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RESEARCH PAPER

BOUNDS ON THE SOLUTION OF A CAUCHY-TYPE PROBLEM INVOLVING A WEIGHTED SEQUENTIAL FRACTIONAL DERIVATIVE

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Abstract

In this paper we establish some bounds for the solution of a Cauchy-type problem for a class of fractional differential equations with a weighted sequential fractional derivative. The bounds are based on a Bihari-type inequality and a bound on the Gauss hypergeometric function.

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1. Introduction

Fractional calculus approach has been increasingly introduced and adopted in many models. It has been proven that fractional models provide a tool for capturing and understanding complex phenomena in many areas. Indeed, some of these models are supported by experimental evidence and yield results that agree with the observed behavior. See for example the surveys in [20, 37] and the collection of applications in [17].

Some recent applications include control systems [6, 33], viscoelasticity [18, 43, 27, 26], and nanotechnology [5]. Also fractional models are used

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to model a vibrating string [40], anomalous transport [24], and anomalous diffusion [13, 38, 29, 30].

Another field of applications is random walk and stochastic processes [19, 45, 4] and their applications in financial modeling [41, 42, 9]. Other physical and engineering processes are given in [34, 36].

As a result, there has been a considerable interest in the theory of fractional differential equations. The investigations include the existence and uniqueness of solutions, asymptotic behavior, stability, etc. See for example the books [20, 25, 8] and the articles [1, 2, 21, 7, 44, 3, 12], and the references therein.

In this paper we consider a Cauchy-type problem associated with the equation

$$\mathcal{D}_{r,a}^{\alpha,\beta} u(t) = f(t, u(t)), \quad a < t \leq b, \quad 0 \leq r < \alpha < 1, \quad 0 < \beta < 1, \tag{1.1}$$

where $\mathcal{D}_{r,a}^{\alpha,\beta}$ is the weighted sequential fractional derivative introduced in [11].

In a series of articles, [14, 15, 16], Glushak studied the uniform well-posedness of a Cauchy-type problem with two fractional derivatives and bounded operator. He also proposed a criterion for the uniform correctness of unbounded operator.

In [11] and [10] the author obtained existence and uniqueness results in the weighted spaces of continuous functions and the space of Lebesgue integrable functions, respectively. In this paper we develop some bounds on the weighted sequential integral and use them to find a bound on the solution of the Cauchy-type problem.

Next, we start with some preliminaries in Section **2**. In Section **3** we define the sequential integrals and derivatives and present some properties. In Section **4** we construct some bounds on the sequential fractional integral. In Section **5** we obtain a bound on the solution of the Cauchy problem.

2. Preliminaries

In this section we present some definitions, lemmas, properties and notation which we use later. For more details please see [20].

Let $-\infty < a < b < \infty$. Let $C[a, b]$ and $C(a, b)$ denote the space of continuous functions on $[a, b]$ and (a, b), respectively. We denote by $L^p(a, b)$, $p \geq 1$, the spaces of Lebesgue integrable functions on (a, b), in particular we use $L(a, b)$ to denote $L^1(a, b)$. We denote the subspace $C(a, b) \cap L(a, b)$ by $CL(a, b)$. Furthermore, we introduce the following weighted spaces of continuous functions

$$C_\omega[a, b] = \{f : (a, b] \to \mathbb{R} : (t - a)^\omega f(t) \in C[a, b]\}, \quad 0 \leq \omega < 1,$$

with the norm

$$\|f\|_{C_\omega} = \|(t-a)^\omega f(t)\|_C \, .$$

The following is a special case of Jensen's Inequality.

Lemma **2.1**. *For nonnegative a_i, $i = 1, \dots, k$,*

$$\left(\sum_{i=1}^{k} a_i\right)^p \le k^{p-1} \sum_{i=1}^{k} a_i^p, \quad p \ge 1. \tag{2.1}$$

The following lemma is given in [35], Theorem 2.5.1.

Lemma **2.2**. *Let $0 < T \le \infty$. Let $a(t)$ and $b(t)$ be continuous positive functions defined on $[0,T)$. Let $w : [0,\infty) \to [0,\infty)$ be a continuous monotonic nondecreasing function such that $w(0) = 0$ and $w(t) > 0$ for $t > 0$. If u is a positive differentiable function on $[0,T)$ that satisfies*

$$u'(t) \le a(t)\, w(u(t)) + b(t), \quad t \in [0,T),$$

then we have

$$u(t) \le G^{-1}\left[G\left(u(0) + \int_0^t b(s)ds\right) + \int_0^t a(s)ds\right], \tag{2.2}$$

for the values of $t \in [0,T)$ for which the right-hand side is well-defined, where

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \qquad r > 0, \quad r_0 > 0.$$

The next lemma is a special case of Lemma **2.2**.

