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Fuzzy Logic: Misconceptions and Clarifications

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Abstract. Some commonly accepted statements concerning the basic fuzzy logic proposed by Lotfi Zadeh in 1965, have led to suggestions that fuzzy logic is not a *logic* in the same sense as classical bivalent logic. Those considered herein are: fuzzy logic generates results that contradict classical logic, fuzzy logic collapses to classical logic, there can be no proof theory for fuzzy logic, fuzzy logic is inconsistent, fuzzy logic produces results that no human can accept, fuzzy logic is not proof-theoretic complete, fuzzy logic is too complex for practical use, and, finally, fuzzy logic is not needed. It is either proved or argued herein that all of the these statements are false and are, hence, misconceptions. A fuzzy logic with truth values specified as subintervals of the real unit interval [0.0, 1.0] is introduced. Proofs of the correctness, consistency, and proof theoretic completeness of the truth interval fuzzy logic are either summarized or cited. It is concluded that fuzzy logics deserve the accolade of *logic* to the same degree that the term applies to classical logics.

Keywords: knowledge management, logic inference, logic processing, theorem proving, uncertainty, uncertainty reasoning

1. Introduction

Fuzzy logic, since its introduction in 1965, seems to have generated controversy in many quarters. In the 70's fuzzy logic is referred to by William Kahan as "the cocaine of science" and by Dana Scott as "pornography" (Haack 1996: 230). The climate created by such statements from eminent scientists and philosophers cannot help but generate misconceptions concerning fuzzy logic in all but the relatively sophisticated. At the Pentagon, in the 70's, "if a commander said he had a fuzzy thinker on his staff, his career would be over" (McNeil 1993: 50). In 1991 we have "Fuzzy logic is based on fuzzy thinking" (Konieki 1991).

In has been claimed, for instance, that fuzzy logic generates results that are incompatible with rational human inference and that the complexity of fuzzy logic makes it unusable as a logic. The features of classical logic, such as proof-theoretic completeness and consistency, that have helped make classical logic the de facto standard for human reasoning, are claimed, by some critics, to be missing from fuzzy logic.

In this paper a few of these beliefs are selected, examined, and exposed as misconceptions. In addition, it is argued that fuzzy logic is a truth preserving formal system well suited to extracting valid inferences from imprecise empirical data, a task beyond the capability of classical logic.

2. Definition of Terms and Introduction to Fuzzy Logic

In this section the terms used are defined, the basic concepts of classical logic reviewed, and brief introduction to fuzzy logic presented.

Classical logic uses a two element set of truth values, $\boldsymbol{TV_c}$ = {***True, False***} ≡ {***T, F***} ≡ {**1, 0**}. The ***Principle of Bivalence (PB)*** states that the elements of $\boldsymbol{TV_c}$ are mutually exclusive and jointly exhaustive (Haack 1996). Hence, if $t(A) = True$, then $t(\neg A) = False$. The ***Law of the Excluded Middle (LEM)*** requires that $(A \vee \neg A)$ be a theorem of the logic system.

The major principle of classical set theory is the ***Axiom of Specification (AS)***, which can be stated:

> ***To every set A and to every sentence S(x) there corresponds a set B whose elements are exactly those elements x of A for which S(x) holds*** (Halmos 1960).

In classical logic and classical set theory (an application of classical logic), then, either $(t(S(x)) = True$, when $x \in B$; or $t(S(x)) = False$, when $x \notin B$; an example of the Law of the Excluded Middle.

Fuzzy sets, as presented by Lotfi Zadeh in his classic work (Zadeh 1965), allow individuals to have ***degrees of membership*** in a set, degree of membership of x in B being specified by the truth value of $S(x)$. If $t(S(x)) = 1.0$, where 1.0 is equivalent to classical truth, x possesses full membership in B; if $t(S(x)) = 0.0$, where 0.0 is equivalent to classical falsity, $x \notin B$; and $0.0 < t(S(x)) < 1.0$ denote *intermediate* degrees of membership of x in B. $S(x)$ is now a formula of ***fuzzy logic***.

An ***interpretation***, in both classical logic and fuzzy logic, is defined as:

> **Given a formula *G*, if $A_1, A_2, \ldots, A_n$ are the atoms that make up *G* then, if *M* is an assignment of a single truth value to each A_i, *M* is an *interpretation* of *G*.**

The definition of logical equivalence, applicable to both classical and fuzzy logic, is:

> **Two formulae *A* and *B* are *logically equivalent*, written $A \equiv B$, iff $t(A) = t(B)$ in all interpretations.**

This is to say that A and B are, actually, the same formula, even though their appearance may differ.

The axioms of fuzzy logic are (Zadeh 1965):

If A and B are propositions with truth values $t(A)$ and $t(B)$, respectively, then:

Axiom Z1: $0.0 \leq t(A), t(B) \leq 1.0$
Axiom Z2: $t(A \wedge B) = min(t(A), t(B))$
Axiom Z3: $t(A \vee B) = max(t(A), t(B))$
Axiom Z4: $t(\neg A) = 1 - t(A)$

The terms *tautology* and *contradiction* in classical logic are defined as follows:

A *tautology* is a formula *F* for which $t(F) = 1.0$, or *True*, in all interpretations.
A *contradiction* is a formula *F* for which $t(F) = 0.0$, or *False*, in all interpretations.

To construct the fuzzy analogs of the classical terms *True* and *False* we examine, in Theorems 1 and 2, below, the truth values attainable in fuzzy logic for $A \wedge \neg A$, a classic contradiction, and $A \vee \neg A$, a classic tautology.

