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On the Space of BV Functions and a Related Stochastic Calculus in Infinite Dimensions

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Functions of bounded variation (BV functions) are defined on an abstract Wiener space (E, H, μ) in a way similar to that in finite dimensions. Some characterizations are given, which justify describing a BV function as a function in $L(\log L)^{1/2}$ with the first order derivative being an H-valued measure. It is also shown that the space of BV functions is obtained by a natural extension of the Sobolev space $\mathbb{D}^{1,1}$. Moreover, some stochastic formulae related to BV functions are investigated. © 2001 Academic Press

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1. INTRODUCTION

A real valued function ρ defined on an open set $U \subset \mathbb{R}^d$ is said to be of bounded variation ($\rho \in BV(U)$ in notation) if the distributional derivatives $\partial_i \rho$, $1 \leqslant i \leqslant d$, are finite signed measures on U, or equivalently, if

$$V(\rho) = \sup \left\{ \int_U \rho \operatorname{div} \varphi \, dx \,\middle|\, \varphi \in C_0^1(U; \mathbb{R}^d),\ \|\varphi\|_\infty \leqslant 1 \right\} < \infty. \quad (1.1)$$

Thus the space $BV(U)$ is a natural extension of the classical Sobolev space $W^{1,1}(U)$ with $V(\rho) = \|\nabla \rho\|_1$ and it has played important roles in solving diverse fine variational problems in finite dimensions ([9, 15, 8, 3]).

In a recent paper [13], the notion of a BV function on an abstract Wiener space (E, H, μ) was introduced as a function ρ on E for which a quantity analogous to (1.1) is finite. However, ρ was required to be in $\bigcup_{p>1} L^p(E; \mu)$, excluding the Malliavin Sobolev space $\mathbb{D}^{1,1}$ from the space $BV(E)$. Furthermore, in order to formulate an analogue to the Gauss formula holding for ρ and an associated measure $\|D\rho\|$ on E, the closability of a pre-Dirichlet form $\mathscr{E}^\rho$ related to ρ was crucially assumed. Then, under the last assumption, a semimartingale decomposition of the associated distorted Ornstein-Uhlenbeck process living on the support of $\rho \, d\mu$ was also presented in [13].

A purpose of the present paper is to extend the analytical part of [13] considerably by requiring a BV function to sit in a broader Orlicz space $L(\log L)^{1/2}$ and by establishing an associated Green formula without any closability assumption (Section 3). The space $BV(E)$ will now contain the Sobolev space $\mathbb{D}^{1,1}$ as a proper subspace and the structures of those spaces will be characterized in terms of the measures $\|D\rho\|$, $\rho \in BV(E)$. Moreover, we shall give in Section 3 a refinement of a theorem in [13] concerning the support of the measure $\|D\rho\|$ by showing that it vanishes outside a quasi support of $\mathbf{1}_{\{\rho \neq a\}} \cdot \mu$ for every $a \in \mathbb{R}$.

In Section 4, we shall assume that $\rho \in BV(E)$ is non-negative and $\mathscr{E}^\rho$ is closable. The martingale part of the associated distorted Ornstein–Uhlenbeck process $\mathbf{M}^\rho = (X_t, \mathscr{M}_t, P_z)$ will then be shown to be a Brownian motion on E under P_z for quasi-every starting point z. This fact was proven in [13] only under P_γ for smooth probability measures γ. By making use of an analogue to the classical Green formula obtained in Section 3, we shall then show in Section 4 a generalized Itô's formula for $\mathbf{M}^\rho$, which has been formulated in [12] in finite dimensions.

When $\rho \in BV(E)$ is an indicator function of a set A, a Green formula (Theorem 3.12) and a result on the support of $\|D\rho\|$ (Theorem 3.15) indicate that $\|D\rho\|$ and σ_ρ in the formula are regarded as a surface measure and a normal vector field of a "boundary" of A, respectively. Such notions in infinite dimensions have been investigated in various contexts, such as in [16, 28, 17, 18, 19, 2, 4, 10]. Our approach is based on a theory of Dirichlet forms and aims at applications to stochastic analysis on sets whose boundaries do not have good smoothness.

2. PRELIMINARIES

Let (E, H, μ) be an abstract Wiener space. Namely, E is a separable Banach space, H is a separable Hilbert space densely and continuously embedded in E, and μ is a Gaussian measure on E which satisfies that

$$\int_E \exp(\sqrt{-1}\ell(z))\,\mu(dz) = \exp(-\|\ell\|_H^2/2), \qquad \ell \in E^*.$$

Here, we regard the topological dual E^* of E as a subspace of H by the natural inclusion $E^* \subset H^*$ and the identification $H^* \simeq H$. The inner product and the norm of H is denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_H$, respectively. Let for each $k \in \mathbb{N} \cup \{\infty\}$,

$$\mathscr{F}C_b^k = \left\{ u\colon E \to \mathbb{R} \;\middle|\; \begin{array}{l} u(z) = f(\ell_1(z), \ldots, \ell_m(z)),\ \ell_1, \ldots, \ell_m \in E^*,\ f \in C_b^k(\mathbb{R}^m) \\ \text{for some } m \in \mathbb{N} \end{array} \right\}.$$

For $u \in \mathscr{F}C_b^1$, the H-derivative of u, denoted by ∇u, is a map from E to H defined by the relation

$$\langle \nabla u(z), \ell \rangle = \partial_\ell u(z), \qquad \ell \in E^* \subset H,$$

where $\partial_\ell u(z) = \lim_{\varepsilon \to 0}(u(z + \varepsilon\ell) - u(z))/\varepsilon$, $\ell \in E^* \subset H \subset E$.

For a separable Hilbert space K, a Borel measure ν on a metric space X and $p \in [1, \infty]$, $L^p(X \to K; \nu)$ denotes the usual L^p space consisting of K-valued functions on X. We shall often omit each symbol X, K and ν if $X = E$, $K = \mathbb{R}$, and $\nu = \mu$, respectively. Also, $L_+^p(\nu)$ denotes the space of all nonnegative functions belonging to $L^p(\nu)$. The norm $\|\cdot\|_p$ always means $L^p(E \to K; \mu)$-norm. For $\rho \in L^1$, we define a symmetric bilinear form $\mathscr{E}^\rho\colon \mathscr{F}C_b^1 \times \mathscr{F}C_b^1 \to \mathbb{R}$ by

$$\mathscr{E}^\rho(u, v) = \frac{1}{2}\int_E \langle \nabla u(z), \nabla v(z) \rangle \rho(z)\,\mu(dz), \qquad u, v \in \mathscr{F}C_b^1.$$

$\mathscr{E}^\rho$ can be regarded as a bilinear form on $L^2(F^\rho; |\rho|\cdot\mu)$ where F^ρ is a support of $|\rho|\cdot\mu$, since ∇ has the following consistency property by Proposition 7.1.4 in [6, Chapter I]: if $u\in\mathscr{F}C_b^1$ and $v\in\mathscr{F}C_b^1$ coincide on a measurable set A, then $\nabla u=\nabla v$ on A μ-a.e. In the following, a measurable function on E is also regarded as a function on F^ρ by the natural restriction. The set of all functions $\rho\in L^1_+$ such that $(\mathscr{E}^\rho, \mathscr{F}C_b^1)$ is closable on $L^2(F^\rho; \rho\cdot\mu)$ will be denoted by $QR(E)$. Its closure $(\mathscr{E}^\rho, \mathscr{F}^\rho)$ is then automatically a quasi-regular local Dirichlet form by the results of [22, 25] (see also [13, Theorem 2.1]). Functions belonging to $\mathbb{H}$ defined in [13], especially positive L^1-functions bounded away from 0, are elements of $QR(E)$. We denote by $\mathscr{F}_b^\rho$ the set of all bounded functions in $\mathscr{F}^\rho$. Following [27, 14], we denote by $\mathscr{F}_e^\rho$ the extended Dirichlet space of $(\mathscr{E}^\rho, \mathscr{F}^\rho)$: $u\in\mathscr{F}_e^\rho$ if and only if $|u|<\infty$ $\rho\cdot\mu$-a.e. and there exists a sequence $\{u_n\}$ in $\mathscr{F}^\rho$ such that $\mathscr{E}^\rho(u_m-u_n, u_m-u_n)\to 0$ as $n\geqslant m\to\infty$ and $u_n\to u$ $\rho\cdot\mu$-a.e. as $n\to\infty$. For example, a function $\ell(\cdot): z\in E\mapsto \ell(z)\in\mathbb{R}$ belongs to $\mathscr{F}_e^\rho$ for every $\ell\in E^*$. Indeed, when Φ_n is a smooth function on $\mathbb{R}$ such that $0\leqslant\Phi_n'\leqslant 1$ on $\mathbb{R}$, $\Phi_n(x)=x$ on $[-n, n]$ and $|\Phi_n(x)|=n+1$ on $\mathbb{R}\setminus[-n-2, n+2]$, $\{\Phi_n\circ\ell(\cdot)\}_{n\in\mathbb{N}}$ is the desired sequence. $\mathscr{E}^\rho$ extends to a bilinear form on $\mathscr{F}_e^\rho$ in a natural way.