Lemma **2.3**. *Let $a(t)$ and $b(t)$ be continuous positive functions defined on $[0,T)$, $T > 0$. Let u be a positive differentiable function on $[0,T)$ that satisfies*

$$u'(t) \le a(t)\, u^m(t) + b(t), \quad t \in [0,T), \quad m > 0.$$

Let

$$A(t) = \int_0^t a(s)\, ds, \qquad B(t) = u(0) + \int_0^t b(s)\, ds.$$

Then for $m > 0$, $m \ne 1$, we have

$$u(t) \le \left[(1-m)A(t) + B^{1-m}(t)\right]^{\frac{1}{1-m}}, \tag{2.3}$$

for $t \in [0, T_m)$, where

$$T_m = \begin{cases} T, & 0 < m < 1, \\ \max\left\{t \in [0,T) : A(t)\, B^{m-1}(t) < \frac{1}{m-1}\right\}, & m > 1. \end{cases} \tag{2.4}$$

For $m=1$ we have

$$u(t) \le e^{A(t)} B(t),$$

for $t \in [0,T)$.

P r o o f. Applying Lemma **2.2** with $w(u)=u^m$,

$$G(r) = \int_{r_0}^{r} \frac{ds}{s^m} = \begin{cases} \frac{1}{1-m}\left(r^{1-m} - r_0^{1-m}\right), & m>0, m\ne 1, \\ \ln r, & m=1. \end{cases} \tag{2.5}$$

and

$$G^{-1}(y) = \begin{cases} \left[r_0^{1-m} - (m-1)y\right]^{\frac{1}{1-m}}, & m>0, m\ne 1, \\ e^y, & m=1. \end{cases} \tag{2.6}$$

Thus for $m>0$, $m\ne 1$, the right-hand side of (2.2) takes the form

$$\begin{aligned} & G^{-1}\left[G\left(u(0) + \int_0^t b(s)ds\right) + \int_0^t a(s)ds\right] = G^{-1}[G(B(t)) + A(t)] \\ & = \left[r_0^{1-m} - (m-1)[G(B(t)) + A(t)]\right]^{\frac{1}{1-m}} \\ & = \left[r_0^{1-m} - (m-1)A(t) - (m-1)\frac{1}{1-m}\left(B^{1-m}(t) - r_0^{1-m}\right)\right]^{\frac{1}{1-m}} \\ & = \left[r_0^{1-m} - (m-1)A(t) + B^{1-m}(t) - r_0^{1-m}\right]^{\frac{1}{1-m}} \\ & = \left[(1-m)A(t) + B^{1-m}(t)\right]^{\frac{1}{1-m}}. \end{aligned}$$

If $0<m<1$ then the expression inside the bracket is nonnegative and thus the power of this expression is well defined for all $t \in [0,T)$. For $m>1$, the first term in the bracket is negative for $t>0$ and thus the bracket might not be nonnegative for sufficiently large t in general.

For $m=1$, the result follows by substituting the case $m=1$ of (2.5) and (2.6) in the inequality (2.2). □

The left-sided Riemann-Liouville fractional integrals and derivatives are defined as follows.

Definition **2.1**. Let $f \in L(a,b)$. The integral

$$I_{a+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} ds, \qquad t>a, \quad \alpha>0,$$

is called the left-sided Riemann-Liouville (R-L) fractional integral of order α of the function f.

Definition **2.2**. The expression

$$D^{\alpha}_{a^+} f(t) := D I^{\alpha-1}_{a^+} f(t), \quad t > a, \quad 0 < \alpha < 1, \quad D = \frac{d}{dt},$$

is called the left-sided Riemann-Liouville (R-L) fractional derivative of order α of f provided the right-hand side exists.

Lemma **2.4** ([31]). *Let $\lambda,\ \nu,\ \eta > 0$, then*

$$I^{\nu}[t^{\lambda-1} e^{-\eta t}] := \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\eta s}\, ds \le A\, t^{\nu-1}, \quad t > 0,$$

where

$$A = \max\left\{1, 2^{1-\nu}\right\} \frac{\Gamma(\lambda)}{\Gamma(\nu)} \left(1 + \frac{\lambda(\lambda+1)}{\nu}\right) \eta^{-\lambda} > 0. \tag{2.7}$$

Now we introduce some properties of the Gauss hypergeometric function ${}_2F_1$. From formula 5, Table 9.1 in [39], we have the following lemma.