Theorem 1: For all fuzzy propositions A and $\neg A$, $0.0 \leq t(A \wedge \neg A) \leq 0.5$
Proof: $t(A \wedge \neg A) = min(t(A), 1 - t(A))$ according to axioms Z2 and Z4.
If: $t(A) = 0.5$ then: $min(t(A), 1 - t(A)) = 0.5$
If: $t(A) = 0.0$ then: $min(t(A), 1 - t(A)) = 0.0$
If: $t(A) < 0.5$ then: $1 - t(A) > 0.5$ and $min(t(A), 1 - t(A) < 0.5$
If: $t(A) > 0.5$ then: $1 - t(A) < 0.5$ and $min(t(A), 1 - t(A)) < 0.5$
so: $0.0 \geq t(A) \wedge \neg A) \geq 0.5$ □

In a similar manner, as shown in (Kenevan 1992), we can prove Theorem 2, below:

Theorem 2: For all fuzzy propositions A and $\neg A$, $0.5 \leq t(A \vee \neg A) \leq 1.0$
Using fuzzy analogs of classical contradiction and tautology, we define fuzzy truth and fuzzy falsity as follows:

A formula *F* is said to be *true* if $t(F) = 1.0$, *fuzzy true* if $0.5 < t(F) < 1.0$, *false* if $f(F) = 0.0$, *fuzzy false* if $0.0 < f(F) < 0.5$, and *fuzzy indeterminate* if $t(F) = 0.5$.

Proceeding to the next stage of analogy we define *fuzzy tautology* and *fuzzy contradiction*:

A formula F is a *fuzzy tautology* if $0.5 < t(F) \leq 1.0$, or F is *true* or *fuzzy true*, in all interpretations.

A formula F is a *fuzzy contradiction* if $0.0 \leq f(F) < 0.5$, or F is *false* or *fuzzy false*, in all interpretations.

The concept of a ***proof***, or logical consequence, in classical logic is defined as:

Given formulae $F_1, F_2, \ldots, F_n$ and a formula G, G is said to be a logical consequence of $F_1, F_2, \ldots, F_n$ if and only if for any interpretation I in which $F_1, F_2, \ldots, F_n$ is true, G is also true.

The single inference rule of classical propositional logic is ***modus ponens***, or ***MP***, where:

Modus Ponens: B is ***a logical consequence*** of A and $A \Rightarrow B$

Theorem 3, easily proven from the definition of logical consequence, is a useful extension of *MP*.

Theorem 3: ***Given formulae $F_1, F_2, \ldots, F_n$ and a formula G, G is a logical consequence of $F_1, F_2, \ldots, F_n$ if and only if the formula $(F_1 \wedge \ldots \wedge F_n) \Rightarrow G$ is a tautology.***
Modus ponens can, then, be stated as $(A \wedge (A \Rightarrow B)) \Rightarrow B$. If G is a logical consequence of $F_1, F_2, \ldots, F_n$, then $(F_1 \wedge \ldots F_n) \Rightarrow G$ is called a ***theorem*** of classical logic and G is said to be ***provable*** in classical logic.

3. Misconceptions Regarding Fuzzy Logic

Misconception #1: Fuzzy logic generates results that contradict classical logic.
Professor William Kahan stated, in 1975, that "Fuzzy logic is wrong, wrong, and pernicious" (McNeil 1993). If fuzzy logic is wrong then, by definition, a proof performed using fuzzy logic must contradict that using another logic, i.e., classical logic, starting from the same premises. But, if the truth values being used are restricted to $\boldsymbol{TV_c}$ axioms Z1–Z4 generate the same results as do those of classical logic, as shown in the truth table of Figure 1 with the truth values formatted as: <classical truth value/fuzzy truth value>. Zadeh's fuzzy logic is, then, an ***extension*** of classical logic. Classical logic cannot process inferences using the other truth values contained in $\boldsymbol{TV_f} = [0.0, 1.0]$ so there is no possibility for conflict with fuzzy logic.

A	B	$A \wedge B$	$A \vee B$	$\neg A$
$True/1.0$	$True/1.0$	$True/1.0$	$True/1.0$	$False/0.0$
$True/1.0$	$False/0.0$	$False/0.0$	$True/1.0$	
$False/0.0$	$True/1.0$	$False/0.0$	$True/1.0$	$True/1.0$
$False/0.0$	$False/0.0$	$False/0.0$	$False/0.0$	

Figure 1. Classical Logic/Fuzzy Logic Results.

Misconception #2: Fuzzy logic is inconsistent (Haack 1996: 237).
The terms *consistent* and *inconsistent* describe systems, or theories, constructed using a logic (Davis 1989: 7). A theory L is ***consistent*** if there is no formula X in L such that both X and $\neg X$ are provable in L. For example, if the axioms $L.1$ and $L.2$ define a classical system L, where $L.1$: $A \Rightarrow B$ and $L.2$: $C \Rightarrow \neg B$, then in any interpretation I in which both A and C are true, applying modus ponens to $L.1$ proves B; and applying modus ponens to $L.2$ proves $\neg B$. L is, therefore, inconsistent. We can construct the fuzzy analog L' of L, choose an interpretation in which both A and C are either true or fuzzy true, and L', like L, would be shown to be inconsistent. Fuzzy logic is, then, consistent in the same sense as classical logic.

Misconception #3: A fuzzy logic defined using axioms Z1–Z4 collapses to a two valued logic (Elkan 1994).
In 1993, at the AAAI Eleventh National Conference on Artificial Intelligence, Charles Elkan presented a paper entitled "The Paradoxical Success of Fuzzy Logic", claiming that that fuzzy logic collapses into classical bivalent logic (Elkan 1993). The paper won a "Best Paper of Conference" citation. A later version with the same title was published in a special issue of IEEE Expert (Elkan 1994). In both articles Elkan claims that axioms Z1–Z4 define a logic that is, in reality, a classical bivalent logic; and that all of the virtues attributed to the capability of reasoning with a set of truth values defined by the real unit interval are illusory, since there are two, and only two, truth values. The basis for Elkan's claim is Theorem 4 (Elkan 1994), presented below.

