The truth constant $\mathbf{1}$ corresponds to tautology in Classical Propositional Logic. It is therefore natural to assume that the degree of truth of a formula $(\alpha \texttt{ and } \mathbf{1})$ coincides with the degree of truth of the formula α, (V4):

$$\text{For all } \alpha \in \mathcal{F},\ v(\alpha \texttt{ and } \mathbf{1}) = v(\alpha).$$

Natural conditions for the degree of truth of formulas containing the connective and are also the following (V5):

$$\text{For all } \alpha, \beta, \gamma \in \mathcal{F},\ \text{if } v(\alpha) \leq v(\beta),\ \text{then } v(\alpha \texttt{ and } \gamma) \leq v(\beta \texttt{ and } \gamma)$$

and (V6):

$$\text{For all } \alpha, \beta, \gamma \in \mathcal{F},\ v(\alpha \texttt{ and } (\beta \texttt{ and } \gamma)) = v((\alpha \texttt{ and } \beta) \texttt{ and } \gamma).$$

Finally, the degree of truth of a formula $[(\alpha \texttt{ and } \beta) \texttt{ imp } \gamma]$ should coincide with the degree of truth of the formula $[\alpha \texttt{ imp } (\beta \texttt{ imp } \gamma)]$ as well as with the degree of the truth of the formula $[\beta \texttt{ imp } (\alpha \texttt{ imp } \gamma)]$, i.e. (V7):

$$\text{For all } \alpha, \beta, \gamma \in \mathcal{F},$$
$$v((\alpha \texttt{ and } \beta) \texttt{ imp } \gamma) = v(\alpha \texttt{ imp } (\beta \texttt{ imp } \gamma)) = v(\beta \texttt{ imp } (\alpha \texttt{ imp } \gamma)).$$

Theorem 1 *Assume there exists a function* $v : \mathcal{F} \searrow L$ *such that all the conditions mentioned above hold. Then the set* L *of values of truth is a residuated lattice.*

Proof. Omitted.

If we assume that small variation in the value of truth in any argument of a formula of the form $(\alpha \texttt{ imp } \beta)$ should not lead to a large change of value of truth in the whole formula, then this requires *Continuity of Implication* (V8):

For all $\alpha, \beta \in \mathcal{F}$, $v(\alpha \texttt{ imp } \beta)$ *is continous in its arguments.*

Thus, by the consideration in the previous section, to fulfil this postulate, the set of values of truth must form a complete MV-algebra. In particular, as injective MV-algebras on the real unit interval coincide with complete MV-algebras, we have

Theorem 2 *Postulates* (V1) - (V8), (PI), (MI) *presuppose an injective MV-algebra structure on* $[0,1]$*-valued Fuzzy Propositional Logic.*

We set

Definition 4 *A function* $v : \mathcal{F} \searrow L$ *such that, for any formulas* α, β *and for any truth constant* **a**,

$$v(\mathbf{a}) = \mathbf{a},\ v(\alpha \texttt{ imp } \beta) = v(\alpha) \to v(\beta),\ v(\alpha \texttt{ and } \beta) = v(\alpha) \odot v(\beta),$$

is called valuation *or* truth value function.

By the consideration above, if the set of values of truth is an injective MV-algebra, then any valuation v satisfies postulates (V1)-(V8), (PI), (MI). In such a case we introduce a logical connective `or` (read 'or' or 'disjunction') as an abbreviation

$$(\alpha \texttt{ or } \beta) = [\texttt{non-(non-}\alpha \texttt{ and non-}\beta)].$$

This generalizes the state of affairs in Classical Propositional Logic and makes the formalized language of Fuzzy Propositional Logic easier to read.

Remark 2 *For any valuation* v*, any* $\alpha, \beta \in \mathcal{F}$,

$$v(\texttt{non-}\alpha) = v(\alpha)^*,\ v(\alpha \texttt{ or } \beta) = v(\alpha) \oplus v(\beta).$$

Generally, $v(\alpha \texttt{ and } \beta) < v(\alpha) \wedge v(\beta)$ and $v(\alpha) \vee v(\beta) < v(\alpha \texttt{ or } \beta)$. In application we may need, however, disjunctive and conjunctive connectives, denote them by $\overline{\texttt{and}}$ and $\overline{\texttt{or}}$, respectively, such that

$$v(\alpha\ \overline{\texttt{and}}\ \beta) = v(\alpha) \wedge v(\beta),\ v(\alpha\ \overline{\texttt{or}}\ \beta) = v(\alpha) \vee v(\beta).$$

This can be done by abbreviating

$$(\alpha\ \overline{\texttt{or}}\ \beta) = [(\alpha\ \texttt{imp}\ \beta)\ \texttt{imp}\ \beta)]$$
$$(\alpha\ \overline{\texttt{and}}\ \beta) = [\texttt{non-(non-}\alpha\ \overline{\texttt{or}}\ \texttt{non-}\beta)],$$

even if the abbreviation of the logical connective $\overline{\texttt{or}}$ is far from being obvious. Indeed, we then have

$$v(\alpha\ \overline{\texttt{or}}\ \beta) = [v(\alpha) \to v(\beta)] \to v(\beta) = v(\alpha) \vee v(\beta),$$
$$v(\alpha\ \overline{\texttt{and}}\ \beta) = [v(\alpha)^* \vee v(\beta)^*]^* = v(\alpha) \wedge v(\beta).$$

We introduce a logical connective `equiv` (read 'equivalent') by abbreviating

$$(\alpha\ \texttt{equiv}\ \beta) = [(\alpha\ \texttt{imp}\ \beta)\ \overline{\texttt{and}}\ [(\beta\ \texttt{imp}\ \alpha)],$$

thus generalizing the situation in Classical Propositional Logic. Then we have, for any valuation v, any formulas $\alpha, \beta \in \mathcal{F}$,

$$v(\alpha\ \texttt{equiv}\ \beta) = v(\alpha) \leftrightarrow v(\beta),$$

where $\leftrightarrow$ is the *bi-residuum* and defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. We also introduce `xor`, a logical connective called *exclusive or* and defined by an abbreviation

$$(\alpha\ \texttt{xor}\ \beta) = (\alpha\ \texttt{or}\ \beta)\ \overline{\texttt{and}}\ (\alpha\ \texttt{imp non-}\beta)\ \overline{\texttt{and}}\ (\beta\ \texttt{imp non-}\alpha).$$

Then, for any valuation v, any formulas α, β,

$$v(\alpha\ \texttt{xor}\ \beta) = v(\beta\ \texttt{xor}\ \alpha) = v([\alpha\ \texttt{or}\ \beta]\ \overline{\texttt{and}}\ \texttt{non-}[\alpha\ \texttt{and}\ \beta]).$$

In Classical Propositional Logic truth value functions $v : \mathcal{F} \searrow \{0, 1\}$ satisfy the truth table

$v(\alpha)$	$v(\beta)$	$v(\texttt{non}-\alpha)$	$v(\alpha\ \texttt{or}\ \beta$	$v(\alpha\ \texttt{and}\ \beta$	$v(\alpha\ \texttt{imp}\ \beta$
1	1	0	1	1	1
1	0	0	1	0	0
0	1	1	1	0	1
0	0	1	0	0	1

It is easy to verify that fuzzy valuations satisfy this table and that in two valued case the truth value tables of the logical connectives $\overline{\texttt{and}}$ and `and` as well as $\overline{\texttt{or}}$ and `or` coincide. Complete truth tables are not, of course, possible in logic with infinite many values of truth. We may, however, write instances of them.

In logic we are interested in the logical consequences of given statements. From semantic point of view this raises a question *Associating fixed values of truth to a set of well-formed formulas $T \subseteq \mathcal{F}$, what is the least degree of truth, or greatest lower bound of such degrees, of an arbitrary formula $\alpha \in \mathcal{F}$ with respect to T?*

This leads us to the following semantic definitions

Definition 5 *A* fuzzy (sub-)set T of formulas *is a function* $T : \mathcal{F} \searrow L$. *A truth value function* $v : \mathcal{F} \searrow L$ satisfies T *if* $T(\alpha) \leq v(\alpha)$ *for any formula* $\alpha \in \mathcal{F}$. *If there is a valuation* v *which satisfies* T *the* T *is called* satisfiable.

The void set $\emptyset$ can be regarded as a fuzzy subset of formulas by defining $\emptyset(\alpha) = 0$ for all formulas $\alpha \in \mathcal{F}$. The void set is of course satisfiable.