Lemma **2.5**. *Let $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}$, and $0 < s < t$, then*

$$\int_s^t (\tau - s)^{\alpha-1}\, (t-\tau)^{\beta-1}\, \tau^{-r} d\tau = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, (t-s)^{\alpha+\beta-1}\, s^{-r}\, {}_2F_1\left(\alpha, r; \alpha+\beta; 1 - \frac{t}{s}\right).$$

The following bound is given in [28], Ch. 4, Lemma 4.1.

Lemma **2.6**. *Let a, b and c be complex numbers and let $\delta > 0$. Then there is a constant M, independent of $\nu \in (0,1)$, such that*

$$\left| {}_2F_1\left(a, b; c; 1 - \frac{1}{\nu}\right)\right| \le M\, \Gamma(c)\, \nu^{\xi}, \quad \xi = \min\{\operatorname{Re} a, \operatorname{Re} b\} - \delta.$$

Thus from Lemma **2.5** and Lemma **2.6** we have

Lemma **2.7**. *Let $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R}$. Given $\delta > 0$, there exists a constant $M > 0$ such that for $0 < s < t$,*

$$0 \le {}_2F_1\left(\alpha, r; \alpha+\beta; 1 - \frac{t}{s}\right) \le M\, \Gamma(\alpha+\beta)\, \left(\frac{s}{t}\right)^{\xi}, \quad \xi = \min\{\alpha, r\} - \delta. \tag{2.8}$$

3. Weighted sequential integrals and derivatives

Definition **3.1**. Let $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}$. Let $f \in CL(a,b)$. Define the weighted sequential integral $\mathcal{J}_{r,a}^{\alpha,\beta} f$ and the weighted sequential derivative $\mathcal{D}_{r,a}^{\alpha,\beta} f$ by

$$\mathcal{J}_{r,a}^{\alpha,\beta} f(t) = I_a^{\alpha} (x-a)^{-r} I_a^{\beta} f(t), \tag{3.1}$$

and

$$\mathcal{D}_{r,a}^{\alpha,\beta} f(t) = D_a^{\alpha} (x-a)^{r} D_a^{\beta} f(t), \tag{3.2}$$

if the right-hand sides exist.

When $r = 0$, the derivative in (3.2) reduces to the sequential derivative introduced by Miller and Ross [32], considered also by Podlubny [37]. Without loss of generality we take $a = 0$ and drop the subscript a.

Note that compositions of two weighted R-L fractional integrals or derivatives as in Definition **3.1** have been considered in details in Samko, Kilbas and Marichev [39, §10]. The integral (3.1) and derivative (3.2) are also examples of Kiryakova's operators of generalized fractional calculus (GFC), in the case of compositions of two ($m = 2$) Erdélyi-Kober (E-K) operators called hypergeometric fractional integrals/ derivatives. Generally, the GFC ([22], [23]) deals with compositions of $m > 1$ E-K operators (weighted R-L operators), see also [39, §10.3].

As the mentioned name hypergeometric fractional integral suggests, the integral $\mathcal{J}_r^{\alpha,\beta} f$ can be represented in terms of the Gauss hypergeometric function ${}_2F_1$ as follows.

Lemma **3.1**. *Let $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R}$. Let f be a measurable nonnegative function defined on $(0,t)$. If $\mathcal{J}_r^{\beta,\alpha} f(t)$ exists then*

$$\mathcal{J}_r^{\beta,\alpha} f(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}\; s^{-r}\; {}_2F_1\left(\alpha, r; \alpha+\beta; 1-\frac{t}{s}\right)\; f(s)\, ds. \tag{3.3}$$

P r o o f. Practically, this is another form of formula (10.10) from Samko, Kilbas and Marichev [39, §10]. For readers's convenience, we give here its derivation. Using Dirichlet formula, we obtain

$$\begin{aligned} I^{\beta}\left[t^{-r}\, I^{\alpha} f(t)\right] &:= \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \tau^{-r}\, I^{\alpha} f(\tau)\, d\tau \\ &:= \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^t (t-\tau)^{\beta-1} \tau^{-r} \left[\int_0^{\tau} (\tau-s)^{\alpha-1} f(s)\, ds\right] d\tau \end{aligned}$$

$$= \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^t \int_0^\tau (t-\tau)^{\beta-1}\tau^{-r}\,(\tau-s)^{\alpha-1} f(s)\,ds\,d\tau$$

$$= \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^t \left[\int_s^t (\tau-s)^{\alpha-1}\,(t-\tau)^{\beta-1}\,\tau^{-r}\,d\tau\right] f(s)\,ds.$$

Thus, formula (3.3) follows by substitution from Lemma **2.5**. □

Now formula (3.3) and Lemma **2.7** lead to the following bound.