Theorem 4: If $\neg(A \wedge \neg B)$ and $B \vee (\neg A \wedge \neg B)$ are logically equivalent then for any two assertions A and B, either $t(A) = t(B)$ or $t(B) = 1 - t(A)$
Axiom Z4 states that $t(\neg A) = 1 - t(A)$, so Theorem 4 requires that either $t(B) = t(A)$ or $t(B) = t(\neg A)$ ***and these are the only possible truth values***. We show here that $\neg(A \wedge \neg B)$ is not logically equivalent to $B \vee (\neg A \wedge \neg B)$ in fuzzy logic; so the hypothesis of Theorem 4 is not true in fuzzy logic, and, hence,

the conclusion does not apply to fuzzy logic. The proof requires the definition of logical equivalence and the following three lemmas, which can be easily proven by the reader:

Lemma L1: $\boldsymbol{max\ \{a, min\{b, c\}\} \equiv min\{max\{a, b\}, max\{a, c\}\}}$
Lemma L2: $\boldsymbol{-\ min\{a, b\} \equiv max\{-a, -b\}}$
Lemma L3: $\boldsymbol{1 + max\{-a, -1 + b\} \equiv max\{1 - a, b\}}$

The hypothesis of Theorem 4 can be written as:

$$\neg(A \wedge \neg B) \equiv B \vee (\neg A \wedge \neg B) \tag{3.1}$$

If we apply Z4, Z2, and Z4, in that sequence, to the left side of (3.1) we get:

$$t(\neg(A \wedge \neg B)) = 1 - t(A \wedge \neg B) = 1 - min\{t(A), t(\neg B)\} = \\ = 1 - min\{t(A), 1 - t(B)\}$$

Applying L2 and L3 to $1 - min(t(A), 1 - t(B)\}$ gives:

$$1 - min\{t(A), 1 - t(B)\} = 1 + max\{\ -t(A), -\ 1 + t(B)\} = \\ = max\{1 - t(A), t(B)\}$$

so that:

$$t(\neg(A \wedge \neg B)) = max\{1 - t(A), t(B)\} \tag{3.2}$$

Applying Z3, Z2, and Z4 to the right side of (3.1) gives:

$$t(B \vee (\neg A \wedge \neg B)) = max\{t(B), min\{1 - t(A), 1 - t(B)\}\}$$

We then have, after applying L1:

$$max\{t(B), min\{1 - t(A), 1 - t(B)\}\} = \\ = min\{max\{1 - t(A), t(B)\}, max\{t(B), 1 - t(B)\}\}$$

From (3.2), Z2, and Z4 we have:

$$t(B \vee (\neg A \wedge \neg\ B)) = min\{t(\neg(A \wedge \neg B)), t(B \vee \neg B)\}. \tag{3.3}$$

Therefore, from (3.3):

$$t(\neg(A \wedge \neg B)) \neq t(B \vee (\neg A \wedge \neg B))$$

In any interpretation in which $t(\neg(A \wedge \neg B)) > t(B \vee \neg B)$, e.g., $t(A) = 0.2$ and $t(B) = 0.6$, we have $0.8 = t(\neg(A \wedge \neg B)) \neq t(B \wedge (\neg A \wedge \neg B)) = 0.6$

So the conditions of logical equivalence are not fulfilled, since $t(\neg\ (A \wedge \neg B)) \neq t(B \vee (\neg A \wedge \neg B))$ in all interpretations; and, therefore, $\neg(A \wedge \neg B)$ is not logically equivalent to $B \vee (\neg A \wedge \neg B)$ in fuzzy logic. The hypothesis of Theorem 4 clearly does not hold for fuzzy logic; the conclusion does not, therefore, apply to fuzzy logic; and it has not been shown that fuzzy logic collapses to classical logic.

Since, in classical bivalent logic, $\neg(A \wedge \neg B) \equiv B \vee (\neg A \wedge \neg B)$, Theorem 4 does prove that classical bivalent logic, but not fuzzy logic, has two and only two truth values.

Misconception #4: Fuzzy logic "can lead inescapably to conclusions that no human being would accept" (deSilva 1994).

The argument presented below (the line numbers have been added for easy reference) is claimed to be a case in which axioms Z1–Z4 lead to conclusions that no human would accept:

> You know that the airplane on which John Doe was traveling has crashed in some remote location, but you have no information whether anyone on board has survived. In this situation, you might make the following assignment:
>
> t("John Doe is alive") = 0.5. (4.1)
>
> Axiom Z4 would lead you immediately to:
>
> t("John Doe is dead") = 0.5. (4.2)
>
> While this is a reasonable assignment, it would in turn lead you to
>
> t("John Doe is both dead and alive") = 0.5 (4.3)
>
> Thus there is an element of truth in the statement "John Doe is both dead and alive". However, any rational person will argue that it is impossible for John Doe to be both dead and alive, so that the statement "John Doe is both dead and alive" must always be false and have a truth value of zero.

The semantics defined in (Zadeh 1965) would assign to (4.1) the following meaning:

> *John Doe has a degree of membership equal to 0.5 in the set of all living persons.*

John Doe, then, possesses 50% of the characteristics associated with persons who are fully alive. While not an unreasonable assignment of truth value, (4.1) implies a very precise knowledge of John Doe's condition which is inconsistent with the statement "you have no information whether anyone on board has survived". Then (4.2) follows immediately from (4.1) via Axiom Z4, and states that John Doe possesses 50% of the characteristics associated with completely dead persons. The conclusion, (4.3), is that John Doe possesses half of the characteristics associated with persons that are fully alive and half of the characteristics that are associated with persons that are completely dead, as may well befit someone involved in an airplane crash in a remote area, is a direct result of Axiom Z2. Other than the initial assignment of a precise truth value while claiming to know nothing about John Doe's status, there would seem to be nothing in the fuzzy logic inference presented in (deSilva 1994) that contradicts human rationality.