Definition 6 *The* degree of validity *of a formula* $\alpha \in \mathcal{F}$ *(with respect to a fuzzy subset of formulas* T*) is a value*

$$\mathcal{C}^{\mathtt{sem}}(T)(\alpha) = \bigwedge\{v(\alpha) \mid v \textit{ satisfies } T\}. \tag{7}$$

In particular, if T *is the void set we define the* degree of taulology *of a formula* α *by*

$$\mathcal{C}^{\mathtt{sem}}(\alpha) = \bigwedge\{v(\alpha) \mid v \textit{ is a valuation }\}. \tag{8}$$

This definition is very natural and generalizes the consept of tautology in Classical Propositional Logic. Using g.l.b in definitions (7) and (8) explains partly why we assume the truth value set L forms a complete lattice. If $\mathcal{C}^{\mathtt{sem}}(T)(\alpha) = a$ we write $T \models_a \alpha$, in particular $\models_a \alpha$ if T is the void set. Of special interest will be formulas α such that $\models_1 \alpha$. Evidently, if $\models_1 \alpha$, then $T \models_1 \alpha$ for any fuzzy subset T of formulas (even for those T which are not satisfied by any valuation v!)

When counting the degree of validity of a formula α, we are minimizing $v(\alpha)$ for such valuations v that satisfy T. How can this be done? We have already seen that formulas containing logical connectives `non`, $\overline{\texttt{and}}$, $\overline{\texttt{or}}$, `equiv`, `xor` are abbreviations of formulas containing only the logical connectives `imp` and `and` and the truth constant **0**. Moreover, for any valuation v, any well-formed formulas α, β,

$$\begin{aligned} v(\texttt{non-}(\alpha \texttt{ imp non-}\beta)) &= [v(\alpha) \to v(\beta)^*]^* \\ &= [v(\alpha)^* \oplus v(\beta)^*]^* \\ &= v(\alpha) \odot v(\beta) \\ &= v(\alpha \texttt{ and } \beta), \end{aligned}$$

thus a formula (α `and` β) can be regarded as an abbreviation of a formula [`non-(`α `imp non-`β)]. We conclude that any well-formed formula α can be replaced by a formula β such that β contains no other connectives that `imp` and that $v(\alpha) = v(\beta)$ for any valuation v. Therefore, the problem of minimizing $v(\alpha)$ coincides with the problem of minimizing $v(\beta)$. For short formulas β this problem can be solved easily. In general, the problem is the minimize a *McNaughton function.*

Example 1 Let the set of values of truth form the Lukasiewicz structure. We are interested in the degree of tautology of a formula α standing for

Assuming winter is long implies either winter is not cold or damages caused by floods are not serious, then insurence business is profitable.

First we have to formalize α. Consider prositional variables p, q, r, s, where p corresponds to *winter is long*, q corresponds to *winter is cold* , r corresponds to *damages caused by floods are serious*, s corresponds to *incurence business is profitable.* Then α can be expressed by

$$[p \ \texttt{imp} \ (\ \texttt{non-}q \ \overline{\texttt{or}} \ \texttt{non-}r)] \ \texttt{imp} \ s.$$

Now we should minimize $[v(p) \to (v(q)^* \vee v(r)^*)] \to v(s)$. Take $v(p) = v(q) = v(r) = v(s) = 0$. Then $v(\alpha) = [0 \to (0^* \vee 0^*)] \to 0 = 0$. Thus, $\models_0 \alpha$, i.e. the statement α has no general validity. Next consider a fuzzy set T of formulas such that $T(p) = 0.2$ (winter is practically not long at all), $T(q) = 0.8$ (winter is relatively cold), $T(r) = 0.1$ (damages caused by floods are minimal) $T(s) = 0.8$ (insurence business is quite profitable), $T(\beta) = 0$ elsewhere. T is satisfiable; indeed, a valuation v such that $v(p) = 0.2$, $v(q) = 0.8$, $v(r) = 0.1$ $v(s) = 0.8$ satisfies T. Moreover, $v(\alpha) = [0.2 \to (0.2 \vee 0.9)] \to 0.8 = 0.8$. This implies $\mathcal{C}^{\texttt{sem}}(T)(\alpha) \leq 0.8$. On the other hand, since $v(s) \leq v(\beta) \to v(s) = v(\beta \ \texttt{imp} \ s)$ $= v(\alpha)$, we reason that $0.8 \leq \mathcal{C}^{\texttt{sem}}(T)(\alpha)$. Thus, $T \models_{0.8} \alpha$.

Roughly speaking, the meaning of a value $\mathcal{C}^{\texttt{sem}}(T)(\alpha)$ can be illustrated by saying

In all those worlds where any formula β is at least $T(\beta)$-true, a fixed formula α is $\mathcal{C}^{\texttt{sem}}(T)(\alpha)$-true.

A *consequence operation* $\mathcal{C}$ in Classical Propositional Logic, due to Tarski, is an operation $\mathcal{C} : \mathcal{F} \searrow \mathcal{F}$ such that, for any sets $X, Y \subseteq \mathcal{F}$, (i) $X \subseteq \mathcal{C}(X)$, (ii) if $X \subseteq Y$, then $\mathcal{C}(X) \subseteq \mathcal{C}(Y)$, (iii) $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}(X))$. Since different two-valued logical systems such as Classical Propostional Logic, Intuitionistic Logic, Modal Logic, etc. can be characterized and distinguished by means of the sets of taulologies in them, logic is sometimes *defined* to be a consequence operation. The concept of consequence operation can be generalized into fuzzy logic in the following way; first we realize that the collection $\widetilde{\mathcal{F}}$ of all fuzzy subsets of formulas is partially ordered by the relation $X \leq Y$ iff $X(\alpha) \leq Y(\alpha)$ for any $\alpha \in \mathcal{F}$. Then we set

Definition 7 *An operation $\mathcal{C} : \widetilde{\mathcal{F}} \searrow \widetilde{\mathcal{F}}$ is a* fuzzy consequence operation *if, for any $X, Y \in \widetilde{\mathcal{F}}$,* (C1) $X \leq \mathcal{C}(X)$, (C2) *if $X \leq Y$ then $\mathcal{C}(X) \leq \mathcal{C}(Y)$,* (C3) $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}(X))$.

Proposition 1 *The operation $\mathcal{C}^{\texttt{sem}}$ is a fuzzy consequence operation, called* semantic consequence operation.

Proof. Let X, Y be fuzzy subsets of formulas. First we realize that any valuation v can be regarded as an element of $\widetilde{\mathcal{F}}$ and obviously $X \leq \mathcal{C}^{\mathtt{sem}}(X)$. Hence (C1) holds. Clearly, if a valuation v satisfies X and $X \leq Y$ then v satisfies Y, too, thus (C2). Finally, $X \leq v$ iff $\mathcal{C}^{\mathtt{sem}}(X) \leq v$, hence $\mathcal{C}^{\mathtt{sem}}(X) = \mathcal{C}^{\mathtt{sem}}(\mathcal{C}^{\mathtt{sem}}(X))$. □

Let T be a fuzzy set of formulas. A *finite part* of T is a fuzzy set of formulas S such that $T(\alpha) = S(\alpha)$ for finite many formulas $\alpha \in \mathcal{F}$ and $S(\alpha) = \mathbf{0}$ elsewhere. Clearly, any valuation v which satisfies T satisfies also S, thus, for any $\alpha \in \mathcal{F}$, $\bigwedge\{v(\alpha) \mid \ v \textit{ satisfies } S\} \leq \bigwedge\{v(\alpha) \mid \ v \textit{ satisfies } T\}$, i.e. $\mathcal{C}^{\mathtt{sem}}(S) \leq \mathcal{C}^{\mathtt{sem}}(T)$ for any finite part S of T. This implies

$$\bigvee\{\mathcal{C}^{\mathtt{sem}}(S) \mid \ S \textit{ is a finite part of } T\} \leq \mathcal{C}^{\mathtt{sem}}(T).$$

Definition 8 *A fuzzy consequence operation $\mathcal{C}$ is* continuous *if, for any fuzzy set of formulas T,*

$$\bigvee\{\mathcal{C}(S) \mid \ S \textit{ is a finite part of } T\} = \mathcal{C}(T) \tag{9}$$

Then we have

Proposition 2 *Consider $[0,1]$-valued Fuzzy Propositional Logic. Let the set of values of truth form a residuated lattice but not an MV-algebra. Then the semantic consequence operation $\mathcal{C}^{\mathtt{sem}}$ is not continous.*

Outline of the proof: There is an element $a \in [0,1]$ and an infinite subset $\{b_i\}_{i\in\Gamma} \subseteq [0,1]$ such that $\bigvee_{i\in\Gamma}(a \to b_i) = c < a \to (\bigvee_{i\in\Gamma} b_i)$. Let p be a fixed propositional variable and T a fuzzy subset of formulas such that $T(\mathbf{b}_i \ \mathtt{imp} \ p)$ $= 1$ for any $i \in \Gamma$ and $T(\alpha) = 0$ elsewhere. We show that

$$\begin{aligned} \bigvee\{\mathcal{C}^{\mathtt{sem}}(S)(\mathbf{a} \ \mathtt{imp} \ p) \mid \ S \textit{ is a finite part of } T\} &< \mathcal{C}^{\mathtt{sem}}(T)(\mathbf{a} \ \mathtt{imp} \ p) \\ &= a \to (\textstyle\bigvee_{i\in\Gamma} b_i). \end{aligned}$$