For each $\rho\in QR(E)$, there exists an associated diffusion process $\mathbf{M}^\rho=(X_t, \mathscr{M}_t, P_z)$ on F^ρ with $(\mathscr{E}^\rho, \mathscr{F}^\rho)$. We denote by $\mathbf{A}^\rho_+$ the set of all positive continuous additive functionals (PCAF in abbreviation) of $\mathbf{M}^\rho$, and define $\mathbf{A}^\rho=\mathbf{A}^\rho_+-\mathbf{A}^\rho_+$. For $A\in\mathbf{A}^\rho$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}^\rho_0=\{A\in\mathbf{A}^\rho \mid E_{\rho\cdot\mu}(\{A\}_t)<\infty \text{ for all } t>0\}$. Each element in $\mathbf{A}^\rho_+$ has a corresponding positive $\mathscr{E}^\rho$-smooth measure on F^ρ by the Revuz correspondence. The totality of such measures will be denoted by S^ρ_+. Accordingly, $\mathbf{A}^\rho$ has a correspondence with $S^\rho=S^\rho_+-S^\rho_+$, the set of $\mathscr{E}^\rho$-smooth signed measures.

For each $u\in\mathscr{F}_e^\rho$, we have the following decomposition:

$$\tilde{u}(X_t)-\tilde{u}(X_0)=M_t^{[u]}+N_t^{[u]},$$

where $\tilde{u}$ is an $\mathscr{E}^\rho$-quasicontinuous modification of u, $M^{[u]}$ is a martingale AF of finite energy and $N^{[u]}$ is a CAF of zero energy. Also, $M^{[u]}$ and $N^{[u]}$ are uniquely determined. When $u=\ell(\cdot)$ for some $\ell\in E^*$, we also write $M^{[\ell]}$ and $N^{[\ell]}$, instead of $M^{[u]}$ and $N^{[u]}$, respectively.

Let $\rho\in L^1$ and $\ell\in E^*$. Following [13], we say that ρ is of bounded variation in direction ℓ ($\rho\in BV_\ell(E)$ in notation) if there is a positive constant C such that

$$\left|\int_E \partial_\ell v(z)\,\rho(z)\,\mu(dz)\right|\leqslant C\,\|v\|_\infty \qquad \text{for every } v\in\mathscr{F}C_b^1.$$

THEOREM 2.1. *Let* $\rho \in L^1_+$ *and* $\ell \in E^*$.

(i) *The following conditions are mutually equivalent.*

(a) $\rho \in BV_\ell(E)$.

(b) *There exists a finite signed measure* ν_ℓ *on* E *such that for every* $v \in \mathcal{F}C_b^1$,

$$\frac{1}{2}\int_E \partial_\ell v(z)\,\rho(z)\,\mu(dz) = -\int_E v(z)\,\nu_\ell(dz). \tag{2.1}$$

In this case, ν_ℓ *necessarily belongs to* $S^{\rho+1}$.

(ii) *Suppose further that* $\rho \in QR(E)$. *Then the following condition is also equivalent to the above.*

(c) $N^{[\ell]} \in \mathbf{A}_0^\rho$.

In this case, ν_ℓ *in* (b) *necessarily satisfies that* $\nu_\ell|_{E\setminus F^\rho} = 0$, $\nu_\ell|_{F^\rho} \in S^\rho$ *and* $N^{[\ell]}$ *is in Revuz correspondence with it. Furthermore, it holds for any* $\mathcal{E}^\rho$*-quasicontinuous function* v *in* $\mathcal{F}_b^\rho$ *that*

$$\mathcal{E}^\rho(v, \ell(\cdot)) = -\int_{F^\rho} v(z)\,\nu_\ell(dz).$$

Proof. We shall use Theorem 6.2 in [11] for the proof. We remark that Proposition 3.1 in [11] holds for u in the extended Dirichlet space. Then Theorems 3.2, 4.2 and 6.2 in [11] also hold for such u.

Since

$$\frac{1}{2}\int_E \partial_\ell v(z)(\rho(z)+1)\,\mu(dz) = \frac{1}{2}\int_E \partial_\ell v(z)\,\rho(z)\,\mu(dz) + \frac{1}{2}\int_E v(z)\,\ell(z)\,\mu(dz),$$

$\rho \in BV_\ell(E)$ if and only if $\rho + 1 \in BV_\ell(E)$. Obviously, $\rho + 1 \in QR(E)$ for any $\rho \in L^1_+$. The assertion (i) follows from Theorem 6.2 in [11] because $\ell(\cdot) \in \mathcal{F}_e^{\rho+1}$ and $\mathcal{E}^{\rho+1}(v, \ell(\cdot)) = \frac{1}{2}\int_E \partial_\ell v(z)(\rho(z)+1)\,\mu(dz)$. For the proof of the assertion (ii), consider the following condition:

(b′) There exists some $\nu'_\ell \in S^\rho$ such that for every $v \in \mathcal{F}C_b^1$,

$$\mathcal{E}^\rho(v, \ell(\cdot)) = -\int_{F^\rho} v(z)\,\nu'_\ell(dz).$$

From Theorem 6.2 in [11], (a), (b′), and (c) are mutually equivalent and $N^{[\ell]}$ is in Revuz correspondence with ν'_ℓ. Suppose (b′). By considering a measure ν_ℓ on E defined by $\nu_\ell|_{F^\rho} = \nu'_\ell$ and $\nu_\ell|_{E\setminus F^\rho} = 0$, the condition (b) holds. Since ν_ℓ in (b) is uniquely determined if it exists, we get the rest of the assertions. ∎

3. STRUCTURES OF BV SPACE

First, we introduce three function spaces on E. The H-derivative ∇ defined on $\mathscr{F}C_b^1$ is closable as an operator from L^1 to $L^1(E \to H)$. The domain of its closure is denoted by $\mathbb{D}^{1,1}$, equipped with the norm $\|f\|_{1,1} = \|f\|_1 + \|\nabla f\|_1$. The closure of ∇ is denoted by the same symbol.

Let

$$A_{1/2}(x) = \int_0^x (\log(1+s))^{1/2}\, ds, \qquad x \geqslant 0,$$

and let Ψ be its complementary function, namely,

$$\Psi(y) := \int_0^y (A'_{1/2})^{-1}(t)\, dt = \int_0^y (\exp(t^2) - 1)\, dt.$$

Then it holds for $x \geqslant 0$ and $y \geqslant 0$ that

$$xy \leqslant A_{1/2}(x) + \Psi(y). \tag{3.1}$$

Define

$$L(\log L)^{1/2} = \{f \mid A_{1/2}(|f|) \in L^1\},$$
$$L^\Psi = \{g \mid \Psi(c|g|) \in L^1 \text{ for some } c > 0\}.$$

From the general theory of Orlicz spaces (see e.g. [24, Chapter 3]), we have the following properties.

(i) $L(\log L)^{1/2}$ and L^Ψ are Banach spaces under the norms $\|f\|_{L(\log L)^{1/2}} = \inf\{\alpha > 0 \mid \int_E A_{1/2}(|f|/\alpha)\, d\mu \leqslant 1\}$ and $\|g\|_{L^\Psi} = \inf\{\alpha > 0 \mid \int_E \Psi(|g|/\alpha)\, d\mu \leqslant 1\}$, respectively. (Note: we adopt a terminology different from [24]; e.g. $N_\Psi(\cdot)$ is used in [24] instead of $\|\cdot\|_{L^\Psi}$).

(ii) For $f \in L(\log L)^{1/2}$ and $g \in L^\Psi$, we have

$$\|fg\|_1 \leqslant 2\, \|f\|_{L(\log L)^{1/2}}\, \|g\|_{L^\Psi}, \tag{3.2}$$

$$\|fg\|_1 \leqslant (\|A_{1/2}(|f|)\|_1 + 1)\, \|g\|_{L^\Psi}. \tag{3.3}$$

We give only a proof of (ii) here. Taking $x=|f(z)|/\|f\|_{L(\log L)^{1/2}}$ and $y=|g(z)|/\|g\|_{L^\Psi}$ in (3.1) and integrating both sides, we get (3.2). Taking $x=|f(z)|$ and $y=|g(z)|/\|g\|_{L^\Psi}$ in (3.1) and integrating both sides, we obtain (3.3). We state a direct implication of (3.2) as a next lemma.

LEMMA 3.1. *Suppose $\varphi\in L^\Psi$. Then, $\varphi f\in L^1$ for any $f\in L(\log L)^{1/2}$. If a sequence $\{f_n\}$ converges to f in $L(\log L)^{1/2}$, then $\lim_{n\to\infty}\int_E \varphi f_n\,d\mu=\int_E\varphi f\,d\mu$.*

Letting $g\equiv 1$ in (3.2), we see that $L(\log L)^{1/2}$ is continuously embedded in L^1. The following observation is also useful: as a set, $L(\log L)^{1/2}=\{f \mid |f|(\log^+|f|)^{1/2}\in L^1\}$ and $L^\Psi=\{g\mid \exp(c\,|g|^2)\in L^1$ for some $c>0\}$, where $\log^+x=\max(\log x, 0)$. This is because the next estimates hold for some positive constants C_1 and C_2:

$$C_1x(\log^+ x)^{1/2}\leqslant A_{1/2}(x)\leqslant x+x(\log^+ x)^{1/2},\qquad x\geqslant 0,$$

$$\exp(y^2/2)-C_2\leqslant \Psi(y)\leqslant \exp(2y^2),\qquad y\geqslant 0.$$

Also, we have the following embedding theorem.