LEMMA **3.2**. *Let $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R}$. Let f be a measurable nonnegative function defined on $(0,t)$. If $\mathcal{J}_r^{\beta,\alpha} f(t)$ exists then given $\delta > 0$, there exists a constant $M > 0$ such that*

$$\mathcal{J}_r^{\beta,\alpha} f(t) \le M t^{-\xi}\,\Gamma(\alpha+\beta)\, I^{\alpha+\beta}\left[t^{\xi-r} f(t)\right], \quad \xi = \min\{\alpha, r\} - \delta. \quad (3.4)$$

P r o o f.

$$\begin{aligned}
\mathcal{J}_r^{\beta,\alpha} f(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}\, s^{-r}\, {}_2F_1\left(\alpha, r; \alpha+\beta; 1-\frac{t}{s}\right) f(s)\,ds \\
&\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}\, s^{-r}\, M\,\Gamma(\alpha+\beta)\, \left(\frac{s}{t}\right)^{\xi} f(s)\,ds \\
&= M t^{-\xi} \int_0^t (t-s)^{\alpha+\beta-1}\, s^{\xi-r}\, f(s)\,ds, \\
&= M t^{-\xi}\,\Gamma(\alpha+\beta)\, I^{\alpha+\beta}\left[t^{\xi-r} f(t)\right].
\end{aligned}$$

□

4. Bounds for the sequential integral

LEMMA **4.1**. *Let $\alpha > 0$ and $0 \le \omega < 1$. Let $f(.,u(.)) \in L(0,T)$, $0 < T \le \infty$, for any $u \in C_\omega[0,T]$ and satisfies*

$$|f(t,u(t))| \le t^{\mu} e^{-\sigma t}\,\phi(t)\,|u(t)|^m, \quad t \in (0,T), \quad (4.1)$$

with $m \in \mathbb{N}$, $\mu > m\omega - 1$, $\sigma > 0$, and ϕ is a continuous nonnegative function on $(0,T)$. Then

$$|I^\alpha f(t,u(t))| \le \frac{A^{1/p}}{\Gamma(\alpha)}\, t^{\alpha-1} \left(\int_0^t \phi^q(s)\, s^{\omega q m}\, |u(s)|^{qm}\,ds\right)^{1/q}, \quad t \in (0,T), \quad (4.2)$$

for any

$$q > \max\left\{\frac{1}{\mu-\omega m+1}, \frac{1}{\alpha}, 1\right\}, \quad (4.3)$$

where $A > 0$ is given by (2.7) with

$$\nu = p(\alpha-1)+1, \qquad \lambda = p(\mu-\omega m)+1, \qquad \eta = p\sigma, \tag{4.4}$$

and p is the conjugate of q, i.e. $\frac{1}{p}+\frac{1}{q}=1$.

P r o o f. Let q be as in (4.3) and p its conjugate. Then by the Hölder inequality we obtain

$$\begin{aligned}
&\Gamma(\alpha)\,|I^{\alpha} f(t,u(t))| \leq \int_0^t (t-s)^{\alpha-1}|f(s,u(s))|\,ds\\
&\leq \int_0^t (t-s)^{\alpha-1}\, s^{\mu}\, e^{-\sigma s}\, \phi(s)\, |u(s)|^m\, ds\\
&\leq \int_0^t (t-s)^{\alpha-1}\, s^{\mu-\omega m}\, e^{-\sigma s}\, \phi(s)\, s^{\omega m}\, |u(s)|^m\, ds\\
&\leq \left(\int_0^t (t-s)^{p(\alpha-1)}\, s^{p(\mu-\omega m)}\, e^{-p\sigma s} ds\right)^{1/p} \left(\int_0^t \phi^q(s)\, s^{\omega q m}\, |u(s)|^{qm}\, ds\right)^{1/q}.
\end{aligned}$$

From (4.3) we have

$$p(\alpha-1) = \frac{q}{q-1}(\alpha-1) = \frac{q\alpha-q}{q-1} > \frac{1-q}{q-1} = -1,$$

and

$$p(\mu-\omega m) = \frac{q(\mu-\omega m)}{q-1} > \frac{1-q}{q-1} = -1.$$

Thus we can apply Lemma **2.4** with

$$\nu = p(\alpha-1)+1 > 0, \quad \lambda = p(\mu-\omega m)+1 > 0, \quad \eta = p\sigma > 0,$$

to obtain

$$\int_0^t (t-s)^{p(\alpha-1)}\, s^{p(\mu-\omega m)}\, e^{-p\sigma s} ds \leq A\, t^{p(\alpha-1)}, \tag{4.5}$$

where A is given by (2.7) with ν, λ and η as in (4.4). Finally, the inequality (4.2) follows by substituting (4.5) into (4.5). □

Remark **4.1**. As for the second integral in (4.5), note that from the hypothesis we have $t^{\omega q m}\ |u|^{qm} \in C[0,T]$. So if $\phi \in L^q(0,T)$, then the integral is finite.