Proposition 3 *Let α, β, γ, α_1, β_1, α_2, β_2 be formulas and $\mathbf{c}$ a truth constant. Then the following forms of formulas are universally valid at the degree $\mathbf{1}$, except for, of course, the inner truth value $\mathbf{c}$, which is universaly valid at the degree c, i.e.*

$$\models_1 \alpha \ \mathtt{imp} \ \alpha, \tag{10}$$

$$\models_1 (\alpha \ \mathtt{imp} \ \beta) \ \mathtt{imp} \ [(\beta \ \mathtt{imp} \ \gamma) \ \mathtt{imp} \ (\alpha \ \mathtt{imp} \ \gamma)], \tag{11}$$

$$\models_1 (\alpha_1 \mathtt{imp} \beta_1) \ \mathtt{imp} \ \{(\beta_2 \mathtt{imp} \alpha_2) \ \mathtt{imp} \ [(\beta_1 \mathtt{imp} \beta_2) \ \mathtt{imp} \ (\alpha_1 \mathtt{imp} \alpha_2)]\}, \tag{12}$$

$$\models_1 \alpha \ \mathtt{imp} \ \mathbf{1}, \tag{13}$$

$$\models_1 \mathbf{0} \ \mathtt{imp} \ \alpha, \tag{14}$$

$$\models_1 (\alpha \ \mathtt{and} \ \ \mathtt{non}\text{-}\alpha) \ \mathtt{imp} \ \mathbf{0}, \tag{15}$$

$$\models_c \mathbf{c}, \tag{16}$$

$$\models_1 \alpha \ \mathtt{imp} \ (\beta \ \mathtt{imp} \ \alpha), \tag{17}$$

$$\models_1 (\mathbf{1} \texttt{ imp } \alpha) \texttt{ imp } \alpha, \quad (18)$$

$$\models_1 [(\alpha \texttt{ imp } \beta) \texttt{ imp } \beta] \texttt{ imp } [(\beta \texttt{ imp } \alpha) \texttt{ imp } \alpha], \quad (19)$$

$$\models_1 (\texttt{non-}\alpha \texttt{ imp } \texttt{non-}\beta) \texttt{ imp } (\beta \texttt{ imp } \alpha). \quad (20)$$

Proof. Follows by properties valid in Wajsberg algebras.□

Example 2 Consider the following statements of natural language
(α) If wages rise or prices rise there will be inflation.
(β) If there will be inflation, then the Government has to stop it or people will suffer.
(γ) If people will suffer the Government will lose its popularity.
(δ) The Government will not stop the inflation and the Government will not lose its popularity.
(ϵ) Wages will not rise.
Formalize them by writing $[(p \texttt{ or } q) \texttt{ imp } r] = \alpha$, $r \texttt{ imp } (s \texttt{ or } t) = \beta$, $t \texttt{ imp } w = \gamma$, $\texttt{non-}s \texttt{ and } \texttt{non-}w = \delta$, $\texttt{non-}p = \epsilon$, where p stands for *Wages rise,* q stands for *Prices rise,* r stand for *There is inflation,* s stands for *The Government stops inflation,* t stands for *People suffer,* w stands for *The Government loses its popularity.* Then we define a fuzzy subset of formulas T by $T(\alpha) = 1$, $T(\beta) = 0.9$, $T(\gamma) = 0.8$ and $T(\delta) = 1$, $T(\mu) = 0$ elsewhere. Our task now is to calculate the degree of validity of the formula ϵ in Lukasiewicz valued Fuzzy Propositional Logic useng `and` and `or` connectives. Any valuation v which satisfies T has a property

$$v(\texttt{non-}s \texttt{ and } \texttt{non-}w) = v(s)^* \odot v(w)^* = 1,$$

thus $v(s) \oplus v(w) = 0$ and therefore $v(s) = v(w) = 0$. Since $v(t \texttt{ imp } w) = v(t) \to v(w) = v(t) \to 0 \geq 0.8$, we conclude $0 \leq v(t) \leq 0.2$ for any v satisfying T. Study first the case $v_1(t) = 0.2$. Then

$$\begin{aligned} v_1(r \texttt{ imp } (s \texttt{ or } t)) &= v_1(r) \to [v_1(s) \oplus v_1(t)] \\ &= v_1(r) \to (0 \oplus 0.2) \\ &= 1 - v_1(r) + 0.2 \\ &\geq 0.9. \end{aligned}$$

Thus, $v_1(r) \leq 0.3$. Now $v(\texttt{non-}p)$ obtains the smallest value when $v(p)$ obtains the largest value, and, by assumption, $v(p) \oplus v(q) \leq v(r)$. Take therefore $v_1(q) = 0$, $v_1(r) = 0.3$, then $v_1(p) \leq 0.3$, in particular, take $v_1(p) = 0.3$. Then $v_1(\texttt{non-}p) = 0.7$. Clearly, v_1 satisfies T. Next, study the case $v_2(t) = 0$. Since $0.9 \leq v_2(r) \to [v_2(s) \oplus v_2(t)] = v_2(r)^*$, we have $v_2(r) \leq 0.1$. Again, $v_2(p) \oplus v_2(q) \leq v_2(r)$, so take $v_2(r) = 0.1$, $v_2(q) = 0$, then $v_2(p) \leq 0.1$, thus $0.9 \leq v_2(\texttt{non-}p)$.

We conclude $\mathcal{C}^{\texttt{sem}}(T)(\texttt{non-}p) = v_1(\texttt{non-}p) = 0.7$.

3 Axiom System for Fuzzy Propositional Logic

The definitions of valuation and the degree of validity of formula α are natural and relatively easy generalizations of the corresponding concepts in two valued logic. Now we consider the following related non-trivial problem

Knowing that a formula α is valid at a certain degree, do there exist a fuzzy subset of axioms and fuzzy rules of inference by which we can infer α at the same degree?

In other words, is there a syntactic consequence operation which coincides with the semantic consequence operation, i.e. is Fuzzy Propositional Logic axiomatizable? To find an answer to this question, we start by defining on what we mean by fuzzy axiom, fuzzy rule of inference, fuzzy proof, etc.

A *rule of inference* in Classical Propositional Logic is an n-ary operation on the set of well-formed formulas which with a finite sequence of formulas $\alpha_1, \cdots, \alpha_n$ $(1 \leq n)$ in a formalized language associates another formula β in this language in such a way that β is a logical consequence of the formulas $\alpha_1, \cdots, \alpha_n$. This fact is usually denoted as follows

$$\frac{\alpha_1, \cdots, \alpha_n}{\beta}$$

Formulas $\alpha_1, \cdots, \alpha_1$ are called *premises* and β the *conclusion* of this rule of inference. For example,

$$\frac{\mathtt{non} - (\mathtt{non} - \alpha)}{\alpha}$$

and

$$\frac{\alpha, (\alpha \ \mathtt{imp} \ \beta)}{\beta}$$

are rules of inference in Classical Propositional Logic, called *Rule of Double Negation* and *Modus Ponens,* respectively. By saying that a formula β is a *logical consequence* of a set S of formulas we mean that if every formula α belonging to S is acknowledged to be true, then β must be accepted as true. Thus, the most important property of rule of inference is soundness, i.e. rule of inference preserves truth.