PROPOSITION 3.2. *The space $\mathbb{D}^{1,1}$ is continuously embedded in $L(\log L)^{1/2}$.*

Proof. The proof is based on the argument in [21, p. 272]. Let $\Phi(r)=(2\pi)^{-1/2}\int_{-\infty}^r\exp(-t^2/2)\,dt$, $r\in\mathbb{R}$ and $\mathcal{U}(x)=\Phi'\circ\Phi^{-1}(x)$, $0<x<1$. Then $\lim_{x\downarrow 0}\mathcal{U}(x)/(x\sqrt{2\log 1/x})=1$ (cf. [21, p. 271]). We can take a constant $\delta>0$ such that $\mathcal{U}(x)\geqslant x\sqrt{\log(1+1/x)}$ for all $x\in(0,\delta]$. We may also take $\delta\leqslant 1/(e-1)$.

Suppose $f\in\mathscr{F}C_b^1$ and $\|f\|_{1,1}\leqslant 1/\sqrt{\log(1+1/\delta)}(\leqslant 1)$. The isoperimetric inequality for Gaussian measure implies that

$$\|\nabla f\|_1\geqslant\int_0^\infty \mathcal{U}(\mu(\{|f|\geqslant s\}))\,ds.$$

If $s\geqslant 1/\delta$, then $\mu(\{|f|\geqslant s\})\leqslant\|f\|_1/s\leqslant 1/s\leqslant\delta$ and

$$\begin{aligned}\mathcal{U}(\mu(\{|f|\geqslant s\}))&\geqslant\mu(\{|f|\geqslant s\})\sqrt{\log(1+1/\mu(\{|f|\geqslant s\}))}\\&\geqslant\mu(\{|f|\geqslant s\})\sqrt{\log(1+s)}.\end{aligned}$$

Therefore,

$$1 \geqslant \sqrt{\log(1+1/\delta)}\ \|f\|_{1,1} \geqslant \sqrt{\log(1+1/\delta)}\ \|f\|_1 + \|\nabla f\|_1$$

$$\geqslant \sqrt{\log(1+1/\delta)} \int_0^\infty \mu(\{|f| \geqslant s\})\, ds + \int_{1/\delta}^\infty \mu(\{|f| \geqslant s\})\ \sqrt{\log(1+s)}\, ds$$

$$\geqslant \int_0^\infty \mu(\{|f| \geqslant s\})\ (\sqrt{\log(1+1/\delta)} + \sqrt{\log(1+s)} \cdot \mathbf{1}_{\{s \geqslant 1/\delta\}})\, ds$$

$$\geqslant \int_0^\infty \mu(\{|f| \geqslant s\})\ \sqrt{\log(1+s)}\, ds$$

$$= \int_E d\mu \int_0^\infty ds\, \mathbf{1}_{\{|f| \geqslant s\}}\ \sqrt{\log(1+s)}$$

$$= \int_E d\mu \int_0^{|f|} \sqrt{\log(1+s)}\, ds = \|A_{1/2}(|f|)\|_1.$$

Therefore, $\|f\|_{L(\log L)^{1/2}} \leqslant 1$. This concludes the claim. ∎

We denote by $(\mathscr{F}C_b^1)_{E^*}$ the set of all E^*-valued functions on E expressed as $\sum_{j=1}^m g_j(z)\,\ell_j$ with $g_j \in \mathscr{F}C_b^1$ and $\ell_j \in E^*$, $j=1, ..., m$ for some $m \in \mathbb{N}$. We also denote by ∇^* the (formal) dual operator with domain $(\mathscr{F}C_b^1)_{E^*}$ of ∇. When $G(z) = g(z)\,\ell$, $g \in \mathscr{F}C_b^1$, $\ell \in E^*$, we have $\nabla^* G(z) = -\partial_\ell g(z) + g(z)\,\ell(z)$.

For $\rho \in L(\log L)^{1/2}$, define

$$V(\rho) = \sup\left\{ \int_E (\nabla^* G)\,\rho\, d\mu \;\middle|\; G \in (\mathscr{F}C_b^1)_{E^*},\ \|G(z)\|_H \leqslant 1 \text{ for every } z \in E \right\} \qquad (\leqslant \infty).$$

Since $\nabla^* G \in L^\Psi$ for each $G \in (\mathscr{F}C_b^1)_{E^*}$, the integral above is well-defined from Lemma 3.1.

Definition 3.3. Let $BV(E) = \{\rho \in L(\log L)^{1/2} \mid V(\rho) < \infty\}$. We say that ρ in $BV(E)$ is of bounded variation.

By Proposition 3.2 and Lemma 3.1, we see that the next duality relation holds for any $\rho \in \mathbb{D}^{1,1}$,

$$\int_E \nabla^* G(z)\,\rho(z)\,\mu(dz) = \int_E \langle G(z), \nabla\rho(z)\rangle \mu(dz), \qquad G \in (\mathscr{F}C_b^1)_{E^*},$$

and hence, we have as in [13],

LEMMA 3.4. *For* $\rho \in \mathbb{D}^{1,1}$, $V(\rho) = \|\nabla \rho\|_1$. *In particular*, $\mathbb{D}^{1,1} \subset BV(E)$.

The Ornstein–Uhlenbeck semigroup $\{T_t\}$ is defined as usual: $T_t f(x) = \int_E f(e^{-t}x + \sqrt{1-e^{-2t}}y)\, \mu(dy)$, where f is a function on E taking values in a separable Hilbert space.

PROPOSITION 3.5. *For every* $t > 0$, *the operator* $\nabla T_t \colon \mathscr{F}C_b^1 \to L^1(E \to H)$ *extends uniquely to a bounded operator from* $L(\log L)^{1/2}$ *to* $L^1(E \to H)$.

Proof. This result is implicitly proved in [20], but we give a proof for readers' convenience. Let $\varphi \in \mathscr{F}C_b^1$ with $\|\varphi\|_{L(\log L)^{1/2}} \leqslant 1$, $\ell \in E^*$ with $\|\ell\|_H = 1$, and $t > 0$. Note that $\|\ell(\cdot)\|_{L^\Psi}$ is independent of the choice of ℓ. From a direct computation (see e.g. [29]),

$$\partial_\ell T_t \varphi(x) = \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \int_E \varphi(e^{-t}x + (1-e^{-2t})^{1/2} y)\, \ell(y)\, \mu(dy).$$

Set $\theta = \arccos(e^{-t})$ and $R_\theta(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$. Then,

$$\begin{aligned} |\partial_\ell T_t \varphi(x)| &= \cot\theta \left| \int_E (\varphi \otimes 1)(R_\theta(x, y))\, \ell(y)\, \mu(dy) \right| \\ &\leqslant c(\|A_{1/2}(|(\varphi \otimes 1)(R_\theta(x, \cdot))|)\|_1 + 1). \end{aligned}$$

Here, (3.3) was used in the second line and c is a constant depending only on t. Then,

$$\begin{aligned} \|\nabla T_t \varphi(x)\|_H &= \sup_{\ell \in E^*,\, \|\ell\|_H = 1} |\partial_\ell T_t \varphi(x)| \\ &\leqslant c(\|A_{1/2}(|(\varphi \otimes 1)(R_\theta(x, \cdot))|)\|_1 + 1). \end{aligned}$$

From the rotational invariance of $\mu \otimes \mu$,

$$\begin{aligned} \|\nabla T_t \varphi\|_1 &\leqslant c\left(\iint_{E \times E} A_{1/2}(|(\varphi \otimes 1)(R_\theta(x, y))|)\, \mu(dx)\, \mu(dy) + 1 \right) \\ &= c\left(\iint_{E \times E} A_{1/2}(|(\varphi \otimes 1)(x, y)|)\, \mu(dx)\, \mu(dy) + 1 \right) \\ &= c\left(\int_E A_{1/2}(|\varphi(x)|)\, \mu(dx) + 1 \right) \leqslant 2c. \end{aligned}$$

Since $\mathscr{F}C_b^1$ is dense in $L(\log L)^{1/2}$, we get the conclusion. ∎

PROPOSITION 3.6. *Take any* $f \in L(\log L)^{1/2}$.

(i) $T_t f \in \mathbb{D}^{1,1}$, $t > 0$.

(ii) $T_t f$ *converges to* f *in* $L(\log L)^{1/2}$ *as* $t \downarrow 0$.

(iii) $V(T_t f) \leqslant e^{-t} V(f) \leqslant \infty$, $t > 0$.

(iv) $\lim_{t \downarrow 0} V(T_t f) = V(f)$.

Proof. (i) Take a sequence of functions $\{f_n\}$ in $\mathscr{F}C_b^1$ such that f_n converges to f in $L(\log L)^{1/2}$. From Proposition 3.5, $\{\nabla T_t f_n\}$ converges in $L^1(E \to H)$. On the other hand, $\{T_t f_n\}$ converges to $T_t f$ in L^1. By the closedness of ∇ on $\mathbb{D}^{1,1}$, we conclude that $T_t f \in \mathbb{D}^{1,1}$.