Lemma **4.2**. *Let $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}$, and $0 \leq \omega < 1$. Let $f(.,u(.)) \in L(0,T)$, $0 < T \leq \infty$, for any $u \in C_\omega[0,T]$ and satisfies*

$$|f(t,u(t))| \leq t^{\mu} e^{-\sigma t}\, \phi(t)\, |u(t)|^m, \quad t \in (0,T), \tag{4.6}$$

with $m \in \mathbb{N}$, $\mu > \omega m - 1 - \min\{\alpha - r, 0\}$, $\sigma > 0$, and ϕ is a continuous nonnegative function on $(0,T)$.

If $\mathcal{J}_r^{\beta,\alpha} f(t,u(t))$ *exists then given*

$$0 < \delta < \mu - \omega m + 1 + \min\{\alpha - r, 0\}, \tag{4.7}$$

there exists $M > 0$ *such that*

$$\left|\mathcal{J}_r^{\beta,\alpha} f(t,u(t))\right| \le A^{1/p}\, M\, t^{\alpha+\beta-\xi-1} \left(\int_0^t \phi^q(s)\, s^{qmw}\, |u(s)|^{qm}\, ds\right)^{1/q}, \tag{4.8}$$

for any

$$q > \max\left\{\frac{1}{\mu+\xi-r-\omega m+1}, \frac{1}{\alpha+\beta}, 1\right\}, \tag{4.9}$$

where $A > 0$ *is given by (2.7) with*

$$\nu = p(\alpha+\beta-1)+1, \quad \lambda = p(\mu+\xi-r-\omega m)+1, \quad \eta = p\sigma, \tag{4.10}$$

p *is the conjugate of* q*, and* $\xi = \min\{\alpha, r\} - \delta$*.*

P r o o f. From Lemma **3.2**, given $0 < \delta < \mu - \omega m + 1 + \min\{\alpha - r, 0\}$, there exists $M > 0$ such that

$$\left|\mathcal{J}_r^{\beta,\alpha} f(t,u(t))\right| \le \mathcal{J}_r^{\beta,\alpha}\, |f(t,u(t))| \le M t^{-\xi}\, \Gamma(\alpha+\beta)\, I^{\alpha+\beta} \left|t^{\xi-r} f(t,u(t))\right|, \tag{4.11}$$

where $\xi = \min\{\alpha, r\} - \delta$. Moreover, for any δ that satisfies (4.7) we have

$$\mu + \xi - r - \omega m + 1 > \delta - \min\{\alpha - r, 0\} + \xi - r = 0,$$

since $\xi - r = \min\{\alpha - r, 0\} - \delta$. Also from hypothesis (4.6),

$$t^{\xi-r} |f(t,u(t))| \le t^{\mu+\xi-r}\, e^{-\sigma t}\, \phi(t)\, |u(t)|^m.$$

Thus we can apply Lemma **4.1** with α and μ replaced by $\alpha+\beta$ and $\mu+\xi-r$, respectively, to obtain the bound

$$I^{\alpha+\beta} \left|t^{\xi-r} f(t,u(t))\right| \le \frac{A^{1/p}}{\Gamma(\alpha+\beta)}\, t^{\alpha+\beta-1} \left(\int_0^t \phi^q(s)\, s^{qmw}\, |u(s)|^{qm}\, ds\right)^{1/q}, \tag{4.12}$$

where A is given by (2.7) with ν, λ and η as in (4.10). Now (4.8) follows by combining (4.11) and (4.12). □

An alternative bound can be obtained using Dirichlet formula.

LEMMA **4.3**. *Let* $0 \le r < \alpha$*,* $\beta > 0$*, and* $0 \le \omega < 1$*. Let* $f(.,u(.)) \in L(0,T)$*,* $0 < T \le \infty$*, for any* $u \in C_\omega[0,T]$ *and satisfies*

$$|f(t,u(t))| \le t^{\mu} e^{-\sigma t}\, \phi(t)\, |u(t)|^m, \quad t \in (0,T), \tag{4.13}$$

with $m \in \mathbb{N}$*,* $\mu - m\omega > -1$*,* $\sigma > 0$*, and* ϕ *is a continuous nonnegative continuous function on* $(0,T)$*. Then*

$$|\mathcal{J}_r^{\beta,\alpha} f(t,u(t))| \le K\ t^{\alpha+\beta-r-1} \left(\int_0^t \phi^q(s)\ s^{\omega q m}\, |u(s)|^{qm}\ ds \right)^{1/q}, \qquad (4.14)$$

for any

$$q > \max\left\{ \frac{1}{\mu - \omega m + 1}, \frac{1}{\alpha}, 1, \frac{1}{\beta}, \frac{1}{\alpha - r} \right\}, \qquad (4.15)$$

where

$$K = \frac{A^{1/p}}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\Gamma(p(\beta-1)+1)\ \Gamma(p(\alpha-r-1)+1)}{\Gamma(p(\alpha-r+\beta-2)+2)} \right)^{1/p}, \qquad (4.16)$$

$A > 0$ is given by (2.7) with

$$\nu = p(\alpha-1)+1, \qquad \lambda = p(\mu - \omega m)+1, \qquad \eta = p\sigma, \qquad (4.17)$$

and p is the conjugate of q.