We define a fuzzy rule of inference as consisting of two componets. The first component operates on formulas and is, in fact, a rule of inference in the usual sense; the second component operates on truth values and says how the truth value of the conclusion is to be computed from the truth values of the premises such that the degree of truth is preserved. More accurately, we set

Definition 9 *An n-ary* fuzzy rule of inference *is a scheme*

$$R \quad : \quad \frac{\alpha_1, \cdots, \alpha_n}{r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n) = \beta}, \frac{a_1, \cdots, a_n}{r^{\mathtt{sem}}(a_1, \cdots, a_n) = b}$$

where the well-formed formulas $\alpha_1, \cdots, \alpha_n$ are the premises *and the well-formed formula β is the* conclusion. *The values $a_1, \cdots, a_n, b \in L$ are the corresponding truth values. The mapping $r^{\mathtt{sem}} : L^n \searrow L$ is* semi-continuous *on each variable, i.e. it holds always that*

$$r^{\mathtt{sem}}(a_1, \cdots, \bigvee_{j \in \Gamma} a_{k_j}, \cdots, a_n) = \bigvee_{j \in \Gamma} r^{\mathtt{sem}}(a_1, \cdots, a_{k_j}, \cdots, a_n), 1 \leq k \leq n.$$

We assume the fuzzy rule of inference is sound, *i.e. for each valuation v holds*

$$r^{\mathtt{sem}}(v(\alpha_1), \cdots, v(\alpha_n)) \leq v(r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n)).$$

Proposition 4 *The following schemes are fuzzy rules of inference in complete MV-algebra valued Fuzzy Propositional Logic.*

Generalized Modus Ponens:

$$R_{GMP} \quad : \quad \frac{\alpha, (\alpha \ \mathtt{imp} \ \beta)}{\beta}, \frac{a, b}{a \odot b}$$

a-Consistency-testing rules:

$$R_{a-CTR} \quad : \quad \frac{\mathbf{a}}{\mathbf{0}}, \frac{b}{c}$$

where **a** *is a truth constant, and $c = \mathbf{0}$ if $b \leq a$ and $c = \mathbf{1}$ elsewhere.*
a-Lifting Rules:

$$R_{a-LR} \quad : \quad \frac{\alpha}{(\mathbf{a} \ \mathtt{imp} \ \alpha)}, \frac{b}{a \to b}$$

where **a** *is a truth constant.*
Rule of Bold Conjunction:

$$R_{RBC} \quad : \quad \frac{\alpha, \beta}{(\alpha \ \mathtt{and} \ \beta)}, \frac{a, b}{a \odot b}$$

Proof. Omitted.

Remark 3 *In general residuated lattice valued Fuzzy Propositional Logic, the semi-continuous condition ceases to hold for the* **a**-*Lifting Rules.*

Definition 10 *A fuzzy (sub-)set* $\mathtt{A}$ *of* logical axioms *in Fuzzy Propositional Logic is a finite set of forms of formulas each being a truth constant* $\mathbf{a}$*, then* $\mathtt{A}(\mathbf{a}) = a$*, or a tautology* α *at the degree* $\mathbf{1}$*, then* $\mathtt{A}(\alpha) = \mathbf{1}$*. Elsewhere* $\mathtt{A}(\alpha) = \mathbf{0}$*.*

For example, forms of formulas in (10)-(20) form a set of logical axioms. Values $\mathtt{A}(\delta)$ are obvious.

Definition 11 *Let* $\mathtt{A}$ *be a fixed set of logical axioms,* $\mathtt{R}$ *a fixed finite set of fuzzy rules of inference and* T *fuzzy set of formulas called* non-logical axioms*. Then a (zero-order)* fuzzy theory *is a triplet* $\langle \mathtt{A}, \mathtt{R}, T\rangle$*. In particular, if the set of logical axioms* $\mathtt{A}$ *is composed of forms of formulas (10) - (20), and the set of fuzzy rules of inference* $\mathtt{R}$ *contains* R_{GMP}*,* R_{a-CTR}*,* R_{a-LR}*,* R_{RBC}*, we denote a fuzzy theory simply by* T*, and if* T *is the void set we talk about* Fuzzy Propositional Calculus.

A *metaproof* of a well-formed formula α in a fuzzy theory $\langle \mathtt{A}, \mathtt{R}, T\rangle$, denoted by w, is a finite sequence

$$\begin{array}{ccc} \alpha_1 & , & a_1 \\ \vdots & & \vdots \\ \alpha_m & , & a_m \end{array}$$

of pairs $(\alpha_i, a_i) \in \mathcal{F} \times L$ such that the following holds: (i) $\alpha_m = \alpha$, (ii) for each $i, 1 \leq i \leq m, \alpha_i$ is a logical axiom, or α_i is a non-logical axiom, or there are a fuzzy rule of inference in $\mathtt{R}$ and formulas $\alpha_{i_1}, \cdots, \alpha_{i_n}$ with $i_1, \cdots, i_n < i$ such that $\alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n})$, (iii) for each $i, 1 \leq i \leq m$, the value a_i is given by

$$a_i = \begin{cases} a & \text{if } \alpha_i \text{ is the axiom } \mathbf{a} \\ \mathbf{1} & \text{if } \alpha_i \text{ is some other logical axiom} \\ T(\alpha_i) & \text{if } \alpha_i \text{ is a non-logical axiom} \\ r^{\mathtt{sem}}(a_{i_1}, \cdots, a_{i_n}) & \text{if } \alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n}). \end{cases}$$

The value a_m is denoted by $Val_{\langle \mathtt{A}, \mathtt{R}, T\rangle}(w)$ and is called the *degree* of the metaproof w. Because a formula α may have many metaproofs with different degrees, we define the *degree of deduction* of the formula α in fuzzy theory $\langle \mathtt{A}, \mathtt{R}, T\rangle$ by

$$C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha) = \bigvee \{ Val_{\langle \mathtt{A}, \mathtt{R}, T\rangle}(w) | w \textit{ is a metaproof for } \alpha \textit{ in } \langle \mathtt{A}, \mathtt{R}, T\rangle\}.$$

The case $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha) = a$ is denoted by $\langle \mathtt{A}, \mathtt{R}, T\rangle \vdash_a \alpha$, in particular, $\vdash_a \alpha$ if the set of logical axioms $\mathtt{A}$ is composed of (10)-(20), the set of fuzzy rules of inference $\mathtt{R}$ contains R_{GMP}, R_{a-CTR}, R_{a-LR}, R_{RBC} and T is the void set.

The following two propositions hold for any complete residuated lattice valued Fuzzy Propositional Logic.

Proposition 5 *The operation* $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}$ *is a fuzzy consequence operation, called* syntactic consequence operation.

Proof. Clearly, $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle} : \widetilde{\mathcal{F}} \searrow \widetilde{\mathcal{F}}$. Let X, Y be fuzzy sets of formulas such that $X \leq Y$. Since, for any $\alpha \in \mathcal{F}$,

$$\alpha \quad , \quad X(\alpha) \leq Y(\alpha)$$

can be regarded as a metaproof for α in the fuzzy theory $\langle \mathtt{A}, \mathtt{R}, X\rangle$ or $\langle \mathtt{A}, \mathtt{R}, Y\rangle$, respectively, we have $X(\alpha) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)(\alpha) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(Y)(\alpha)$. By a similar argument, $\mathtt{A}(\alpha) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)(\alpha)$. In particular,

$$C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)).$$

Let $\alpha \in \mathcal{F}$. Any metaproof w for α in the fuzzy theory $\langle \mathtt{A}, \mathtt{R}, C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)\rangle$ is of form

$$\begin{array}{ccc} \alpha_1 & , & a_1 \\ \vdots & & \vdots \\ \alpha_m & , & a_m \end{array}$$

For each $i \leq m$, if α_i is a logical or non-logical axiom, then obviously $a_i \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)(\alpha_i)$, and if $\alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n})$, then $a_i = r^{\mathtt{sem}}(a_{i_1}, \cdots, a_{i_n})$, where, for $1 \leq j \leq n$,

$$a_{i_j} \leq \bigvee\{Val_{\langle \mathtt{A},\mathtt{R},X\rangle}(w_{i_j}) \mid w_{i_j} \text{ is a metaproof for } \alpha_{i_j} \text{ in } \langle \mathtt{A}, \mathtt{R}, X\rangle\}.$$

By semi-continuity,

$$\begin{array}{rcl} a_i & \leq & \bigvee\{r^{\mathtt{sem}}(Val_{\langle \mathtt{A},\mathtt{R},X\rangle}(w_{i_1}), \cdots, Val_{\langle \mathtt{A},\mathtt{R},X\rangle}(w_{i_n}))\} \\ & = & C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)(\alpha_i). \end{array}$$

In particular, $Val_{\langle \mathtt{A},\mathtt{R},C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}\rangle}(w) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)(\alpha)$. Therefore

$$\bigvee\{Val_{\langle \mathtt{A},\mathtt{R},C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}\rangle}(w) \mid w \text{ is a metaproof for } \alpha \text{ in } \langle \mathtt{A}, \mathtt{R}, C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)\rangle\} \\ \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)(\alpha).$$

This proves $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)$. Thus,

$$C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X) = C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(X)).$$

The proof is complete. □

Proposition 6 *The consequence operation $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}$ is a continuous.*

Proof. Obviously, if S is a finite part of T, then

$$C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(S)(\alpha) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha).$$

Thus,

$$\bigvee\{C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(S) \mid \; S \textit{ is a finite part of } T\} \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T).$$