(ii) This is clear when f is bounded continuous. For a general f, we have by the Jensen inequality

$$A_{1/2}(|T_t f(x)|/\alpha) \leqslant \int_E A_{1/2}(|f(e^{-t}x + \sqrt{1-e^{-2t}}y)|/\alpha)\, \mu(dy), \qquad \alpha > 0.$$

Integrating the both hand sides by $\mu(dx)$, we get

$$\|A_{1/2}(|T_t f|/\alpha)\|_1 \leqslant \|A_{1/2}(|f|/\alpha)\|_1, \qquad \alpha > 0,$$

which means that $\|T_t f\|_{L(\log L)^{1/2}} \leqslant \|f\|_{L(\log L)^{1/2}}$. The claim follows from a usual approximation argument.

(iii) By (i), we get the following formula in the same way as in [13]:

$$\int_E \nabla^* G(z)\, T_t f(z)\, \mu(dz)$$

$$= e^{-t} \int_E \nabla^*(T_t G)(z) f(z)\, \mu(dz), \qquad G \in (\mathscr{F}C_b^1)_{E^*}, \quad f \in L(\log L)^{1/2},$$

which immediately implies (iii).

(iv) As in Lemma 4.1 in [13], we can prove that $V(f) \leqslant \underline{\lim}_{t \downarrow 0} V(T_t f)$. By combining (iii), the claim follows. ∎

Now we can give a characterization of the space $BV(E)$ as follows.

THEOREM 3.7. *It holds that*

$$BV(E) = \left\{ \rho \in L^1 \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{\rho_n\} \subset \mathbb{D}^{1,1} \text{ such that} \\ \rho_n \to \rho \text{ in } L^1 \text{ and } \sup_n \|\nabla \rho_n\|_1 < \infty. \end{array} \right\}. \tag{3.4}$$

Moreover, if ρ_n *and* ρ *are as in the right hand side of* (3.4), *then* $V(\rho) \leqslant \underline{\lim}_{n \to \infty} \|\nabla \rho_n\|_1$.

Proof. Let the right-hand side of (3.4) be denoted by BV_1. First, we prove $BV(E) \subset BV_1$. This is proved in the same way as in Proposition 4.1 in [13]. Let $\rho_n = T_{1/n}\rho$, $n \in \mathbb{N}$. Then $\rho_n \to \rho$ in L^1 as $n \to \infty$, and from Proposition 3.6 and Lemma 3.4, $\rho_n \in \mathbb{D}^{1,1}$, $\|\nabla \rho_n\|_1 = V(\rho_n) \leqslant V(\rho)$ for every n. Therefore, $\rho \in BV_1$.

Next, we prove $BV_1 \subset BV(E)$. For $\rho \in BV_1$, let $\{\rho_n\}$ be as in the definition of BV_1. Let $M = \underline{\lim}_{n \to \infty} \|\nabla \rho_n\|_1 < \infty$. From Proposition 3.2, $\{\rho_n\}$ is bounded in $L(\log L)^{1/2}$. By taking a subsequence, we may assume that ρ_n converges to ρ μ-a.e. From Fatou's lemma, $\rho \in L(\log L)^{1/2}$.

Let Φ_m be a smooth function on $\mathbb{R}$ such that $0 \leqslant \Phi'_m \leqslant 1$ on $\mathbb{R}$, $\Phi_m(x) = x$ on $[-m, m]$ and $|\Phi_m(x)| = m+1$ on $\mathbb{R} \backslash [-m-2, m+2]$. Then for $G \in (\mathscr{F}C_b^1)_{E^*}$,

$$\begin{aligned}\left|\int_E (\nabla^* G)\, \Phi_m \circ \rho \, d\mu\right| &= \lim_{n \to \infty} \left|\int_E (\nabla^* G)\, \Phi_m \circ \rho_n \, d\mu\right| \\ &= \lim_{n \to \infty} \left|\int_E \langle G, \nabla(\Phi_m \circ \rho_n)\rangle \, d\mu\right| \\ &\leqslant \underline{\lim}_{n \to \infty} \|G\|_\infty \, \|\nabla(\Phi_m \circ \rho_n)\|_1 \\ &\leqslant \underline{\lim}_{n \to \infty} \|G\|_\infty \, \|\nabla \rho_n\|_1 \leqslant M \, \|G\|_\infty.\end{aligned}$$

Since $\Phi_m \circ \rho \to \rho$ as $m \to \infty$ in $L(\log L)^{1/2}$, $\int_E (\nabla^* G)\, \Phi_m \circ \rho \, d\mu \to \int_E (\nabla^* G)\, \rho \, d\mu$ by Lemma 3.1. Therefore, $|\int_E (\nabla^* G)\, \rho \, d\mu| \leqslant M \, \|G\|_\infty$. Hence, $\rho \in BV(E)$ and $V(\rho) \leqslant M$. ∎

COROLLARY 3.8. *Let Λ be a function on $\mathbb{R}$ so that $|\Lambda(x) - \Lambda(y)| \leqslant |x-y|$ for all $x, y \in \mathbb{R}$. Then for every $\rho \in BV(E)$, $\Lambda \circ \rho$ also belongs to $BV(E)$ and $V(\Lambda \circ \rho) \leqslant V(\rho)$. In particular, the space $BV(E)$ is a vector lattice.*

Proof. Let $\rho_n = T_{1/n}\rho$, $n \in \mathbb{N}$. Then $\{\Lambda \circ \rho_n\} \subset \mathbb{D}^{1,1}$, $\Lambda \circ \rho_n \to \Lambda \circ \rho$ in L^1 as $n \to \infty$, and

$$\sup_n \|\nabla(\Lambda \circ \rho_n)\|_1 \leqslant \sup_n \|\nabla \rho_n\|_1 \leqslant V(\rho)$$

by Lemma 3.4 and Proposition 3.6. This implies that $\Lambda \circ \rho \in BV(E)$ and $V(\Lambda \circ \rho) \leqslant V(\rho)$ by virtue of Theorem 3.7. ∎

We now extend Theorem 3.1(ii) in [13] together with a uniqueness statement.

THEOREM 3.9. *For each* $\rho \in BV(E)$, *there exists a positive finite measure* ν *on* E *and an* H*-valued Borel function* σ *on* E *such that* $\|\sigma\|_H = 1$ ν-a.e. *and for every* $G \in (\mathscr{F}C_b^1)_{E^*}$,

$$\int_E (\nabla^* G)\, \rho \, d\mu = \int_E \langle G, \sigma \rangle \, d\nu. \tag{3.5}$$

The measure ν *belongs to* $S^{|\rho|+1}$. *If moreover* $\rho \in QR(E)$, *then* $\nu|_{E \setminus F^\rho} = 0$ *and* $\nu|_{F^\rho} \in S^\rho$. *Also,* ν *and* σ *are uniquely determined; namely, if* ν' *and* σ' *are another pair satisfying the relation* (3.5) *for all* $G \in (\mathscr{F}C_b^1)_{E^*}$, *then* $\nu = \nu'$ *and* $\sigma = \sigma'$ ν-a.e.

Proof. First, suppose $\rho \geqslant 0$ μ-a.e. By Theorem 2.1, for each $\ell \in E^*$, there exists a signed measure ν_ℓ on E which belongs to $S^{\rho+1}$ (resp. such that $\nu_\ell|_{E \setminus F^\rho} = 0$ and $\nu_\ell|_{F^\rho} \in S^\rho$ if $\rho \in BV(E) \cap QR(E)$) satisfying

$$\tfrac{1}{2} \int_E \partial_\ell v(z)\, \rho(z)\, \mu(dz) = -\int_E v(z)\, \nu_\ell(dz), \qquad v \in \mathscr{F}C_b^1.$$

Define $D_\ell \rho = 2\nu_\ell + \ell(\cdot)\, \rho \cdot \mu$. Then for an H-valued function G expressed as $G(z) = g(z)\, \ell$ with $g \in \mathscr{F}C_b^1$, $\ell \in E^*$ and $\|\ell\|_H = 1$, we have

$$\int_E (\nabla^* G)\, \rho \, d\mu = \int_E (-\partial_\ell g + g\ell(\cdot))\, \rho \, d\mu = \int_E g(z)\, D_\ell \rho(dz).$$

Therefore, $V(D_\ell \rho)$, the total variation measure of $D_\ell \rho$, satisfies $V(D_\ell \rho)(E) \leqslant V(\rho)$.

For a general ρ, let ρ_+ and ρ_- be the positive part and the negative part of ρ, respectively. Then from Corollary 3.8, $\rho_\pm \in BV(E)$ and $V(\rho_\pm) \leqslant V(\rho)$. Therefore, $V(D_\ell \rho_\pm)(E) \leqslant V(\rho)$. Define $D_\ell \rho = D_\ell \rho_+ - D_\ell \rho_-$. Then $V(D_\ell \rho)(E) \leqslant 2V(\rho)$.