It seems likely, however, that different semantics were intended in justifying the conclusion that "*any rational person will argue that it is impossible for John Doe to be both dead and alive, so that the statement 'John Doe is both dead and alive' must always be false and have a truth value of zero*" (deSilva 1994). If the statement "John Doe is alive" was construed as meaning that "John Doe is fully alive" then t("John Doe is alive") = 1.0 $\neq$ 0.5, and axiom Z4 gives t("John Doe is dead") = 0.0. Then it follows immediately from axiom Z2 that t("John Doe is both dead and alive") = 0.0. The problem thus stated is one of classical bivalent logic, not fuzzy logic, and the initial truth value assignment of t("John Doe is alive") = 0.5 is erroneous.

Alternatively, the following meaning may have been intended:

[t("John Doe is alive") = 0.5] $\equiv$ [the probability that John Doe is fully alive = 0.5] (4.4)

[t("John Doe is dead") = 0.5] $\equiv$ [the probability that John Doe is completely dead = 0.5] (4.5)

Once again, these are reasonable estimates of the probability that John Doe has full membership in either the set of living persons or the set of dead persons, respectively. But, these semantics involve only classical sets and probability theory, and the problem statement has nothing to do with fuzzy logic. The Law of the Excluded Middle applies; John Doe cannot be both fully alive and completely dead; so we must have:

t("John Doe is both dead and alive") $\equiv$ [the probability that John Doe is both fully alive and fully dead] = 0.0

The claim that "fuzzy logic can lead inescapably to conclusions that no human being would accept" (deSilva 1994) is not supported by any of these arguments.

The use of the Law of the Excluded Middle when reasoning with empirical data deserves additional attention. If the sentences "The glass is full" and

"The glass is empty" had been used above to describe a case in which a glass is half full, we would have *t*("The glass is full") = 0.5 and *t*("The glass is empty") = 0.5. The resulting conclusion,

$$t(\text{“The glass is full”} \wedge \text{“The glass is empty”}) = 0.5 \tag{4.6}$$

indicates that the glass is both half full and half empty, i.e., it possess 50% of the characteristics of a full glass and 50% of the characteristics of an empty glass, a result acceptable to rational humans. Classical logic, in which *t*("The glass is full") = 0.0, since the glass is not full; and *t*("The glass is empty') = 0.0, since the glass is half full; generates, in accord with LEM, the conclusion: *t*(The glass is full" ∧ "The glass is empty") = 0.0. This is a correct result, the glass is neither full nor empty, but, in comparison to (4.6), relatively uninformative.

Even when dealing with emotionally charged issue of human life and death it is not clear that the Law of the Excluded Middle always applies. Consider the case of a person in a "persistent vegetative state", i.e., one who exhibits no brain activity and and whose heart and lung functions are controlled by external devices. To assign to this person the same degree of membership in the set of all living persons as that accorded to the physician attending the patient would seem to be a gross miscategorization (Cranford 1987) (Wikler 1988).

4. Introduction to the Kenevan Truth Interval Fuzzy Logic

Axioms Z1–Z4 provide us with a mechanism by which, if we know $t(A)$ and $t(B)$, we can determine $t(A \wedge B)$. But, if we are given $t(A \wedge B)$ we cannot infer $t(A)$ or $t(B)$. If $t(A)$ is completely unknown, we can only write $t(A) = x$. In what follows we define a fuzzy logic that uses a subinterval of the real unit interval, to denote the degree of truth of a proposition that eliminates these shortcomings.

We assume that the single truth value a of any formula A exists but that it is known only as a closed subinterval of the real unit interval that encloses it. The subinterval will be designed by its endpoints and written as A: $[a_0, a_1]$, where $a_0 \leq a \leq a_1$. If the single truth value is known, we write $A : [a, a]$. For ease of exposition we denote $max(a, b, \ldots, z)$ by $\overline{ab \ldots z}$ and $min(a, b, \ldots, z)$ by $\underline{ab \ldots z}$.

We will state the inference rules in tableau format; introduced by Bethe, enhanced by Smullyan; and described in (Elfrink 1989). The premises for the inference are written above the line, the conclusions below. The vertical line used in the inference rules for conjunction and disjunction separates two worlds, each of which represents a possible conclusion inferable from the hypotheses. In an example from classical logic we represent a logical formula S conjoined with the conjunction $A \wedge B$ in an interpretation in which $A \wedge B$ has truth value *False* as $S[F(A \wedge B)]$. The inference is then stated as:

$$\frac{S[F(A \wedge B)]}{FA, S|FB, S}$$

Two worlds, or interpretations, are possible in this case; one in which the truth value of A must be *False* and that of B can be either *True* or *False*; and another in which the truth value of B must be *False* and that of A can be either *True* or *False*.