Let $\alpha \in \mathcal{F}$ and let w be a metapropof for α in the fuzzy theory $\langle \mathtt{A}, \mathtt{R}, T\rangle$. Since w is finite we may construct a finite part S_w of T by setting $S_w(\alpha_k) = T(\alpha_k)$ if in w occurs

$$\alpha_k \quad , \quad T(\alpha_k)$$

and $S_w(\beta) = \mathbf{0}$ elsewhere. Then $Val_{\langle \mathtt{A},\mathtt{R},T\rangle}(w) \leq C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(S_w)(\alpha)$. Therefore

$$\begin{array}{rcl} C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha) & \leq & \bigvee\{C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(S_w)(\alpha) \mid w \textit{ metaproof for } \alpha \text{ in } \langle \mathtt{A}, \mathtt{R}, S_w\rangle\} \\ & \leq & \bigvee\{C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(S)(\alpha) \mid \; S \textit{ is a finite part of } T\} \\ & \leq & C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha). \end{array}$$

We conclude $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T) = \bigvee\{C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(S) \mid S \textit{ is a finite part of } T\}$. The proof is complete. □

Fix the truth value set L of Fuzzy Propostional Logic. If $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}$ is continuous while $C^{\mathtt{sem}}$ is not then, of course, the corresponding fuzzy teories are not complete. Now, necessary condition for continuity of semantic consequence operation $C^{\mathtt{sem}}$ in $[0,1]$-valued Fuzzy Propositional Logic is that the truth value set $[0,1]$ forms a complete MV-algebra. Thus, we have

Theorem 3 (Incomleteness Theorem for Fuzzy Logic). *Assume the truth value set in $[0,1]$-valued Fuzzy Propositional Logic forms a residuated lattice but does not form an complete MV-algebra. Then there exist fuzzy theories T which are not axiomatizable, i.e. $C^{\mathtt{sem}}(T) \neq C^{\mathtt{syn}}(T)$.*

For now on, we assume that the set of logical axioms $\mathtt{A}$ is composed of forms of formulas (10) - (20) (and denoted now by Ax.10 - Ax.20) and the set of fuzzy rules of inference $\mathtt{R}$ contains the fuzzy rules of inference R_{GMP}, R_{a-CTR}, R_{a-LR}, R_{RBC}. Fuzzy theories are thus identified by means of their sets T of non-logical axioms. We will write $C^{\mathtt{syn}}$ instead of $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}$. Obviously, for any fuzzy theory T, if $\vdash_1 \alpha$ then $T \vdash_1 \alpha$, and by Ax.16, for any inner truth value $\mathbf{a}$, $a \leq C^{\mathtt{syn}}(T)(\mathbf{a})$. This leads us to the following

Definition 12 *A fuzzy theory T is* consistent *if, for any truth constant* $\mathbf{a}$*, $a = C^{\mathtt{syn}}(T)(\mathbf{a})$, and otherwise T is* contradictory.

Proposition 7 *A fuzzy theory T is contradictory iff $T \vdash_1 \alpha$ holds for each $\alpha \in \mathcal{F}$.*

Proof. Assume T is contradictory. Then there exists a truth constant $\mathbf{a}$ such that $a \neq C^{\mathtt{syn}}(T)(\mathbf{a})$. If for each metaproof w for $\mathbf{a}$ holds $Val_T(w) \leq a$, then $a \leq C^{\mathtt{syn}}(T)(\mathbf{a}) \leq a$, hence $C^{\mathtt{syn}}(T)(\mathbf{a}) = a$, which is not the case. Therefore there exist a metaproof w for $\mathbf{a}$ such that $Val_T(w) \not\leq a$. For every formula $\alpha \in \mathcal{F}$, we have now the following metaproof:

$\mathbf{a}$	,	$Val_T(w)$	,	*assumption*
$\mathbf{0}$	,	$\mathbf{1}$	,	R_{a-CTR}
$\mathbf{0}$ $\mathtt{imp}$ α	,	$\mathbf{1}$	,	$Ax.14$
α	,	$\mathbf{1}$	,	R_{GMP}

We conclude that $T \vdash_1 \alpha$ holds for each $\alpha \in \mathcal{F}$. Conversely, if $T \vdash_1 \alpha$ holds for each $\alpha \in \mathcal{F}$ then, in particular, $T \vdash_1 \mathbf{0}$, i.e. $C^{\mathtt{syn}}(T)(\mathbf{0}) = \mathbf{1} \neq \mathbf{0}$. □

Proposition 8 *A fuzzy theory T is contradictory iff the following condition holds: There is a formula α and metaproofs w, w' for α, $\mathtt{non}$-α, respectively, such that $Val_T(w) = a$, $Val_T(w') = b$ and $\mathbf{0} < a \odot b$.*

Proof. Omitted.

Let T be a fuzzy theory. The choice of the logical axioms Ax.10 - Ax.20 and soundness of fuzzy rules of inference guarantee, for each formula α, each metaproof w for α in T, each valuation v which satisfies T, that $Val_T(w) \leq v(\alpha)$. Thus,

$$\bigvee\{Val_T(w) \mid w \textit{ is a metaproof for } \alpha \textit{ in } T\} \leq \bigwedge\{v(\alpha) \mid v \textit{ satisfies } T\},$$

by symbols, $\mathcal{C}^{\mathtt{syn}}(T)(\alpha) \leq \mathcal{C}^{\mathtt{sem}}(T)(\alpha)$. (This (in-)equality holds even if T is not satisfiable as $\bigwedge\{\emptyset\} = \mathbf{1}$.) We write

Theorem 4 (Soundness Theorem for Fuzzy Propositional Logic) *Let T be a fuzzy theory. For each formula α, if $T \vdash_a \alpha$, $T \models_b \alpha$, then $a \leq b$.*

We also have

Proposition 9 *Any satisfiable fuzzy theory T is consistent, in particular, Fuzzy Propositional Calculus is consistent.*

Proof. Let v satisfy T, $\alpha \in \mathcal{F}$ and $v(\alpha) = c$. Then $v(\texttt{non-}\alpha) = c^*$. Assume $T \vdash_a \alpha$, $T \vdash_b \texttt{non-}\alpha$. Then

$$a = \bigvee \{Val_T(w) \mid w \textit{ is a metaproof for } \alpha \textit{ in } T\} \leq v(\alpha) = c.$$

and

$$b = \bigvee \{Val_T(w) \mid w \textit{ is a metaproof for } \texttt{non-}\alpha \textit{ in } T\} \leq v(\texttt{non} - \alpha) = c^*.$$

Thus, for any metaprooves w, w' for α, $\texttt{non} - \alpha$, respectively, holds

$$Val_T(w) \odot Val_T(w') \leq a \odot b \leq c \odot c^* = \mathbf{0}.$$

We conclude that T is not contradictory and is therefore consistent. □

Proposition 10 *Let T be a fuzzy theory, α a well-formed formula and $a, b \in L$. If $T \vdash_1 (\mathbf{a}\ \texttt{imp}\ \alpha)$, $T \vdash_b \alpha$, then $a \leq b$.*

Proof. Let w be a metaproof for the formula $(\mathbf{a}\ \texttt{imp}\ \alpha)$ and $Val_T(w) = c$. We write the following metaproof w' for α:

$$\begin{array}{llllll} \mathbf{a}\ \texttt{imp}\ \alpha & , & c & , & \textit{assumption} \\ \mathbf{a} & , & a & , & Ax.16 \\ \alpha & , & c \odot a & , & R_{GMP} \end{array}$$

Hence, we have

$$\begin{array}{rcl} a = a \odot 1 & = & a \odot \bigvee\{Val_T(w) \mid w \textit{ is a metaproof for } (\mathbf{a}\ \texttt{imp}\ \alpha)\} \\ & = & \bigvee\{a \odot Val_T(w) \mid w \textit{ is a metaproof for } (\mathbf{a}\ \texttt{imp}\ \alpha)\} \\ & \leq & \bigvee\{Val_T(w') \mid w' \textit{ is a metaproof for } \alpha\} \\ & = & b. \end{array}$$

□

Proposition 11 *Assume T be a fuzzy theory, α a well-formed formula, and $T \vdash_a \alpha$. Then $T \vdash_1 (\mathbf{a}\ \texttt{imp}\ \alpha)$.*

Proof. Let w be a metaproof for the formula α and $Val_T(w) = c$. We write the following metaproof w' for $(\mathbf{a}\ \texttt{imp}\ \alpha)$:

$$\begin{array}{llllll} \alpha & , & c & , & \textit{assumption} \\ \mathbf{a}\ \texttt{imp}\ \alpha & , & a \to c & , & R_{\mathbf{a}-LR} \end{array}$$

Hence, we have

$$\begin{array}{rcl} 1 = a \to a & = & a \to \bigvee\{Val_T(w) \mid w \textit{ is a metaproof for } \alpha\} \\ & = & \bigvee\{a \to Val_T(w) \mid w \textit{ is a metaproof for } \alpha\} \\ & \leq & \bigvee\{Val_T(w') \mid w' \textit{ is a metaproof for } (\mathbf{a}\ \texttt{imp}\ \alpha)\}. \end{array}$$

We conclude $T \vdash_1 (\mathbf{a}\ \texttt{imp}\ \alpha)$. □

Finiteness Theorem in Classical Propositional Logic states that if a formula α is deducible from a set T of formulas, i.e. if there is a metaproof for α in T, then α is deducible from a finite part S of T. As one may assume, this is not, in general, the case in Fuzzy Propositional Logic as we shall now see.