P r o o f. Let $v(t) = t^{\omega}|u(t)|$, then $v \in C[0,T]$. Since the hypothesis of Lemma **4.1** hold, using Hölder inequality we obtain the following bound.

$$\begin{aligned}
&|\mathcal{J}_r^{\beta,\alpha} f(t,u(t))| \\
&= \left| I^{\beta} t^{-r} I^{\alpha} f(t,u(t)) \right| \le I^{\beta} t^{-r}\, |I^{\alpha} f(t,u(t))| \\
&\le \frac{A^{1/p}}{\Gamma(\alpha)} I^{\beta} \left[t^{-r} t^{\alpha-1} \left(\int_0^t \phi^q(s)\ v^{qm}(s)\ ds \right)^{1/q} \right] \\
&= \frac{A^{1/p}}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\ s^{\alpha-r-1} \left(\int_0^s \phi^q(\tau)\ v^{qm}(\tau)\ d\tau \right)^{1/q} ds \\
&= \frac{A^{1/p}}{\Gamma(\alpha)\,\Gamma(\beta)} \left(\int_0^t (t-s)^{p(\beta-1)}\ s^{p(\alpha-r-1)} ds \right)^{1/p} \\
&\qquad \times \left(\int_0^t \int_0^s \phi^q(\tau)\ v^{qm}(\tau)\ d\tau\ ds \right)^{1/q}. \qquad (4.18)
\end{aligned}$$

From (4.15), the powers inside the first integral satisfy

$$p(\beta-1) > p\left(\frac{1}{q} - 1\right) = \frac{p}{q} - p = -1,$$

and

$$p(\alpha-r-1) > p\left(\frac{1}{q} - 1\right) = -1.$$

Thus this integral is finite and given by

$$\begin{aligned}\int_0^t (t-s)^{p(\beta-1)}\ s^{p(\alpha-r-1)} ds &= \Gamma(p(\beta-1)+1)\ I^{p(\beta-1)+1}\ t^{p(\alpha-r-1)} \\ &= \frac{\Gamma(p(\beta-1)+1)\ \Gamma(p(\alpha-r-1)+1)}{\Gamma(p(\alpha-r-1)+p(\beta-1)+2)}\ t^{p(\alpha-r-1)+p(\beta-1)+1} \\ &= \frac{\Gamma(p(\beta-1)+1)\ \Gamma(p(\alpha-r-1)+1)}{\Gamma(p(\alpha-r+\beta-2)+2)}\ t^{p(\alpha-r+\beta-2)+1}. \end{aligned} \tag{4.19}$$

Regarding the second integral, if it is infinite then (4.14) is true. If the integral is finite then using Dirichlet formula we obtain

$$\begin{aligned}\int_0^t \int_0^s \phi^q(\tau)\ v^{qm}(\tau)\ d\tau\ ds &= \int_0^t \int_\tau^t \phi^q(\tau)\ v^{qm}(\tau)\ ds\ d\tau \\ &= \int_0^t (t-\tau)\ \phi^q(\tau)\ v^{qm}(\tau)\ d\tau \\ &\leq t \int_0^t \phi^q(\tau)\ v^{qm}(\tau)\ d\tau. \end{aligned} \tag{4.20}$$

Substituting (4.19) and (4.20) into (4.18) yields

$$\begin{aligned}|\mathcal{J}_r^{\beta,\alpha} f(t,u(t))| &\leq \frac{A^{1/p}}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\Gamma(p(\beta-1)+1)\ \Gamma(p(\alpha-r-1)+1)}{\Gamma(p(\alpha-r+\beta-2)+2)} \right)^{1/p} \\ &\quad \times t^{\alpha-r+\beta-2+1/p+1/q} \left(\int_0^t \phi^q(\tau)\ v^{qm}(\tau)\ d\tau \right)^{1/q} \\ &= \frac{A^{1/p}}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\Gamma(p(\beta-1)+1)\ \Gamma(p(\alpha-r-1)+1)}{\Gamma(p(\alpha-r+\beta-2)+2)} \right)^{1/p} \\ &\quad \times t^{\alpha+\beta-r-1} \left(\int_0^t \phi^q(\tau)\ v^{qm}(\tau)\ d\tau \right)^{1/q}. \end{aligned}$$