Given these conventions the inference rules for disjunction, conjunction, negation, and intersection (Kenevan 1992: 147) are shown in tableaux format in Figure 2, below.

$$\begin{array}{c|c} \multicolumn{2}{c}{A : [a_0, a_1]} \\ \multicolumn{2}{c}{B : [b_0, b_1]} \\ \multicolumn{2}{c}{A \vee B : [p_0, p_1]} \\ \hline A : [a_0, \underline{p_1 a_1}] & A : [\overline{p_0 a_0}, \underline{p_1 a_1}] \\ B : [\overline{p_0 b_0}, \underline{p_1 b_1}] & B : [b_0, \underline{p_1 b_1}] \\ A \vee B : [\overline{p_0 a_0 b_0}, \underline{p_1 \overline{a_1 b_1}}] & A \vee B : [\overline{p_0 a_0 b_0}, \underline{p_1 \overline{a_1 b_1}}] \end{array}$$

Inference rule for disjunction ($\vee$)

$$\begin{array}{c} A : [a_0, a_1] \\ \hline \neg A : [1 - a_1, 1 - a_0] \end{array}$$

Negation Inference rule ($\neg$)

$$\begin{array}{c|c} \multicolumn{2}{c}{A : [a_0, a_1]} \\ \multicolumn{2}{c}{B : [b_0, b_1]} \\ \multicolumn{2}{c}{A \wedge B : [q_0, q_1]} \\ \hline A : [\overline{q_0 a_0}, a_1] & A : [\overline{q_0 a_0}, \underline{q_1 a_1}] \\ B : [\overline{q_0 b_0}, \underline{q_1 b_1}] & B : [\overline{q_0 b_0}, b_1] \\ A \wedge B : [\overline{q_0 \underline{a_0 b_0}}, \underline{q_1 a_1 b_1}] & A \wedge B : [\overline{q_0 \underline{a_0 b_0}}, \underline{q_1 a_1 b_1}] \end{array}$$

Inference rule for conjunction ($\wedge$)

$$\begin{array}{c} A : [x_0, x_1] \\ A : [y_0, y_1] \\ \hline A : [\overline{x_0 y_0}, \underline{x_1 y_1}] \end{array}$$

Intersection Inference rule

Figure 2. Inference Rules for Truth Interval Fuzzy Logic.

The inference rule for conjunction states that; if a world, i.e., an interpretation, exists in which the single truth value for A is known to lie in the interval $[a_0, a_1]$, that for B in the interval $[b_0, b_1]$, and that for $A \wedge B$ in the interval $[q_0, q_1]$; then two worlds are possible, one of which must occur even though we do not know which. In one, the truth value of A is contained by the interval $[\overline{q_0 a_0}, a_1]$, that of B by $[\overline{q_0 b_0}, \underline{q_1 b_1}]$, and that of $A \wedge B$ by $[\overline{q_0 \underline{a_0 b_0}}, \underline{q_1 a_1 b_1}]$. The second world differs from the first in that the truth value of A is contained by the interval $[\overline{q_0 a_0}, \underline{q_1 a_1}]$ and that of B by $[\overline{q_0 b_0}, b_1]$. The other inference rules operate in a similar manner.

The intersection inference rule deserves additional explanation. It is possible that a formula or proposition *F* may appear more than once, with different truth intervals associated with each occurrence, in a single node of the proof tree resulting from successive applications of inference rules. The multiple appearances of *F* can be compressed into a single occurrence by using the inference rule for intersection which requires that the single truth interval associated with *F* be the intersection of all of the truth intervals associated with the multiple occurrences of *F*. If, in any leg of the proof tree, a null interval results from an application of the intersection inference rule, that leg is considered to be closed.

The correctness of the truth interval boundaries for the disjunctive rule is demonstrated in (Kenevan 1992: 148–150) and proofs for the remaining inference rules are similar. The truth interval fuzzy logic defined by these inference rules includes the classical logic as a special case (Entemann 2000: 169) and the truth intervals specified in the inference rules are as small as they can be (Entemann 2000: 166).

The simplest proof method of the fuzzy interval logic, the *truth interval refinement proof method*, starts by creating a tableau with the premises, above the line. The inference rules are then applied until the resulting truth intervals cease to contract, or until either an impossible truth interval (one in which the lower boundary is greater than the upper boundary) or a null interval is generated. Any such interval designates a closed leg of the proof tree, one which contributes nothing to the final truth intervals. At the termination of the proof each non-closed leaf of the proof tree, a tableau, describes a possible set of truth values for each logical expression in the hypothesis, i.e., a possible *world*. If all branches of the proof tree close then it has been inferred that the original premises constitute a contradiction.

Each application of an inference rule results in a truth interval that is smaller than or equal to the original, so if one starts with X: $[x, y]$, $0.5 < x, y \leq 1.0$, it is impossible to prove X: $[w, z]$, $0.0 \leq w, z, < 0.5$. The truth interval fuzzy logic is, then, consistent in the same sense that classical logic is consistent.

To demonstrate the truth interval refinement proof method we return to Misconception #4 and let:

A = "John Doe is alive" and
$\neg A$ = ¬("John Doe is alive") = "John Doe is dead"

Since we are totally ignorant of John doe's condition, we set A: [0.0, 1.0]. By this choice we are asserting that, since we have absolutely no knowledge of John Doe's degree of membership in the set of living persons, the smallest interval that is guaranteed to contain the correct value is the entire real unit

Original Tableau	A : [0.0, 1.0] $\neg A$: [0.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]	
∧ Inference Rule	A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]	A : [0.0, 0.0] $\neg A$: [0.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]
¬ Inference Rule	A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $\neg A$: [0.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]	A : [0.0, 0.0] $\neg A$: [0.0, 1.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]
Intersection Inference Rule	A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]	A : [0.0, 0.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]

Figure 3. First Step in the Proof of $A \wedge \neg A$.

interval. The negation inference rule then requires that $\neg A$: [0.0, 1.0]. If we are certain of nothing else, then we must also assert that $A \wedge \neg A$: [0.0, 1.0]. The conjunctive inference rule then infers that one of two worlds must exist, each of which is identical to the initial world, or that, since we know nothing, we can infer nothing.