Consider Lukasiewicz-valued Fuzzy Propositional Logic. Let T be a fuzzy theory such that $T(\mathbf{a}_n \texttt{ imp } p) = 1$, where p is a fixed propositional variable, $a_n = \frac{n-1}{n}$, $n \in \mathcal{N}$, and $T(\alpha) = 0$ elsewhere. Obviously, any valuation v such that $v(p) = 1$ satisfies T, thus T is consistent. Let $0 \leq b < 1$ and $\frac{1}{1-b} < n$. Then $b < a_n$. We write the following metaproof w for p:

$\mathbf{a}_n \texttt{ imp } p$	,	$\mathbf{1}$	,	*non-logical axiom*
$\mathbf{a}_n$	,	a_n	,	$Ax.16$
p	,	a_n	,	R_{GMP}

i.e. $Val_T(w) = a_n$. Hence, $\bigvee\{Val_T(w) \mid w \textit{ is a metaproof for } p \textit{ in } T\} = 1$. Thus $T \vdash_1 p$. Assume now S is a finite part of T and

$$a_i = \max\{a_k \mid S(\mathbf{a}_k \texttt{ imp } p) = 1\} < 1.$$

Then a valuation v such that $v(p) = a_i$ satisfies S, thus S is consistent and $S \models_c p$, where $c \leq a_i$. By Soundness Theorem, $S \vdash_d p$, where $d \leq c < 1$. We conclude that $S \not\vdash_1 p$

Example 3 Let p stand for *It is raining enough* and q for *Potato is growing fast.* Study a fuzzy theory T such that $T(\texttt{non}-p \texttt{ imp non}-\texttt{q}) = 1$ standing for *If it is not raining enough then potato is not growing fast* and $T(q) = 0.7$ standing loosely for *Potato is grouwing more or less fast.* We are interested in the degree of deduction of p. The following is a metaproof for p:

$(\texttt{non}-p \texttt{ imp non}-\texttt{q}) \texttt{ imp } (q \texttt{ imp } p)$	,	1	,	$Ax.20$
$(\texttt{non}-p \texttt{ imp non}-\texttt{q})$	,	1	,	*non-logical axiom*
$(q \texttt{ imp } p)$	,	1	,	R_{GMP}
q	,	0.7	,	*non-logical axiom*
p	,	0.7	,	R_{GMP}

Therefore $0.7 \leq \mathcal{C}^{\texttt{syn}}(p)$. Since a valuation v such that $v(p) = v(q) = 0.7$ satisfies T, we have $0.7 \leq \mathcal{C}^{\texttt{syn}}(T)(p) \leq \mathcal{C}^{\texttt{sem}}(T)(p) \leq 0.7$. Thus, the degree of deduction of p is 0.7. Freely speeking, *Is is raining more or less enough.*

4 Completeness of Fuzzy Propositional Logic

In this section we establish the most important result concerning Fuzzy Propositional Logic; if the set L of values of truth is an injective MV-algebra, in particular, in the unit interval if, and only if $[0, 1]$ forms a complete MV-algebra then the semantic consequence operation $\mathcal{C}^{\mathtt{sem}}$ and the syntactic consequence operation $\mathcal{C}^{\mathtt{syn}}$ coincide. In other words, for any fuzzy theory T, for each formula $\alpha \in \mathcal{F}$ and for any value $a \in L$, holds

$$T \vdash_a \alpha \text{ if, and only if } T \models_a \alpha.$$

Since this trivially holds for inconistent fuzzy theories, we assume the fuzzy theory T under consideration, for now on fixed, is consistent. We start by constructing the *Lindenbaum algebra* of Fuzzy Propositional Logic.

Define on the set $\mathcal{F}$ a binary relation $\preceq$ in the following way:

$$\alpha \preceq \beta \text{ if, and only is } T \vdash_1 (\alpha \ \mathtt{imp} \ \beta).$$

By Ax.10, $\preceq$ is reflexive and if $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then by Ax.11, $\alpha \preceq \gamma$, thus $\preceq$ is transitive, and therefore, a quasi-order. Hence, by defining a binary operation $\sim$ via

$$\alpha \sim \beta \text{ iff } T \vdash_1 (\alpha \ \mathtt{imp} \ \beta) \text{ and } T \vdash_1 (\beta \ \mathtt{imp} \ \alpha)$$

we obtain an equivalence relation on $\mathcal{F}$. We denote by $|\alpha|$ the equivalence class defined by α and the set of all equivalenvce classes by $\mathcal{F}/\sim$. We can demostrate that the relation $\sim$ is a concruence with respect to the logical connectives `imp` and `non`. Accordingly, the equation

$$|\alpha| \rightarrow |\beta| = |\alpha \ \mathtt{imp} \ \alpha|$$

defines a binary operation $\rightarrow$ on $\mathcal{F}/\sim$, and

$$|\alpha|^* = |\mathtt{non}\text{-}\alpha|$$

defines a unary operation * on $\mathcal{F}/\sim$. One can show that $|\mathbf{0}|$ and $|\mathbf{1}|$ are the least element and the largest element, respectively, in $\mathcal{F}/\sim$ with respect to an order relation given by

$$|\alpha| \leq |\beta| \text{ iff } T \vdash_1 (\alpha \ \mathtt{imp} \ \beta).$$

Proposition 12 *For each $\alpha \in \mathcal{F}$, $T \vdash_1 \alpha$ iff $|\alpha| = |\mathbf{1}|$*

Proof. Omitted.

Proposition 13 *The algebra* $\langle \mathcal{F}/\sim, \rightarrow, ^*, |\mathbf{1}|\rangle$ *is a Wajsberg algebra.*

Proof. We demostrate that the Wajsberg axioms (1) - (4) hold in the set $\mathcal{F}/\sim$. Let $|\alpha|$, $|\beta|$, $|\gamma| \in F/\sim$. By Ax.17, we have $|\alpha| \leq |\mathbf{1}| \rightarrow |\alpha|$, and by Ax.18, $|\mathbf{1}| \rightarrow |\alpha| \leq |\alpha|$. Therefore $|\mathbf{1}| \rightarrow |\alpha| = |\alpha|$. Thus, (1) holds. (2) follows from Ax.11. By Ax.19, we have

$$(|\alpha| \rightarrow |\beta|) \rightarrow |\beta| \leq (|\beta| \rightarrow |\alpha|) \rightarrow |\alpha|$$

and, by changing the roles of $|\alpha|$ and $|\beta|$,

$$(|\beta| \rightarrow |\alpha|) \rightarrow |\alpha| \leq (|\alpha| \rightarrow |\beta|) \rightarrow |\beta|.$$

Therefore $(|\beta| \rightarrow |\alpha|) \rightarrow |\alpha| = (|\alpha| \rightarrow |\beta|) \rightarrow |\beta|$, thus (3) holds. Finally, by Ax.20, we have $(|\alpha|^* \rightarrow |\beta|^*) \rightarrow (|\beta| \rightarrow |\alpha|) = |\mathbf{1}|$, whence (4) holds. □

In the corresponding lattice, the lattice operations $\wedge$ and $\vee$ are defined by

$$|\alpha| \wedge |\beta| = |\alpha \ \overline{\texttt{and}}\ \beta|,\ |\alpha| \vee |\beta| = |\alpha \ \overline{\texttt{or}}\ \beta|,$$

Remark 4 *Assume* $h : [\mathcal{F}/\sim] \searrow L$ *is an MV-homomorphism and* $T \vdash_a \alpha$. *Then* $h(|\mathbf{a}|) \leq h(|\alpha|)$.

Proof. Omitted

Proposition 14 *Let* T *be a consistent fuzzy theory. For each* $a, b \in L$, $a \neq b$ *iff* $|\mathbf{a}| \neq |\mathbf{b}|$.