This completes the proof. □

5. Bounds for the solution of the Cauchy problem

In this section we obtain a bound on the solution of the Cauchy-type problem

$$\begin{aligned} &\mathcal{D}_r^{\alpha,\beta} u(t) = f(t,u(t)), \qquad t>0, \quad 0 \leq r < \alpha < 1, \quad 0 < \beta < 1, \\ &\lim_{t\to 0^+} I^{1-\beta} u(t) = c_0, \\ &\lim_{t\to 0^+} I^{1-\alpha} \left[t^r D^\beta u \right](t) = c_1, \end{aligned} \tag{5.1}$$

where c_0 and c_1 are real numbers.

The proof of the following lemma is similar to the proof of Theorem 27 in [11].

Lemma **5.1**. *Let $0 \le r < \alpha < 1$, $0 < \beta < 1$, and $0 \le \omega < 1$. Let $f : (0,T] \times \mathbb{R} \to \mathbb{R}$ be such that $f(.,u(.)) \in C_\omega[0,T)$ for any $u \in C_\omega[0,T)$.*

If $u \in C_\omega[0,T)$ with $D^\beta u \in C(0,T)$ satisfies the Cauchy-type problem (5.1), then u satisfies the integral equation

$$u(t) = \frac{c_0}{\Gamma(\beta)}\, t^{\beta-1} + \frac{c_1\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-r)}\, t^{\alpha+\beta-r-1} + \mathcal{J}_r^{\beta,\alpha} f(t,u(t)), \quad (5.2)$$

for $0 < t < T$.

The integral form (5.2) leads to the following result.

Theorem **5.1**. *Let $0 \le r < \alpha < 1$, $0 \le \beta \le 1$, and $0 \le \omega < 1$. Let $f(.,u(.)) \in C_\omega[0,T)$, $0 < T \le \infty$, for any $u \in C_\omega[0,T)$ and satisfies*

$$|f(t,u(t))| \le t^\mu e^{-\sigma t} \phi(t)\, |u|^m, \quad \mu \ge 0, \quad m \in \mathbb{N}, \quad \sigma > 0, \qquad (5.3)$$

with $m \in \mathbb{N}$, $\mu > m\omega - 1$, and $\sigma > 0$. Here ϕ is a nonnegative function in $C_\eta[0,T]$ such that ϕ^q is integrable at 0, where

$$q > \max\left\{\frac{1}{\mu-\omega m+1}, \frac{1}{\beta}, \frac{1}{\alpha-r}\right\}, \qquad (5.4)$$

and

$$\eta = m(\beta - 1 + \omega). \qquad (5.5)$$

If $u \in C_\omega[0,T)$ with $D^\beta u \in C(0,T)$ is a solution of (5.1) then on $(0,T)$, u satisfies the inequality

$$|u(t)| \le \qquad (5.6)$$
$$\begin{cases} K\ \ t^{\alpha+\beta-r-1}\ \left[(1-m)A(t) + B^{1-m}(t)\right]^{\frac{1}{q(1-m)}}, & m > 0, m \ne 1, \\ \frac{c_0}{\Gamma(\beta)}\, t^{\beta-1} + \frac{c_1\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-r)}\, t^{\alpha+\beta-r-1} + K\ \ t^{\alpha+\beta-r-1}\, B^{1/q}(t)\, e^{A(t)/q}, & m = 1, \end{cases}$$

where K is given by (4.16),

$$A(t) = A_2 \int_0^t s^{qm(\alpha-r)}\ s^{q\eta}\ \phi^q(s)\, ds, \qquad (5.7)$$

$$B(t) = A_0 \int_0^t s^{q\eta}\ \phi^q(s)\, ds + \frac{A_1}{A_2}\, A(t), \qquad (5.8)$$

and

$$A_0 = 3^{q-1}\left(\frac{c_0}{\Gamma(\beta)}\right)^{qm}, \qquad A_1 = 3^{q-1}\left(\frac{c_1\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-r)}\right)^{qm},$$
$$A_2 = 3^{q-1}\,K^{qm}. \tag{5.9}$$

P r o o f. Since the hypothesis of Lemma **5.1** are satisfied, the solution u of the problem (5.1) satisfies the nonlinear integral equation (5.2). From the bound (4.14) in Lemma **4.3** we obtain the inequality

$$\begin{aligned}|u(t)| \;\leq\; & \frac{c_0}{\Gamma(\beta)}\,t^{\beta-1} + \frac{c_1\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-r)}\,t^{\alpha+\beta-r-1} \\ & +K\,t^{\alpha+\beta-r-1}\left(\int_0^t \phi^q(s)\,|s^w\,u(s)|^{qm}\,ds\right)^{1/q}.\end{aligned} \tag{5.10}$$