If we are sure that the Law of the Excluded Middle applies, we may also assert that $A \wedge \neg A$: [0.0, 0.0], or that John Doe cannot have any degree of membership in both the set of living persons and the set of dead persons. Figure 3 shows the first steps in the inference. First, the conjunctive inference rule is applied to the premises, the result being given as the first row below the line. The negation inference rule is then applied to A to generate the second row, containing two occurrences of $\neg A$ with different truth intervals. An application of the negation inference rule to the two occurrences of $\neg A$ completes the first three steps in the proof. We continue by applying the appropriate inference rules to the bottom left tableau of Figure 3 to generate the next step in the proof process, shown in Figure 4. Here the right leg of the proof tree terminates with a tableau that contains $\neg A$: [0.0, 0.0] and $\neg A$: [1.0, 1.0] which would, after an application of the intersection inference rule, generate the null interval, closing that leg. The tableau on the lower left is identical to the original, so another application of the inference rules will simply duplicate Figure 4. This is, then the final step in the left leg of the proof tree started in Figure 3.

In a similar manner we now apply inference rules to the bottom right tableau of Figure 3, with the result shown as Figure 5, to complete the proof. The left subtree in Figure 5 is closed due to the nonexistent truth interval [1.0, 0.0] attached to $\neg A$. The final tableau of the right subtree duplicates the

Leftmost Tableau of Fig. 3		A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]	
$\wedge$ Inference Rule	A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]		A : [0.0, 0.0] $\neg A$: [0.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]
$\neg$ Inference Rule	A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $\neg A$: [0.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]		A : [0.0, 0.0] $\neg A$: [0.0, 0.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]
Intersection Inference Rule	A : [0.0, 1.0] $\neg A$: [0.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]		***CLOSED***

Figure 4. Left Leg of the Proof Tree for $A \wedge \neg A$

	A : [0.0, 0.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]		Rightmost Tableau of Fig. 3
A : [0.0, 0.0] $\neg A$: [1.0, 0.0] $A \wedge \neg A$: [0.0, 0.0]		A : [0.0, 0.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]	$\wedge$ Inference Rule
CLOSED		A : [0.0, 0.0] $\neg A$: [1.0, 1.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]	$\neg$ Inference Rule
		A : [0.0, 0.0] $\neg A$: [1.0, 1.0] $A \wedge \neg A$: [0.0, 0.0]	

Figure 5. Right Leg of the Proof Tree Started in Figure 3.

premises, so further application of inference rules will not produce any new results, and this leg of the proof tree is terminated.

Figure 6 shows, without the intermediate steps, the only survivors of the proof procedure, the terminal tableaux of both the right and left subtrees of Figure 3. That for the left describes a world in which one has no knowledge of the John Doe's degree of membership in the set of living persons, but he is not a member of the set of dead persons. In the second, John Doe is clearly dead. In both worlds the Law of the Excluded Middle is satisfied, in agreement with popular dogma, since John Doe has a zero degree of membership in the set of those who are both living and dead.

Original Tableau	$A : [0.0, 1.0]$ $\neg A : [0.0, 1.0]$ $A \wedge \neg A : [0.0, 0.0]$	
	$A : [0.0, 1.0]$ $\neg A : [0.0, 0.0]$ $A \wedge \neg A : [0.0, 0.0]$	$A : [0.0, 0.0]$ $\neg A : [1.0, 1.0]$ $A \wedge \neg A : [0.0, 0.0]$

Figure 6. Final Tableau in the Proof of $A \wedge \neg A$.

5. Misconceptions Regarding Fuzzy Logic Proof Theory

Misconception #5: There can be no proof theory for fuzzy logic (Pelletier 1994).
A proof theory for the Kenevan truth interval fuzzy logic has been presented above. To completely invalidate this claim a proof theory for a logic defined only by axioms Z1–Z4 is presented below.

By analogy with the classical concept, we define *fuzzy logical consequence* as follows:

> **Given formulae $F_1, F_2, \ldots, F_n$ and a formula G, G is said to be a *fuzzy logical consequence* of $F_1, F_2, \ldots, F_n$ (or, G *logically follows from* $F_1, F_2, \ldots, F_n$ *using fuzzy logic*) if and only if for any interpretation in which $F_1 \wedge F_2 \wedge F_3 \wedge \ldots \wedge F_n$ is true or fuzzy true, G is also true or fuzzy true, respectively.**

Theorem 5: Given formulae $F_1, F_2, \ldots, F_n$ and a formula G, G is a fuzzy logical consequence of $F_1, F_2, \ldots, F_n$ if and only if the formula $(F_1 \wedge \ldots \wedge F_n) \Rightarrow G$ is a *fuzzy tautology*.
Proof: Recall that: $(F_1 \wedge \ldots \wedge F_n) \Rightarrow G \equiv \neg(F_1 \wedge \ldots \wedge F_n) \vee G$.

Assume that G is a fuzzy logical consequence of $(F_1 \wedge \ldots \wedge F_n)$ so, in any interpretation I in which $t(F_1 \wedge \ldots \wedge F_n) > 0.5$, we must have $t(G) > 0.5$ in I. When $t(F_1 \wedge \ldots \wedge F_n) > 0.5$ and $t(G) > 0.5$, we have from Z3, $t(\neg(F_1 \wedge \ldots \wedge F_n) \vee G) > 0.5$ in I. If $t(F_1 \wedge \ldots \wedge F_n) < 0.5$ in I then $t(\neg(F_1 \wedge \ldots \wedge F_n)) > 0.5$ in I and, from Z2, $t(\neg(F_1 \wedge \ldots \wedge F_n) \vee G) > 0.5$ in I. Therefore, $t((F_1 \wedge \ldots \wedge F_n) \Rightarrow G) > 0.5$ in all interpretations and is a fuzzy tautology.