Proof. Assume $a \neq b$, in particular, $a \not\leq b$ but $|\mathbf{a}| = |\mathbf{b}|$ (case $b \not\leq a$ is symmetric). Then $T \vdash_1 (\mathbf{a}\ \texttt{imp}\ \mathbf{b})$. If for every metaproof w for $(\mathbf{a}\ \texttt{imp}\ \mathbf{b})$ holds $Val_T(w) \leq a \rightarrow b$, then

$$\mathbf{1} = \bigvee\{Val_T(w) \mid w \text{ is a metaproof for } (\mathbf{a}\ \texttt{imp}\ \mathbf{b})\} \leq a \rightarrow b,$$

consequently, $a \leq b$, which is not the case. Thus, there exists a metaproof w' for $(\mathbf{a}\ \texttt{imp}\ \mathbf{b})$ such that $Val_T(w') = c$ and $c \not\leq a \rightarrow b$, or equivalently, $a \odot c \not\leq b$. Let α be a fixed formula. We have the following metaproof for α:

$\mathbf{a}\ \texttt{imp}\ \mathbf{b}$,	c ,	*assumption*
$\mathbf{a}$,	a ,	$Ax.16$
$\mathbf{b}$,	$c \odot a$,	R_{GMP}
$\mathbf{0}$,	$\mathbf{1}$,	R_{b-CTR}
$\mathbf{0}\ \texttt{imp}\ \alpha$,	$\mathbf{1}$,	$Ax.14$
α ,	$\mathbf{1}$,	R_{GMP}

Thus, $T \vdash_1 \alpha$. This contradicts the assumption T is consistent. We conclude $|\mathbf{a}| \neq |\mathbf{b}|$. The converse is trivial. □

Theorem 5 (Completeness Theorem for Fuzzy Propositional Logic) *Any consistent fuzzy theory T is semantically complete.*

Proof. Let $a, b \in L$, α a well-formed formula and $T \vdash_a \alpha$, $T \models_b \alpha$. By Soundness Theorem, $a \leq b$. We construct a truth value function v such that v satisfies T and $v(\alpha) = a$. This shows that $a = b$.

Let $A = \langle A, \rightarrow, ^*, |\mathbf{1}|\rangle$ be the subalgebra of the Wajsberg algebra $\mathcal{F}/\sim$ generated by the set $A_\circ = \{|\alpha|\} \cup \{|\mathbf{c}| \; |c \in L\}$. Obviously, A is a Wajsberg algebra such that the operations $\rightarrow, ^*$ coincide with those of $\mathcal{F}/\sim$. Define a mapping $h_\circ : A_\circ \searrow L$ by $h_\circ(|\alpha|) = a$ and $h_\circ(|\mathbf{c}|) = c$ for each $c \in L$. The mapping $h_\circ$ is well-defined and can be extended in natural way on A and, since L is injective, on the whole Wajsberg algebra $\mathcal{F}/\sim$. Denote the extended map by h. Let k be the *natural homomorphism,* i.e. $k(\beta) = |\beta|$ for any $\beta \in \mathcal{F}$. Then the mapping $h \circ k$: $\mathcal{F} \searrow L$ is a truth value function. Indeed, $h \circ k(\beta)$ is defined for all $\beta \in \mathcal{F}$, $h \circ k(\mathbf{c}) = h(|\mathbf{c}|) = h_\circ(|\mathbf{c}|) = c$ for each $c \in L$ and, for each $\beta, \gamma \in \mathcal{F}$,

$$\begin{aligned} h \circ k(\beta \texttt{ imp } \gamma) &= h(|\beta \texttt{ imp } \gamma|) \\ &= h(|\beta|) \rightarrow h(|\gamma|) \\ &= h \circ k(\beta) \rightarrow h \circ k(\gamma). \end{aligned}$$

Assume $\beta \in \mathcal{F}$ is a non-logical axiom of T and $T(\beta) = c$, $T \vdash_d \beta$. Then $c \leq d$ $= h(|\mathbf{d}|) \leq h(|\beta|) = h \circ k(\beta)$, hence $T(\beta) \leq h \circ k(\beta)$, accordingly, $h \circ k$ satisfies T. Finally, $h \circ k(\alpha) = h(|\alpha|) = h_\circ(|\alpha|) = a$. Consequently, $h \circ k$ is the truth value function we are looking for. □

Corollary Any satisfiable fuzzy theory T is complete.

In Classical Propositional Logic holds the following *Deduction Theorem*

$$\text{If } \alpha \vdash \beta \text{ then } \vdash (\alpha \texttt{ imp } \beta).$$

In Fuzzy Propositional Logic the situation is different. Consider Lukasiewicz valued fuzzy theory T with one non-logical axiom $T(\mathbf{a} \texttt{ imp } [\mathbf{a} \texttt{ and } p]) = 1$, where p is a propositional variable and $a = \frac{1}{4}$. For any valuation v which satisfies T holds $v(\mathbf{a} \texttt{ imp } [\mathbf{a} \texttt{ and } p]) = 1$, or equivalently, $v(\mathbf{a}) \leq v(\mathbf{a}) \odot v(p)$ implying $v(p)$= 1. Thus, $T \models_1 p$ and by Completeness Theorem, $T \vdash_1 p$. However, for a valuation v such that $v(p) = 0.45$ it holds that $v(\{\mathbf{a} \texttt{ imp } [\mathbf{a} \texttt{ and } p]\} \texttt{ imp } p) =$ 0.70. Therefore

$$\not\models_1 (\{\mathbf{a} \texttt{ imp } [\mathbf{a} \texttt{ and } p]\} \texttt{ imp } p),$$

whence

$$\not\vdash_1 (\{\mathbf{a} \texttt{ imp } [\mathbf{a} \texttt{ and } p]\} \texttt{ imp } p).$$

Thus, by taking $\alpha = (\mathbf{a}\ \texttt{imp}\ [\mathbf{a}\ \texttt{and}\ p])$, $\beta = p$ we have a case such that

$$\alpha \vdash_1 \beta \text{ holds, while } \not\vdash_1 (\alpha\ \texttt{imp}\ \beta).$$

Fuzzy rules of inference R_{GMP}, R_{a-CTR}, R_{a-LR} and R_{RBC} are sufficient to establish the Completeness Theorem of Fuzzy Propositional Logic. Since fuzzy rules of inference are sound, introducing new fuzzy rules of inference into a fuzzy theory does not have any effect on the degree of deduction of any formula α. Metaprooves, however, are easier to be found.

Proposition 15 *In complete MV-algebra valued Fuzzy Propositional Logic, the following schemas are fuzzy rules of inference.*

Generalized Modus Tollendo Tollens:

$$R_{GMTT} \quad : \quad \frac{\texttt{non-}\beta, (\alpha\ \texttt{imp}\ \beta)}{\texttt{non-}\alpha}, \frac{a, b}{a \odot b}$$

Generalized Hypothetical Syllogism:

$$R_{GHS} \quad : \quad \frac{(\alpha\ \texttt{imp}\ \beta), (\beta\ \texttt{imp}\ \gamma)}{(\alpha\ \texttt{imp}\ \gamma)}, \frac{a, b}{a \odot b}$$

Generalized Commutative Law 1:

$$R_{GCL1} \quad : \quad \frac{(\alpha\ \texttt{and}\ \beta)}{(\beta\ \texttt{and}\ \alpha)}, \frac{a}{a}$$

Generalized Commutative Law 2:

$$R_{GCL2} \quad : \quad \frac{(\alpha\ \texttt{or}\ \beta)}{(\beta\ \texttt{or}\ \alpha)}, \frac{a}{a}$$

Generalized Equivalence Law 1:

$$R_{GEL1} \quad : \quad \frac{(\alpha\ \texttt{equiv}\ \beta)}{(\alpha\ \texttt{imp}\ \beta)}, \frac{a}{a}$$

Generalized Equivalence Law 2:

$$R_{GEL2} \quad : \quad \frac{(\alpha\ \texttt{equiv}\ \beta)}{(\beta\ \texttt{imp}\ \alpha)}, \frac{a}{a}$$

Generalized Equivalence Law 3:

$$R_{GEL3} \quad : \quad \frac{(\alpha \texttt{ imp } \beta), (\beta \texttt{ imp } \alpha)}{(\alpha \texttt{ equiv } \beta)}, \; \frac{a, b}{a \wedge b}$$

Generalized Simplification Law 1:

$$R_{GS1} \quad : \quad \frac{(\alpha \texttt{ and } \beta)}{\alpha}, \; \frac{a}{a}$$

Generalized Simplification Law 2:

$$R_{GS2} \quad : \quad \frac{(\alpha \texttt{ and } \beta)}{\beta}, \; \frac{a}{a}$$

Generalized Rule of Introduction of Douple Negation:

$$R_{GIDN} \quad : \quad \frac{\alpha}{\texttt{non-(non-}\alpha)}, \; \frac{a}{a}$$

Generalized Rule of Elimination of Douple Negation:

$$R_{GEDN} \quad : \quad \frac{\texttt{non-(non-}\alpha)}{\alpha}, \; \frac{a}{a}$$