Take $\{h_j\}_{j=1}^\infty \subset E^*$ as a c.o.n.s. of H. Let $\gamma = \sum_{j=1}^\infty 2^{-j} V(D_{h_j} \rho)$ and $v_j(z) = (dD_{h_j} \rho / d\gamma)(z)$, $j \in \mathbb{N}$. Then γ is a positive finite measure, $\gamma(E) \leqslant 2V(\rho)$, and $\gamma \in S^{|\rho|+1}$ (resp. $\gamma|_{E \setminus F^\rho} = 0$ and $\gamma|_{F^\rho} \in S^\rho$ if $\rho \in BV(E) \cap QR(E)$). We may assume that each v_j is Borel measurable. From the same argument as in the proof of Theorem 3.1(ii) in [13], we can construct ν and σ from γ and v_j so that (3.5) holds for all G expressed as $G(z) = \sum_{j=1}^n g_j(z)\, h_j$ with $g_j \in \mathscr{F}C_b^1$, $j = 1, ..., n$ for some $n \in \mathbb{N}$. In order to finish the proof of the first claim, it suffices to prove the validity of (3.5) for $G(z)=g(z)\,\ell$, where $g\in\mathscr{F}C_b^1$ and $\ell\in E^*$ with $\|\ell\|_H=1$. The relation to prove is

$$-\int_E (\partial_\ell g)\,\rho\,d\mu+\int_E g\ell(\cdot)\,\rho\,d\mu=\int_E g\langle \ell,\sigma\rangle\,d\nu. \tag{3.6}$$

Let $\ell_n=\sum_{j=1}^n \langle \ell, h_j\rangle\, h_j$, $n\in\mathbb{N}$. Denote the linear span of $\{\ell_n,\ell\}$ in H by H_n, and its orthogonal complement by $H_n^\perp$. Take a unitary operator U_n on H satisfying that H_n is U_n-invariant, $U_n(\ell_n)=\ell$, and $U_n|_{H_n^\perp}$ is an identity mapping. U_n can be extended to a continuous operator on E, leaving μ invariant. Set $g_n(z)=g(U_n(z))$, $z\in E$. We already know that (3.6) holds if ℓ and g are replaced by ℓ_n and g_n, respectively. We shall observe that each term converges appropriately as $n\to\infty$. Since $\partial_{\ell_n}g_n(z)=(\partial_\ell g)(U_n(z))$, it holds by the dominated convergence theorem that

$$\lim_{n\to\infty}\int_E (\partial_{\ell_n}g_n)\,\rho\,d\mu=\int_E (\partial_\ell g)\,\rho\,d\mu$$

and

$$\lim_{n\to\infty}\int_E g_n\langle \ell_n,\sigma\rangle\,d\nu=\int_E g\langle \ell,\sigma\rangle\,d\nu.$$

From the U_n-invariance of μ, it holds that $\rho\circ U_n^{-1}\to\rho$ in $L(\log L)^{1/2}$ as $n\to\infty$. Indeed, this is proved by approximating ρ by bounded continuous functions and using a triangle inequality. Then from Lemma 3.1,

$$\int_E g_n\ell_n(\cdot)\,\rho\,d\mu=\int_E g\ell(\cdot)(\rho\circ U_n^{-1})\,d\mu\to\int_E g\ell(\cdot)\,\rho\,d\mu\quad\text{as } n\to\infty.$$

Therefore, (3.6) holds.

We shall proceed to the proof of uniqueness. Suppose that σ' and ν' are another pair. Then,

$$\int_E \langle G,\gamma\rangle\,d\xi=0\qquad\text{for every}\quad G\in(\mathscr{F}C_b^1)_{E^*},$$

where $\xi=\nu+\nu'$ and $\gamma=\sigma\,\frac{d\nu}{d\xi}-\sigma'\,\frac{d\nu'}{d\xi}$. Taking a uniformly bounded sequence $\{G_n\}\subset(\mathscr{F}C_b^1)_{E^*}$ so that $\langle G_n,\gamma\rangle\to\|\gamma\|_H$ ξ-a.e., we get $\gamma=0$ ξ-a.e. Therefore, $\|\sigma\|_H\,\frac{d\nu}{d\xi}=\|\sigma'\|_H\,\frac{d\nu'}{d\xi}$ ξ-a.e. Since $\|\sigma\|_H=1$ ν-a.e., $\|\sigma\|_H\,\frac{d\nu}{d\xi}=\frac{d\nu}{d\xi}$ ξ-a.e. Similarly, $\|\sigma'\|_H\,\frac{d\nu'}{d\xi}=\frac{d\nu'}{d\xi}$ ξ-a.e. Then, $\frac{d\nu}{d\xi}=\frac{d\nu'}{d\xi}$ ξ-a.e., which implies $\nu=\nu'$. Also, it follows that $\sigma=\sigma'$ ν-a.e. from $\gamma=0$ ξ-a.e. and $\nu=\nu'$. ∎

We shall hereafter write $\|D\rho\|$ and σ_ρ in place of ν and σ, respectively.

COROLLARY 3.10. *It holds that*

$$BV(E)=\left\{\rho\in L(\log L)^{1/2}\;\middle|\;\begin{array}{l}\text{there exist a sequence } \{\rho_n\}\subset\mathbb{D}^{1,1}\text{, a positive}\\ \text{finite measure } \nu \text{ on } E \text{ and an } H\text{-valued Borel}\\ \text{function } \sigma \text{ on } E \text{ such that } \rho_n\to\rho \text{ in}\\ L(\log L)^{1/2},\ \sup_n\|\nabla\rho_n\|_1<\infty,\ \|\sigma\|_H=1\ \nu\text{-a.e.,}\\ \text{and } \lim_{n\to\infty}\int_E\langle G,\nabla\rho_n\rangle\,d\mu=\\ \int_E\langle G,\sigma\rangle\,d\nu \text{ for all } G\in(\mathscr{F}C_b^1)_{E^*}.\end{array}\right\}. \tag{3.7}$$

Furthermore, σ and ν in the right-hand side coincide with σ_ρ and $\|D\rho\|$, respectively.

Proof. Let the right-hand side in (3.7) be denoted by BV_2. Take $\rho\in BV(E)$, and let $\rho_n=T_{1/n}\rho$ for $n\in\mathbb{N}$. Then $\rho_n\in\mathbb{D}^{1,1}$, $\rho_n\to\rho$ in $L(\log L)^{1/2}$, and $V(\rho_n)\leqslant V(\rho)$ by Proposition 3.6. For any $G\in(\mathscr{F}C_b^1)_{E^*}$,

$$\int_E\langle G,\nabla\rho_n\rangle\,d\mu=\int_E(\nabla^*G)\,\rho_n\,d\mu\to\int_E(\nabla^*G)\,\rho\,d\mu$$

$$=\int_E\langle G(z),\sigma_\rho(z)\rangle\,\|D\rho\|\,(dz)\qquad\text{as}\quad n\to\infty.$$

Therefore, $\rho\in BV_2$. The inverse inclusion is trivial from Theorem 3.7. The latter assertion follows from Theorem 3.9. ∎

Remark 3.11. We can also replace $\mathbb{D}^{1,1}$ by $\mathscr{F}C_b^1$ and two $L(\log L)^{1/2}$'s by L^1 in the right-hand side of (3.7). Compare (3.7) with characterizations of $\mathbb{D}^{1,1}$:

$$\mathbb{D}^{1,1}=\left\{\rho\in L^1\;\middle|\;\begin{array}{l}\text{there exist a sequence } \{\rho_n\}\subset\mathscr{F}C_b^1\\ \text{and } J\in L^1(E\to H) \text{ such that}\\ \rho_n\to\rho \text{ in } L^1 \text{ and } \nabla\rho_n\to J \text{ in } L^1(E\to H).\end{array}\right\}$$

$$=\left\{\rho\in L(\log L)^{1/2}\;\middle|\;\begin{array}{l}\text{there exist a sequence } \{\rho_n\}\subset\mathscr{F}C_b^1 \text{ and}\\ J\in L^1(E\to H) \text{ such that}\\ \rho_n\to\rho \text{ in } L(\log L)^{1/2}\\ \text{and } \nabla\rho_n\to J \text{ in } L^1(E\to H).\end{array}\right\}.$$

We can now present a formula analogous to the classical Green formula. We consider the Ornstein–Uhlenbeck operator $L = -\nabla^*\nabla$, which can be expressed as

$$Lu(z) = \sum_{i=1}^{m} \partial_i^2 f(\ell_1(z), ..., \ell_m(z)) - \sum_{i=1}^{m} \partial_i f(\ell_1(z), ..., \ell_m(z))\, \ell_i(z)$$

if $u(z) = f(\ell_1(z), ..., \ell_m(z)) \in \mathscr{F}C_b^2$ and $\ell_1, ..., \ell_m \in E^*$ are orthonormal in H.