Now assume that $(F_1 \wedge \ldots \wedge F_n) \Rightarrow G$ is a fuzzy tautology. Therefore $t(\neg(F_1 \wedge \ldots \wedge F_n) \vee G) > 0.5$ in all interpretations I. Then, from Z4, if $t(F_1 \wedge \ldots \wedge F_n) > 0.5$ in I, $t(\neg(F_1 \wedge \ldots \wedge F_n)) < 0.5$ in I, and, according to Z3, $t(G) > 0.5$ in I. Hence, G is a fuzzy logical consequence of $(F_1 \wedge \ldots \wedge F_n)$. □

Whenever: $(F_1 \wedge \ldots \wedge F_n) \Rightarrow G$ is a fuzzy tautology,
$\neg((F_1 \wedge \ldots \wedge F_n) \Rightarrow G)$ must be a fuzzy contradiction.
$\neg((F_1 \wedge \ldots \wedge F_n) \Rightarrow G) \equiv \neg(\neg(F_1 \wedge F_2 \wedge \ldots \wedge F_n) \vee \neg G) \equiv$
$(F_1 \wedge F_2 \wedge \ldots \wedge F_n) \wedge \neg G$
and we have proven Theorem 6, the fuzzy analog of the classical ***proof by contradiction***.

Theorem 6: Given formulae $F_1, F_2, \ldots, F_n$ and a formula G, G is a fuzzy logical consequence of $F_1, \ldots, F_n$ iff the formula $(F_1 \wedge \ldots \wedge F_n) \wedge \neg G$ is a fuzzy contradiction.

So, if we restrict ourselves to fuzzy propositions A for which

$$0.5 < t(A) \leq 1.0 \text{ or } 0.0 \leq t(A) < 0.5$$

fuzzy logic is amenable to the concept of logical consequence and, hence, a proof theory. This subset of the entire fuzzy logic includes Zadeh's *dispositional logic*, where a disposition is "a proposition which is preponderantly, but not necessarily always, true" (Zadeh 1988).

Misconception #6: Fuzzy logic can never be proven to be proof theoretic complete (Pelletier 1994).
The term *proof theoretic complete*, as applied to a logical theory, indicates that any theorem that can be composed using the syntax of a theory can also be proven using the inference rules of the theory. Formal definitions of these terms are:

> **A logical theory *L* is said to be *proof theoretic complete*, or *complete*, if every tautology in *L* is provable in *L*.**
>
> **A computation method is *complete* if, for every sentence *S*, the alogrithm will, with input *S*, return an indication that *S* is a tautology, or that it is not, in a finite amount of time.**

The second definition ties the term *complete* to a computational recipe that determines whether or not a theorem is a tautology. In this case the term *decideable* should, perhaps, be used instead of *complete*.

Truth interval fuzzy dispositional logic is proof theoretic complete (Entemann 2000: 174–176). The proof involves showing that all fuzzy tautologies involving only one disjunction, e.g., $A \vee \neg A$, or two disjunctions and one conjunction, e.g., $(A \vee \neg A) \wedge (B \vee \neg B)$, can be proven in the theory. This set of formulae contains all simple tautologies. A simple inductive argument extends this result to include all tautologies.

Misconception #7: "Fuzzy logic introduces enormous complexities" (Haack 1996: 238).

The term ***complexity***, when applied to a logic, embraces multiple meanings. The first involves ***computational complexity*** or, very informally, a measure of the time required by a theorem proving algorithm to process a formula containing *N* propositions. In another sense, the complexity of a logic refers to the difficulty of formulating, then solving, a problem using the syntax of that logic. A third case, probably that intended by Susan Haack, concerns the size of the formal rule base that is required to assign truth values to logical formulae, i.e. the process of logical inference.

The first, that of computational complexity, can be dealt with quickly. The worst case complexity of the truth interval refinement method in the Kenevan Truth Interval Fuzzy Logic is $\mathbf{O}\ (2^{3*N})$, that of the proof methods involving fuzzy logical consequence is $\mathbf{O}(2^{4*N})$, where *N* is the number of connectives in the formula being processed (Entemann 2000: 178–179). In contrast, the complexity of proof methods associated with classical propositional logic is $\mathbf{O}(2^N)$, where *N* is the number of symbols in the formula. The computational complexity of fuzzy logic is, then, greater than that of classical logic. Nonetheless, PROLOG, an automatic theorem prover for the Horn clause subset of classical predicate logic (which is, in the worst case, undecideable), has proven to be quite useful for solving empirical problems. Fuzzy logic theorem provers are likely to be equally useful. If the increased complexity proves to be a problem, heuristics of considerably lower complexity are available (Cox 1994). An automatic theorem proving program implementing the truth interval refinement proof method for the truth interval fuzzy logic has been implemented (Entemann 2000: 180–181).

As an illustration of the second sense of complexity, we consider the control logic of an autonomous mobile robot and contrast implementations in fuzzy logic and classical logic. The speed of the robot and the distance separating it from the nearest obstacle on its current trajectory are, at any instant in time, crisp values known to the limits of precision of the sensors employed, their maxima, and the CPU being used. The set all discrete values for such parameters is, usually, impossibly large. It would seem to be desirable, even necessary, to categorize discrete parameter values and define the control logic for each category rather than for each individual parameter value. Speed, for example, could be categorized by *SLOW, MODERATE*, and *FAST*; each category being typified by the observed behavior of the robot at a speed deserving full membership in that category. At any speed fully deserving the designation *SLOW*, for example, the robot can execute a right angle turn without rolling over. In contrast, at any speed fully deserving the appellation of *MODERATE* the sharpest turning angle that can be successfully negotiated is 60°. It is, futhermore, very unlikely that the behavior of the robot at a speed of 5.63 cm/sec will be significantly different than at 5.64 cm/sec. Hence, categorizing 5.63 cm/sec as *SLOW* and 5.64 cm/sec. as *MODERATE* is not justified.