Generalized De Morgan Law 1:

$$R_{GDeM1} \quad : \quad \frac{(\texttt{non-}\alpha) \texttt{ and } (\texttt{non-}\beta)}{\texttt{non-(}\alpha \texttt{ or } \beta)}, \; \frac{a}{a}$$

Generalized De Morgan Law 2:

$$R_{GDeM2} \quad : \quad \frac{\texttt{non-(}\alpha \texttt{ or } \beta)}{(\texttt{non-}\alpha) \texttt{ and } (\texttt{non-}\beta)}, \; \frac{a}{a}$$

Generalized De Morgan Law 3:

$$R_{GDeM3} \quad : \quad \frac{(\texttt{non-}\alpha) \texttt{ or } (\texttt{non-}\beta)}{\texttt{non-(}\alpha \texttt{ and } \beta)}, \; \frac{a}{a}$$

Generalized De Morgan Law 4:

$$R_{GDeM4} \quad : \quad \frac{\texttt{non-}(\alpha \texttt{ and } \beta)}{(\texttt{non-}\alpha) \texttt{ or } (\texttt{non-}\beta)}, \frac{a}{a}$$

Generalized Addition Law:

$$R_{GAL} \quad : \quad \frac{\alpha}{(\alpha \texttt{ or } \beta)}, \frac{a}{a}$$

Generalized Modus Tollendo Ponens:

$$R_{GMTP} \quad : \quad \frac{\texttt{non-}\beta, (\alpha \texttt{ or } \beta)}{\alpha}, \frac{a, b}{a \odot b}$$

Generalized Disjunctive Syllogism:

$$R_{GDS} \quad : \quad \frac{(\alpha \texttt{ or } \beta), (\alpha \texttt{ imp } \gamma), (\beta \texttt{ imp } \delta)}{(\gamma \texttt{ or } \delta)}, \frac{a, b, c}{a \odot b \odot c}$$

Generalized Rule of Introduction of Implication:

$$R_{II} \quad : \quad \frac{(\texttt{non-}\alpha \texttt{ or } \beta)}{(\alpha \texttt{ imp } \beta)}, \frac{a}{a}$$

Generalized Rule of Elimination of Implication:

$$R_{EI} \quad : \quad \frac{(\alpha \texttt{ imp } \beta)}{(\texttt{non-}\alpha \texttt{ or } \beta)}, \frac{a}{a}$$

Rule of Bold Disjunction:

$$R_{RBD} \quad : \quad \frac{\alpha, \beta}{(\alpha \texttt{ or } \beta)}, \frac{a, b}{a \oplus b}$$

Rule of Conjunction:

$$R_{RC} \quad : \quad \frac{\alpha, \beta}{(\alpha\ \overline{\texttt{and}}\ \beta)}, \frac{a, b}{a \wedge b}$$

Rule of Exclusive or:

$$R_{XOR} \quad : \quad \frac{(\alpha \texttt{ xor } \beta),\ \texttt{non-}\alpha}{\beta}, \frac{a, b}{a \odot b}$$

Proof. Omitted.

Example 4 Recall Example 1 where we studied the following

Assuming winter is long implies either winter is not cold or damages caused by floods are not serious, then insurence business is profitable.

This was expressed formally by $\{[p \texttt{ imp } (\texttt{ non-}q \overline{\texttt{or}} \texttt{ non-}r)] \texttt{ imp } s\}$. In particular, we set $T(p) = 0.2$, $T(q) = 0.8$, $T(r) = 0.1$, $T(s) = 0.8$, $T(\beta) = 0$ elsewhere. Then we reasoned

$$T \models_{0.8} \{[p \texttt{ imp } (\texttt{non} - q \overline{\texttt{or}} \texttt{ non-}r)] \texttt{ imp } s\}.$$

Now we are looking for metaprooves for this same formula. For example, we have w_1:

s	,	0.8	,	*non-logical axiom*
$(\texttt{non-}\beta \texttt{ or } s)$	,	0.8	,	R_{GAL}
$(\beta \texttt{ imp } s)$	,	0.8	,	R_{II}

where β stands for $[p \texttt{ imp } (\texttt{non} - q \overline{\texttt{or}} \texttt{ non-}r)]$. Thus, $Val_T(w_1) = 0.8$. Since the fuzzy theory T is satisfiable it is complete. Thus,
$T \vdash_{0.8} \{[p \texttt{ imp } (\texttt{non} - q \overline{\texttt{or}} \texttt{ non-}r)] \texttt{ imp } s\}$. Freely expressed, *The statement is, under given conditions, mostly true.*

Example 5 In Example 2 we studied such a fuzzy theory T that
$T([(p \texttt{ or } q) \texttt{ imp } r]) = 1$, $T(r \texttt{ imp } (s \texttt{ or } t)) = 0.9$, $T(t \texttt{ imp } w) = 1$,
$T(\texttt{non-}s \texttt{ and } \texttt{non-}w) = 1$, where p stands for *Wages rise*, q stands for *Prices rise*, r stand for *There is inflation*, s stands for *The Government stops inflation*, t stands for *People suffer*, w stands for *The Government loses its popularity*. We showed that $\mathcal{C}^{\texttt{sem}}(T)(\texttt{non-}p) = 0.7$. Now we are looking for a metaproof for the formula $\texttt{non-}p$. We find the following:

(1)	$[(p \texttt{ or } q) \texttt{ imp } r]$	,	1	,	*non-logical axiom*
(2)	$[r \texttt{ imp } (s \texttt{ or } t)]$	,	0.9	,	*non-logical axiom*
(3)	$(t \texttt{ imp } w)$	,	0.8	,	*non-logical axiom*
(4)	$(\texttt{non-}s \texttt{ and } \texttt{non-}w)$	,	1	,	*non-logical axiom*
(5)	$\texttt{non-}w$	,	1	,	(4) , R_{GS2}
(6)	$\texttt{non-}s$	,	1	,	(4) , R_{GS1}
(7)	$\texttt{non-}t$	,	0.8	,	(5) , (3) , R_{GMTT}
(8)	$(\texttt{non-}s \texttt{ and } \texttt{non-}t)$	,	0.8	,	(6) , (7) , R_{RBC}
(9)	$\texttt{non-}(s \texttt{ or } t)$	,	0.8	,	(8) , R_{GDeM1}
(10)	$\texttt{non-}r$	,	0.7	,	(9) , (2) , R_{GMTT}
(11)	$\texttt{non-}(p \texttt{ or } q)$	,	0.7	,	(10) , (1) , R_{GMTT}
(12)	$(\texttt{non-}p \texttt{ and } \texttt{non-}q)$	,	0.7	,	(11) , R_{GDeM2}
(13)	$\texttt{non-}p$	,	0.7	,	(12) , R_{GS1}

We conclude that $\mathcal{C}^{\texttt{syn}}(T)(\texttt{non-}p) = 0.7$, a fact which, of course, follows also by Completeness Theorem.

Evidently, the longer a deduction from partial true premises is, the more unsure is the truth of the conclusion. Due to non-idempotency of the operations $\odot$ and $\oplus$ in MV-algebras, Fuzzy Propositional Logic obeys this property. A conclusion of partially true statements may sometimes have a greater value of truth than any of the premises. Consider

Example 6 *A technical system needs control if any of its four subsystems is performing badly. Assume all the four subsystems are performing* more or less *well. Does the system need technical control?*

Define a Lukasiewicz valued fuzzy theory T by setting $T([p_1 \texttt{ or } p_2 \texttt{ or } p_3 \texttt{ or } p_4] \texttt{ imp } q) = 1$, $T(p_i) = 0.3$ for $i = 1, \cdots, 4$ where p_i stands for *The subsystem i is performing badly,* and q stands for *The system needs technical control.* This fuzzy theory is satisfiable, indeed, a valuation v such that $v(p_i) = v(q) = 1$ for $i = 1, \cdots, 4$ satisfies T. We have the following metaproof for the formula q:

p_1	,	0.3	,	*non-logical axiom*
p_2	,	0.3	,	*non-logical axiom*
$p_1 \texttt{ or } p_2$	,	0.6	,	R_{RBD}
p_3	,	0.3	,	*non-logical axiom*
$p_1 \texttt{ or } p_2 \texttt{ or } p_3$	,	0.9	,	R_{RBD}
p_4	,	0.3	,	*non-logical axiom*
$p_1 \texttt{ or } p_2 \texttt{ or } p_3 \texttt{ or } p_4$	,	1	,	R_{RBD}
$[p_1 \texttt{ or } p_2 \texttt{ or } p_3 \texttt{ or } p_4] \texttt{ imp } q$	,	1	,	*non-logical axiom*
q	,	1	,	R_{GMP}

Our conclusion is *The system needs a technical control.*

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