THEOREM 3.12. *Let $\rho \in BV(E)$. For any $u \in \mathscr{F}C_b^2$ and any $v \in \mathscr{F}C_b^1$,*

$$\mathscr{E}^\rho(u, v) = -\tfrac{1}{2} \int_E v(Lu)\, \rho\, d\mu - \tfrac{1}{2} \int_E v(z) \langle \nabla u(z), \sigma_\rho(z) \rangle\, \|D\rho\|\, (dz). \tag{3.8}$$

If further $\rho \in QR(E)$, then the equation obtained by replacing E by F^ρ in (3.8) holds for any $u \in \mathscr{F}C_b^2$ and any $\mathscr{E}^\rho$-quasicontinuous function $v \in \mathscr{F}_b^\rho$.

Proof. Take ρ_n as in the right hand side of (3.7). An integration by part gives

$$\int_E \langle \nabla u, \nabla v \rangle \rho_n\, d\mu = -\int_E v(Lu)\, \rho_n\, d\mu - \int_E v \langle \nabla u, \nabla \rho_n \rangle\, d\mu.$$

Letting n tend to infinity, we get the desired formula. ∎

The next theorem is a converse to Lemma 3.4 and characterizes the space $\mathbb{D}^{1,1}$ as a subspace of $BV(E)$.

THEOREM 3.13. *Let $\rho \in BV(E)$. If $\|D\rho\| \ll \mu$, Then $\rho \in \mathbb{D}^{1,1}$ and*

$$\|D\rho\| = \|\nabla \rho\|_H \cdot \mu, \qquad \sigma_\rho = \frac{\nabla \rho}{\|\nabla \rho\|_H} \cdot \mathbf{1}_{\{\nabla \rho \neq 0\}}.$$

Proof. It suffices to prove that if there exists $J \in L^1(E \to H)$ such that

$$\int_E (\nabla^* G)\, \rho\, d\mu = \int_E \langle G, J \rangle\, d\mu, \qquad \text{for all} \quad G \in (\mathscr{F}C_b^1)_{E^*},$$

then $\rho \in \mathbb{D}^{1,1}$ and $\nabla\rho = J$. Since $\rho \in L(\log L)^{1/2}$, $T_t\rho$ belongs to $\mathbb{D}^{1,1}$ and for any $G \in (\mathscr{F}C_b^1)_{E^*}$,

$$\int_E \langle G, \nabla T_t\rho\rangle \, d\mu = \int_E e^{-t}(\nabla^* T_t G)\, \rho \, d\mu$$

$$= e^{-t} \int_E \langle T_t G, J\rangle \, d\mu = e^{-t} \int_E \langle G, T_t J\rangle \, d\mu.$$

Therefore, $\nabla T_t\rho = e^{-t} T_t J$. This converges to J in L^1 as $t \to 0$, which implies that $\rho \in \mathbb{D}^{1,1}$ and $\nabla\rho = J$. ∎

Remark 3.14. We have a coarea formula

$$V(\rho) = \int_{-\infty}^{\infty} V(\mathbf{1}_{\{\rho > t\}})\, dt, \qquad \rho \in BV(E),$$

just as Theorem 4.1 in [13]. Then for every $\rho \in BV(E)$, $\mathbf{1}_{\{\rho > t\}}$ belongs to $BV(E)$ for a.e. t with respect to the Lebesgue measure. However, $\mathbf{1}_A \in \mathbb{D}^{1,1}$ if and only if $\mu(A) = 0$ or 1. Indeed, if $\mathbf{1}_A \in \mathbb{D}^{1,1}$, then $\nabla \mathbf{1}_A = \nabla(\mathbf{1}_A)^2 = 2 \cdot \mathbf{1}_A \nabla \mathbf{1}_A$, which implies that $\nabla \mathbf{1}_A = 0$ μ-a.e. Then for all $t > 0$, $\nabla T_t \mathbf{1}_A = e^{-t} T_t \nabla \mathbf{1}_A = 0$ μ-a.e. Since $T_t \mathbf{1}_A \in \mathscr{F}^1$, it is well-known that $T_t \mathbf{1}_A =$ constant μ-a.e., therefore, $\mathbf{1}_A =$ constant μ-a.e. Hence, there are many functions which belong to $BV(E)$ but do not belong to $\mathbb{D}^{1,1}$.

Finally in this section, we study the support of $\|D\rho\|$. Let $\rho \in QR(E)$. We denote the $\mathscr{E}^\rho$-quasi support of $\nu \in S_+^\rho$ by $\mathscr{E}^\rho$-q. Supp ν. When A is a measurable subset of E, we define

$$\bar{A}^\rho = \mathscr{E}^\rho\text{-q.Supp}(\mathbf{1}_A \cdot \rho \cdot \mu), \qquad \partial^\rho A = \bar{A}^\rho \cap \overline{E \setminus A}^\rho.$$

THEOREM 3.15. *Let* $\rho \in BV(E)$. *Then for every* $a \in \mathbb{R}$, *we have* $\|D\rho\|(E \setminus \overline{\{\rho \neq a\}}^{|\rho|+1}) = 0$, *namely*,

$$\mathscr{E}^{|\rho|+1}\text{-q.Supp}\, \|D\rho\| \subset \overline{\{\rho \neq a\}}^{|\rho|+1} \qquad \mathscr{E}^{|\rho|+1}\text{-q.e.}$$

When $\rho = \mathbf{1}_A \in BV(E)$ *for a certain set* A, *we have*

$$\mathscr{E}^1\text{-q.Supp}\, \|D\rho\| \subset \partial^1 A \quad \mathscr{E}^1\text{-q.e.}$$

Proof. Let $a \in \mathbb{R}$. From the way of construction of $\|D\rho\|$ in the proof of Theorem 3.9, it is enough to prove that $D_\ell \rho(E \setminus F_a) = 0$ for each $\ell \in E^*$, where $F_a := \overline{\{\rho \neq a\}}^{|\rho|+1}$. By Lemma 4.6.1 in [14], there is a nonnegative and $\mathscr{E}^{|\rho|+1}$-quasicontinuous function u in $\mathscr{F}_b^{|\rho|+1}$ such that $F_a = \{u = 0\}$ $\mathscr{E}^{|\rho|+1}$-q.e. We can take a uniformly bounded sequence $\{u_n\} \subset \mathscr{F}C_b^1$ such that $u_n \to u$ in $\mathscr{F}^{|\rho|+1}$ and $\mathscr{E}^{|\rho|+1}$-q.e. as $n \to \infty$. For any $g \in \mathscr{F}C_b^1$, we have

$$\begin{aligned}\int_E g(z)\, u_n(z)\, D_\ell \rho(dz) = \int_E g(z)\, u_n(z)\, D_\ell(\rho - a)(dz) \\ = -\int_E \partial_\ell (g u_n)(\rho - a)\, d\mu + \int_E g u_n \ell(\cdot)(\rho - a)\, d\mu \\ = -\int_E \{(\partial_\ell g)\, u_n + g\langle \nabla u_n, \ell \rangle\}(\rho - a)\, d\mu \\ + \int_E g u_n \ell(\cdot)(\rho - a)\, d\mu.\end{aligned}$$

Keeping $\mathscr{E}^{|\rho|+1}$-smoothness of $D_\ell \rho$ in mind, we obtain by letting $n \to \infty$ that

$$\begin{aligned}\int_E g(z)\, u(z)\, D_\ell \rho(dz) = -\int_E \{(\partial_\ell g)\, u + g\langle \nabla u, \ell \rangle\}(\rho - a)\, d\mu \\ + \int_E g u \ell(\cdot)(\rho - a)\, d\mu. \qquad (3.9)\end{aligned}$$

Since $u = 0$ on F_a μ-a.e., $\nabla u = 0$ on F_a μ-a.e. by Theorem 7.1.1 in [6, Chapter I]. Therefore, the right-hand side of (3.9) vanishes. This means that $D_\ell \rho(E \backslash F_a) = 0$, which finishes the proof of the first part.

When $\rho = \mathbf{1}_A \in BV(E)$, by applying the first claim with $a = 0$ and $a = 1$, we get

$$\mathscr{E}^1\text{-q.Supp}\, \|D\rho\| \subset \overline{A}^1 \cap \overline{E \backslash A}^1 = \partial^1 A \qquad \mathscr{E}^1\text{-q.e.} \quad \blacksquare$$

4. DISTORTED ORNSTEIN–UHLENBECK PROCESS AND ITÔ'S FORMULA

Since E is separable, E^* is also separable in the weak*-topology (see e.g. [26, p. 90]). Let $\{\ell_n\}$ be a countable dense subset of E^*.