A ***fuzzy logic controller*** would utilize (Entemann 2001):

1. A ***fuzzy rule base***, or a set of rules of the form:
   IF speed is ***MODERATE*** **&** distance is ***MODERATE*** **THEN** turn ***MODERATELY***
   IF speed is ***FAST*** **&** distance is ***MODERATE*** **THEN** turn ***SHARPLY***
2. A set of functions defining the degrees of membership of discrete values in the various categories. If a relatively large number of values deserve full membership in each category C_i a graphical representation of the membership function might resemble Figure 7a, below. Alternatively, if only one value, or a very small number of values, characterizes a category the membership function might resemble that of Figure 7b.
3. A ***fuzzy inference engine*** to:
   a. Calculate the truth value of the premise of each rule according to axiom Z2, invoking the membership functions supplied to do so.
   b. Select the maximum truth value associated with each action in accordance with Axiom Z3.
   c. Compute a weighted average (the weights are the truth values) of action values to generate a crisp value for the designated action.

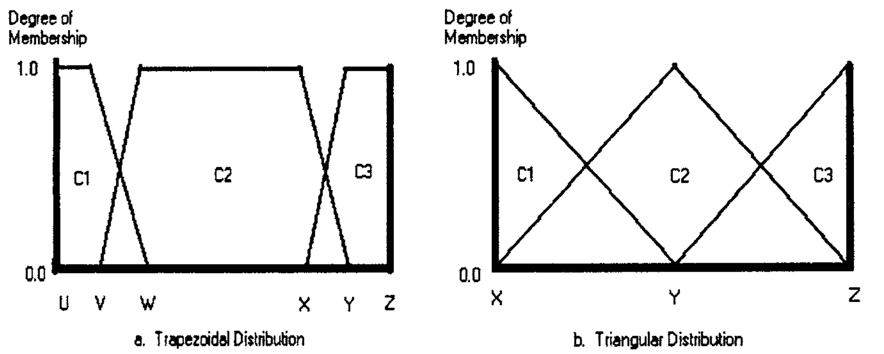

Figure 7. Membership Functions for Fuzzy Categories.

A classical inference engine, using crisp sets, would execute the first rule for which the premises could be evaluated as ***True***. Since very small changes in parameter values should not, usually, cause large changes in the action taken, the precision of the classical inference engine would need to be enhanced by significantly reducing the size of each category. The number of categories is, as a result, increased thereby significantly increasing both the computational complexity of the classical inference engine and the design complexity of its rule base. The complexity of the fuzzy control system is, less than that of the classical under these circumstances.

The third, and final, sense of *complexity* refers to the complexity of the formal rule set required to process a particular system, as in "the complexities introduced by fuzzy logic are such as to nullify the usual definite, mechanical, routine character of formal rules" when *linguistic variables*, e.g., *MODERATE* in "speed is *MODERATE*", are used in statements of a logic (Haack 1996: 239). The rationale for this statement appears to in the statement that follows: "Zadeh concedes that the choice of suitable linguistic approximations is a matter of discretion." Consider, however, a formal logic in which:

1. The definitions of the membership functions, e.g., those of Figure 7, that define the linguistic variables (one for each linguistic variable) used in logical inference are part of the input.
2. Each membership function is invoked as a step in specifying each interpretation.

The membership functions replace the "built in" rules of classical logic that assign values of ***True*** and ***False*** when creating a classical interpretation. The resulting logic system would retain the "*definite, mechanical, and routine character*" of the formal rules of inference. The increase in complexity would be of the order of magnitude of the task performed in classical first order logic when creating interpretations for a universally quantified formula. In PROLOG, Robinson's Unification Algorithm performs this task quite nicely and would, with some modifications, do the same thing for fuzzy lgoic.

Proving theorems in fuzzy logic is, indeed, a more complex task than proving theorems in classical logic, but not impossibly complex. A basis for a system that performs such tasks using rule-based heuristic rules of inference and a limited number of membership functions, is presented in (Chang 1997).

Misconception #8: "We Do Not Need Fuzzy Logic" (Haack 1996: 242).
Classical logic is, clearly, an indispensable tool for mathematicians and philosophers who can postulate worlds in which premises are known precisely and can be legitimately assigned bivalent truth values. In particular, all of the proofs *about* fuzzy logic, i.e., *meta* proofs, presented herein used classical logic. It would seem, then, that philosophers and mathematicians, and anyone else who restricts themselves to inference in theoretical domains, do not need fuzzy logic. But, a theory formulated in classical logic can be expected to have only limited applicability to empirical problems because

"... its use presupposes that the relevant relations and individuals of the theoretical discourse are exact. Relations and individuals of empirical discourse are, however, usually inexact and so cannot be included in the theoretical domain. A correspondence with theoretical entities is achieved by idealization" (Cleave 1991).

So, investigators drawing inferences from empirical data, those who are not content to approximate reality in order to use classical logic, need a logic that processes truth values other than ***True*** or ***False***. The descriptors *degrees of belief*, or *degrees of nearness to truth*, which are acceptable to Susan Haack (Haack 1996: 257), are excellent descriptors for the truth values of fuzzy logic. The use of such designations, in either formal or heuristic inference, requires fuzzy logic.

Conclusions

All of the alleged misconceptions concerning fuzzy logic presented here have been demonstrated to be misconceptions. Most importantly, fuzzy logic is not classical bivalent logic masquerading under a misleading title; nor does it lack the properties of consistency and completeness, which are deemed necessary in a formal logic system; nor do the additional complexities inherent in fuzzy inference justify the statement that "fuzzy logic introduces enormous complexities".

To say that "... fuzzy logic is not, after all, an attempt to represent truth-preserving inferences, and is not, after all, a theory in the same domain as classical logic; in fact, so construed, it is obviously not properly describable as a 'logic' at all" (Haack 1996: 231), is simply not justified. Fuzzy logic as described in this paper is a formal tool eminently suited for dealing with the imprecision inherent in empirical data. Imprecise premises used in formal inference will, of necessity, generate imprecise conclusions. This does not justify withholding the accolade of ***logic*** from fuzzy logic.

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