Multiplying both sides of (5.10) by t^w we can write

$$\begin{aligned}t^w\,|u(t)| \;\leq\; & \frac{c_0}{\Gamma(\beta)}\,t^{\beta-1+\omega} + \frac{c_1\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-r)}\,t^{\alpha+\beta-r-1+\omega} \\ & +K\;t^{\alpha+\beta-r-1+\omega}\left(\int_0^t \phi^q(s)\,|s^w\,u(s)|^{qm}\,ds\right)^{1/q}.\end{aligned}$$

Let $z(t) = t^w\,|u(t)|$, then

$$\begin{aligned}z(t) \leq & \frac{c_0}{\Gamma(\beta)}\,t^{\beta-1+\omega} + \frac{c_1\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-r)}\,t^{\alpha+\beta-r-1+\omega} \\ & +K\;t^{\alpha+\beta-r-1+\omega}\,g^{1/q}(t),\end{aligned} \tag{5.11}$$

where

$$g(t) = \int_0^t \phi^q(s)\,z^{qm}(s)\,ds.$$

Note that it follows from the assumptions on ϕ and u that $g \in AC[0,T]$. From Lemma **2.1** we obtain

$$\begin{aligned}g'(t) &= \phi^q(t)\,z^{qm}(t) \\ &= \phi^q(t)\,\Big[A_0\,t^{q\eta} + A_1\,t^{q(\eta+m(\alpha-r))} + A_2\,t^{q(\eta+m(\alpha-r))}\,g^m(t)\Big].\end{aligned}$$

where A_0, A_1, and A_2 are given by (5.9). Thus we can write

$$g'(t) \leq a(t)\,g^m(t) + b(t),$$

where

$$a(t) = A_2\,t^{q(\eta+m(\alpha-r))}\,\phi^q(t) = A_2\,t^{qm(\alpha-r))}\,[t^\eta\,\phi(t)]^q,$$

and

$$b(t) = \Big[A_0\,t^{q\eta} + A_1\,t^{q(\eta+m(\alpha-r))}\Big]\,\phi^q(t) = A_0\,[t^\eta\,\phi(t)]^q + \frac{A_1}{A_2}\,a(t).$$

Note that $a, b \in C[0,T]$. Note that g is nonnegative and continuous in $(0,T)$. Furthermore, g is also nondecreasing since g' is nonnegative. Thus by Lemma **2.3**, we have two cases.

Case 1: $m > 0$, $m \neq 1$. Then for $t \in [0, T_m)$, where T_m is as given by (2.4),

$$g(t) \leq [(1-m)A(t) + B^{1-m}(t)]^{\frac{1}{1-m}},$$

where $A(t)$ and $B(t)$ are as given in (5.7) and (5.8), respectively.

Thus,

$$\begin{aligned} |u(t)| &= t^{-\omega} z(t) \\ &\leq \frac{c_0}{\Gamma(\beta)} t^{\beta-1} + \frac{c_1 \Gamma(\alpha - r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)} t^{\alpha+\beta-r-1} \\ &\quad + K \ t^{\alpha+\beta-r-1} \left[\left[(1-m)A(t) + B^{1-m}(t) \right]^{\frac{1}{1-m}} \right]^{1/q} \\ &= \frac{c_0}{\Gamma(\beta)} t^{\beta-1} + \frac{c_1 \Gamma(\alpha - r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)} t^{\alpha+\beta-r-1+} \\ &\quad + K \ t^{\alpha+\beta-r-1} \left[(1-m)A(t) + B^{1-m}(t) \right]^{\frac{1}{q(1-m)}}. \end{aligned}$$

Case 2: $m = 1$. In this case, for $t \in [0, T)$

$$g(t) \leq B(t)\, e^{A(t)}.$$

Therefore,

$$\begin{aligned} |u| &= t^{-\omega} z(t) \\ &\leq \frac{c_0}{\Gamma(\beta)} t^{\beta-1} + \frac{c_1 \Gamma(\alpha - r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)} t^{\alpha+\beta-r-1} + K \ t^{\alpha+\beta-r-1} g^{1/q}(t) \\ &\leq \frac{c_0}{\Gamma(\beta)} t^{\beta-1} + \frac{c_1 \Gamma(\alpha - r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)} t^{\alpha+\beta-r-1} \\ &\qquad + K \ t^{\alpha+\beta-r-1} B^{1/q}(t)\, e^{A(t)/q}. \end{aligned}$$

This completes the proof. □

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