LEMMA 4.1. *Let $\{B_t\}$ be an E-valued continuous process starting at* 0. *If $\{\ell_n(B_t)\}$ is an $\{\mathscr{M}_t\}$-martingale with quadratic variation $t\, \|\ell_n\|_H^2$ for every n, then $\{B_t\}$ is an $\{\mathscr{M}_t\}$-Brownian motion on E.*

Proof. By the martingale representation theorem, each $\{\ell_n(B_t)\}$ is a 1-dimensional $\{\mathcal{M}_t\}$-Brownian motion with a constant time change. Take any $\ell \in E^*$ with $\|\ell\|_H = 1$. There exists a subsequence $\{\ell_{n_k}\}$ of $\{\ell_n\}$ converging to ℓ in the weak* sense. Then,

$$\lim_{k\to\infty} \exp(-\|\ell_{n_k}\|_H^2/2) = \lim_{k\to\infty} \int_E \exp(\sqrt{-1}\ell_{n_k}(z))\,\mu(dz)$$
$$= \int_E \exp(\sqrt{-1}\ell(z))\,\mu(dz) = \exp(-1/2),$$

therefore $\lim_{k\to\infty} \|\ell_{n_k}\|_H = 1$. For $\xi \in \mathbb{R}$, $t > s > 0$ and an $\mathcal{M}_s$-measurable bounded function f,

$$\begin{aligned}
&\mathbb{E}[\exp(\sqrt{-1}\xi(\ell(B_t) - \ell(B_s)))f] \\
&= \lim_{k\to\infty} \mathbb{E}[\exp(\sqrt{-1}\xi(\ell_{n_k}(B_t) - \ell_{n_k}(B_s)))f] \\
&= \lim_{k\to\infty} \exp(-(t-s)\,\xi^2\,\|\ell_{n_k}\|_H^2/2)\,\mathbb{E}[f] \\
&= \exp(-(t-s)\,\xi^2/2)\,\mathbb{E}[f].
\end{aligned}$$

Namely, $\{\ell(B_t)\}$ is a 1-dimensional $\{\mathcal{M}_t\}$-Brownian motion. Therefore, $\{B_t\}$ is an $\{\mathcal{M}_t\}$-Brownian motion on E. ▮

By using this lemma, Theorem 3.2 in [13] is improved as follows.

THEOREM 4.2. *Let $\rho \in BV(E) \cap QR(E)$. Then the sample path of the distorted Ornstein–Uhlenbeck process $\mathbf{M}^\rho = (X_t, \mathcal{M}_t, P_z)$ associated with $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ admits the following expression as a sum of three E-valued CAF's:*

$$X_t(\omega) - X_0(\omega) = W_t(\omega) - \tfrac{1}{2}\int_0^t X_s(\omega)\,ds$$
$$+ \tfrac{1}{2}\int_0^t \sigma_\rho(X_s(\omega))\,dA_s^{\|D\rho\|}(\omega), \qquad t \geq 0. \qquad (4.1)$$

Here, the $\mathcal{E}^\rho$-smooth measure $\|D\rho\|$ and the H-valued function σ_ρ are defined as in Theorem 3.9; $A^{\|D\rho\|}$ is a real valued PCAF associated with $\|D\rho\|$ via the Revuz correspondence. Moreover, for $\mathcal{E}^\rho$-q.e. $z \in F^\rho$, $\{W_t(\omega)\}$ is an $\{\mathcal{M}_t\}$-Brownian motion on E under P_z.

Proof. We can define an E-valued CAF W_t by the equation (4.1). As in the same way of the proof of Theorem 3.2 in [13], for every $\ell \in E^*$, for $\mathcal{E}^\rho$-q.e. z, $\{\ell(W_t)\}$ is a martingale under P^z with quadratic variation $t\,\|\ell\|_H^2$. Since a countable union of exceptional sets is also exceptional, Lemma 4.1 completes the proof. ▮

We now turn to a generalized Itô's formula which has been formulated in [12] for the additive functionals of the distorted Brownian motion on $\mathbb{R}^d$. For this purpose, we prepare a lemma for quasi-sure analysis on Hilbert space valued functions. Though it is quite standard and we need it only for the Ornstein–Uhlenbeck semigroup, we shall formulate it under a general framework and give a proof for completeness. Let $(\mathscr{E}, \mathscr{F})$ be a quasi-regular symmetric Dirichlet form on a state space (Ω, m), where Ω is a Hausdorff topological space with a countable base and m is a σ-finite Borel measure on Ω. Let $\{P_t\}$ and (ω_t, Q_z) be a Markovian semigroup on $L^2(\Omega; m)$ and a Markov process on Ω associated with $(\mathscr{E}, \mathscr{F})$, respectively. The expectation with respect to Q_z is denoted by E^{Q_z}. Let K be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|_K$. For a K-valued step function G on Ω expressed as $G = \sum_{i=1}^n \mathbf{1}_{A_i} k_i$ with $k_i \in K$, $m(A_i) < \infty$, and $A_i \cap A_j = \emptyset$ if $i \neq j$, define $P_t G = \sum_{i=1}^n (P_t \mathbf{1}_{A_i}) k_i$, $t > 0$. Then P_t extends uniquely to a bounded operator on $L^2(\Omega \to K; m)$ and satisfies that $\|P_t G\|_K \leqslant P_t(\|G\|_K)$ m-a.e. for all $G \in L^2(\Omega \to K; m)$. In the following lemma, $\tilde{f}$ means an $\mathscr{E}$-quasicontinuous modification of a function f on Ω if it exists. Note that an $\mathscr{E}$-quasicontinuous modification of a K-valued function is uniquely determined up to $\mathscr{E}$-exceptional set like real-valued functions, because of the separability of K.

LEMMA 4.3. (i) *Let $\{G_n\}$ be a sequence of $\mathscr{E}$-quasicontinuous functions in $L^2(\Omega \to K; m)$ and $G \in L^2(\Omega \to K; m)$. If there exists a sequence $\{w_n\} \subset \mathscr{F}$ such that*

$$\|G_n - G\|_K \leqslant w_n \ m\text{-a.e. for all } n \text{ and } w_n \to 0 \text{ in } \mathscr{F} \text{ as } n \to \infty,$$

then G has an $\mathscr{E}$-quasicontinuous modification $\tilde{G}$, and $G_{n_k} \to \tilde{G}$ $\mathscr{E}$-q.e. for some subsequence $\{G_{n_k}\}$.

(ii) *Let G be a Borel measurable function in $L^2(\Omega \to K; m)$ and $t > 0$. Then $E^{Q_\cdot}[G(\omega_t)]$ is an $\mathscr{E}$-quasicontinuous modification of $P_t G$.*

(iii) *Let G_n, $G \in L^2(\Omega \to K; m)$ and $G_n \to G$ in $L^2(\Omega \to K; m)$ as $n \to \infty$. Then for each $t > 0$, there is a subsequence $\{G_{n_k}\}$ such that $\widetilde{P_t G_{n_k}} \to \widetilde{P_t G}$ $\mathscr{E}$-q.e.*

Proof. Let Cap denote the capacity associated with $(\mathscr{E}, \mathscr{F})$.

(i) Let $\varepsilon > 0$. Since $w_n \to 0$ in $\mathscr{F}$, there exists a sequence $\{n_l\} \uparrow \infty$ and an open set U_1 such that $\mathrm{Cap}(U_1) < \varepsilon/2$ and $\widetilde{w_{n_l}}$ converges to 0 uniformly on $E \backslash U_1$. Since $\|G_m - G_n\|_K \leqslant |w_m| + |w_n|$ m-a.e., it holds that $\|G_m - G_n\|_K \leqslant |\widetilde{w_m}| + |\widetilde{w_n}|$ $\mathscr{E}$-q.e. There is an open set U_2 such that $\mathrm{Cap}(U_2) < \varepsilon/2$, $G_n|_{E \backslash U_2}$ is continuous for every n, and the inequality above holds on $E \backslash U_2$ for every m and n. Let $U = U_1 \cup U_2$. Then $\mathrm{Cap}(U) < \varepsilon$ and

$\{G_{n_l}\}$ converges uniformly on $E\backslash U$. By diagonalization argument, we can take a subsequence of $\{G_n\}$ which converges to some $\mathscr{E}^1$-quasicontinuous function $\tilde{G}$ $\mathscr{E}$-q.e., and clearly $G=\tilde{G}$ m-a.e.

(ii) Take K-valued step functions $\{G_n\}$ such that $G_n \to G$ in $L^2(\Omega \to K; m)$ as $n \to \infty$. Each $P_t G_n$ has an $\mathscr{E}$-quasicontinuous modification in view of the result for scalar valued functions. It holds that

$$\|\widetilde{P_t G_n} - P_t G\|_K \leqslant P_t(\|G_n - G\|_K) \qquad m\text{-a.e.},$$

and the right-hand side of the inequality above converges to 0 in $\mathscr{F}$ as $n \to \infty$. By (i), $P_t G$ has an $\mathscr{E}^1$-quasicontinuous modification. On the other hand, $E^{Q_z}[G(\omega_t)]$ exists for $\mathscr{E}$-q.e. z since $E^{Q_z}[\|G(\omega_t)\|_K] < \infty$ $\mathscr{E}$-q.e.

For each $k \in K$, we have

$$\langle \widetilde{P_t G}, k \rangle = P_t(\langle G, k \rangle) = E^{Q_\cdot}[\langle G, k \rangle(\omega_t)] \quad m\text{-a.e.}$$

Therefore, $\langle \widetilde{P_t G}, k \rangle = E^{Q_\cdot}[\langle G, k \rangle(\omega_t)]$ $\mathscr{E}$-q.e. since both are $\mathscr{E}$-quasicontinuous. This implies that $\widetilde{P_t G}(z) = E^{Q_z}[G(\omega_t)]$ for $\mathscr{E}$-q.e. z.

(iii) From (ii), $P_t G_n$ has an $\mathscr{E}$-quasicontinuous modification for every n. Since

$$\|\widetilde{P_t G_n} - P_t G\|_K \leqslant P_t(\|G_n - G\|_K) \qquad m\text{-a.e.}$$

and the right-hand side converges to 0 in $\mathscr{F}$ as $n \to \infty$, the assertion follows from (i). ∎

Recall Theorem 3.12 where the Ornstein–Uhlenbeck operator $L = -\nabla^*\nabla$ appears. Let $\{T_t\}$ be its associated Ornstein–Uhlenbeck semigroup as before. Its corresponding Dirichlet form is nothing but $(2\mathscr{E}^1, \mathscr{F}^1)$. The following theorem is a counterpart of Theorem 3.3 in [12]. Below, all functions are regarded as Borel measurable.

THEOREM 4.4. *Suppose either of the following.*

(a) $\rho \in BV(E) \cap QR(E)$, $u(z) = f(\ell_1(z), ..., \ell_m(z)) \in \mathscr{F}C_b^1$, *and* $u^\varepsilon(z) = f^\varepsilon(\ell_1(z), ..., \ell_m(z))$, *where* f^ε *is an ordinary mollification of* f *on* $\mathbb{R}^m$.

(b) $\rho \in BV(E) \cap QR(E) \cap L^\infty$, $u = w|_{F^\rho}$ *for some* $w \in \mathscr{F}_b^1$ *such that* $\|\nabla w\|_H$ *is* μ*-essentially bounded and* ∇w *has an* $\mathscr{E}^1$*-quasicontinuous modification, and* $u^\varepsilon = T_\varepsilon w$. *In this case,* ∇u *denotes* $\nabla w|_{F^\rho}$.

Then, the next conditions are equivalent.

(i) $N^{[u]} \in \mathbf{A}_0^\rho$.

(ii) *There exists a finite signed measure* $\nu_{u,\rho}$ *on* F^ρ *such that*

$$\lim_{\varepsilon \to 0} \int_{F^\rho} v(Lu^\varepsilon)\, \rho\, d\mu = \int_{F^\rho} v\, d\nu_{u,\rho} \qquad \textit{for every} \quad v \in \mathscr{F}C_b^1.$$

In this case, it holds that for any $v \in \mathscr{F}C_b^1$,

$$\mathscr{E}^\rho(u, v) = -\tfrac{1}{2} \int_{F^\rho} v\, d\nu_{u,\rho} - \tfrac{1}{2} \int_{F^\rho} v(z) \langle \widetilde{\nabla u}(z), \sigma_\rho(z) \rangle \, \|D\rho\|\, (dz),$$

where $\widetilde{\nabla u}$ *is an* $\mathscr{E}^\rho$*-quasicontinuous modification of* ∇u. *Moreover,* $\nu_{u,\rho} \in S^\rho$. *Let* A^{Lu} *and* $A^{\|D\rho\|}$ *denote the CAFs associated with* $\nu_{u,\rho}$ *and* $\|D\rho\|$, *respectively. Then* P_z*-a.e. for* $\mathscr{E}^\rho$*-q.e.* $z \in F^\rho$, *the equation*

$$u(X_t) - u(X_0) = M_t^{[u]} + \tfrac{1}{2} A_t^{Lu} + \tfrac{1}{2} \int_0^t \langle \widetilde{\nabla u}, \sigma_\rho \rangle (X_s)\, dA_t^{\|D\rho\|}$$

holds. Here $M^{[u]}$ *is a continuous martingale AF with quadratic variation*

$$\langle M^{[u]} \rangle_t = \int_0^t \|\nabla u\|_H^2 (X_s)\, ds.$$

Further, for some sequence $\{\varepsilon_n\} \downarrow 0$,

$$\lim_{n \to \infty} \int_0^t (Lu^{\varepsilon_n})(X_s)\, ds = A_t^{Lu} \qquad \textit{locally uniformly in } t.$$

Proof. We shall give a proof only in the case (b). The case (a) is similarly (and more easily) proved. First we remark that $\mathscr{E}^1$-exceptional sets are $\mathscr{E}^\rho$-exceptional sets and $\mathscr{E}^1$-convergence implies $\mathscr{E}^\rho$-convergence because $\rho \in L^\infty$. From the theorem of [7] and an argument in the proof of [1, Theorem 2.4], we can take a sequence $\{v_n\} \subset \mathscr{F}C_b^\infty$ such that $v_n \to w$ μ-a.e. and both $\{\|v_n\|_\infty\}$ and $\{\|\nabla v_n\|_\infty\}$ are uniformly bounded. By the Banach–Saks theorem, a sequence of the Cesàro mean $\{u_n\}$ of some subsequence of $\{v_n\}$ satisfies that $u_n \in \mathscr{F}C_b^\infty$, $u_n \to w$ μ-a.e. and in $\mathscr{F}^1$, and both $\{\|u_n\|_\infty\}$ and $\{\|\nabla u_n\|_\infty\}$ are uniformly bounded.

For each $\varepsilon > 0$, $T_\varepsilon u_n \to u^\varepsilon$ in $\mathscr{F}^1$ and $LT_\varepsilon u_n \to Lu^\varepsilon$ in $L^2(\mu)$ as $n \to \infty$. Also, $\nabla T_\varepsilon u_n = e^{-\varepsilon} T_\varepsilon \nabla u_n$, $\nabla u^\varepsilon = e^{-\varepsilon} T_\varepsilon \nabla w$ and $\nabla u_n \to \nabla w$ in $L^2(E \to H; \mu)$ as $n \to \infty$. By Lemma 4.3, ∇u^ε has an $\mathscr{E}^1$-quasicontinuous modification $\widetilde{\nabla u^\varepsilon}$ and, by taking a subsequence if necessary,

$$\nabla T_\varepsilon u_n \to \widetilde{\nabla u^\varepsilon} \qquad \mathscr{E}^1\text{-q.e.} \qquad \text{as} \quad n \to \infty.$$

Then, letting $n \to \infty$ in our version of the Green formula (Theorem 3.12) for $T_\varepsilon u_n$ in place of u, we have

$$\mathscr{E}^\rho(u^\varepsilon, v) = -\frac{1}{2}\int_{F^\rho} v(Lu^\varepsilon)\,\rho\,d\mu$$

$$-\frac{1}{2}\int_{F^\rho} v(z)\langle \widetilde{\nabla u^\varepsilon}(z), \sigma_\rho(z)\rangle\,\|D\rho\|\,(dz), \qquad v \in \mathscr{F}C_b^1,$$

and accordingly

$$N_t^{[u^\varepsilon]} = \frac{1}{2}\int_0^t (Lu^\varepsilon)(X_s)\,ds + \frac{1}{2}\int_0^t \langle \widetilde{\nabla u^\varepsilon}, \sigma_\rho\rangle(X_s)\,dA_s^{\|D\rho\|}.$$

Let $\widetilde{\nabla w}$ be an $\mathscr{E}^1$-quasicontinuous modification of ∇w. We may assume that $\widetilde{\nabla w}$ is Borel measurable. Then $\widetilde{\nabla u} := \widetilde{\nabla w}|_{F^\rho}$ is an $\mathscr{E}^\rho$-quasicontinuous modification of ∇u and Borel measurable. Also, by Lemma 4.3(ii),

$$\widetilde{\nabla u^\varepsilon}(z) = e^{-\varepsilon}\,\widetilde{T_\varepsilon \nabla w}(z) = e^{-\varepsilon}E_z^{O-U}[\widetilde{\nabla w}(X_\varepsilon^{O-U})] \qquad \mathscr{E}^1\text{-q.e. } z,$$

where (X_t^{O-U}) is the Ornstein–Uhlenbeck process on E and E_z^{O-U} represents an expectation with respect to the distribution of the process starting at z. Since $\widetilde{\nabla w}$ is finely continuous $\mathscr{E}^1$-q.e., which is proved in the same way as in Theorem 4.2.2. in [14], we get

$$e^{-\varepsilon}E_z^{O-U}[\widetilde{\nabla w}(X_\varepsilon^{O-U})] \to \widetilde{\nabla w}(z) \quad \mathscr{E}^1\text{-q.e. } z \quad \text{as} \quad \varepsilon \downarrow 0.$$

Therefore, $\widetilde{\nabla u^{\varepsilon_n}} \to \widetilde{\nabla u}$ $\mathscr{E}^\rho$-q.e. as $n \to \infty$ for an arbitrary sequence $\{\varepsilon_n\}$ decreasing to 0. Keeping the fact that $u^\varepsilon \to u$ in $\mathscr{F}^\rho$ as $\varepsilon \downarrow 0$ in mind, we have

$$\mathscr{E}^\rho(u, v) = -\lim_{\varepsilon \downarrow 0}\frac{1}{2}\int_{F^\rho} v(Lu^\varepsilon)\,\rho\,d\mu$$

$$-\frac{1}{2}\int_{F^\rho} v(z)\langle \widetilde{\nabla u}(z), \sigma_\rho(z)\rangle\,\|D\rho\|\,(dz), \qquad v \in \mathscr{F}C_b^1. \tag{4.2}$$

By Theorem 2.2 in [12], the equivalence of (i) and (ii) and all other assertions hold except for the last one, which in turn follows from Corollary 5.2.1 in [14]. ∎

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