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An optimal logarithmic Sobolev inequality with Lipschitz constants

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Abstract

In this paper, we give an optimal logarithmic Sobolev inequality on $\mathbb{R}^n$ with Lipschitz constants. This inequality is a limit case of the L^p-logarithmic Sobolev inequality of Gentil (2003) [7] as $p \to \infty$. As a result of our inequality, we show that if a Lipschitz continuous function f on $\mathbb{R}^n$ fulfills some condition, then its Lipschitz constant can be expressed by using the entropy of f. We also show that a hypercontractivity of exponential type occurs in the heat equation on $\mathbb{R}^n$. This is due to the Lipschitz regularizing effect of the heat equation.

Keywords: Lipschitz constants; Logarithmic Sobolev inequality; Heat equation; Lipschitz regularizing effect

1. Introduction

Let $n \in \mathbb{N}$. A famous logarithmic Sobolev inequality of Gross [8] for a probability measure μ on $\mathbb{R}^n$ asserts that

$$\rho\, \mathrm{Ent}_\mu\big(f^2\big) \leqslant 2 \int |Df|^2\, d\mu \tag{1.1}$$

for some constant $\rho > 0$ and all smooth enough functions f on $\mathbb{R}^n$, where

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$$\mathrm{Ent}_\mu(f^2) = \int f^2 \log f^2 \, d\mu - \int f^2 \, d\mu \log \int f^2 \, d\mu$$

and $|Df|$ is the Euclidean length of the gradient Df of f; in this paper, the integral without its domain is always understood as the one over $\mathbb{R}^n$. The canonical Gaussian measure with respect to the Lebesgue measure on $\mathbb{R}^n$ is the basic example of measure μ satisfying (1.1) with $\rho = 1$.

On the other hand, Gentil [7, Theorem 1.1] gave a logarithmic Sobolev inequality for the Lebesgue measure. For a smooth enough function $f \geqslant 0$ on $\mathbb{R}^n$, we define the entropy of f by

$$\mathrm{Ent}(f) = \int f \log f \, dx - \int f \, dx \log \int f \, dx.$$

Let $p \geqslant 1$. We denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions f on $\mathbb{R}^n$ such that f and $|Df|$ are in $L^p(\mathbb{R}^n)$. Then,

$$\mathrm{Ent}(|f|^p) \leqslant \frac{n}{p} \int |f|^p \, dx \log\left(L_p \frac{\int |Df|^p \, dx}{\int |f|^p \, dx}\right) \quad \text{for } f \in W^{1,p}(\mathbb{R}^n). \tag{1.2}$$

Here,

$$L_p = \begin{cases} \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-p/2} \left(\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\frac{p-1}{p}+1)}\right)^{p/n}, & p > 1, \\ \frac{1}{n}\pi^{-1/2}[\Gamma(\frac{n}{2}+1)]^{1/n}, & p = 1, \end{cases} \tag{1.3}$$

and this is the best possible constant satisfying (1.2). This inequality was first shown by Del Pino and Dolbeault [3] for $1 \leqslant p \leqslant n$.

These two logarithmic Sobolev inequalities have been studied by many authors, because (1.1) and (1.2) are, respectively, equivalent to a hypercontractivity of the Ornstein–Uhlenbeck semigroup and a hypercontractivity of Hamilton–Jacobi equations (cf. [2,6,7]). Here, a hypercontractivity of a Cauchy problem (resp. a hypercontractivity of a semigroup) means that a solution of this Cauchy problem (resp. this semigroup) gets more integrability than its initial data in some sense.

In the following, we denote by $\|\cdot\|_\infty$ the $L^\infty(\mathbb{R}^n)$-norm with respect to the Lebesgue measure on $\mathbb{R}^n$. Hence, if f is Lipschitz continuous on $\mathbb{R}^n$, $\|Df\|_\infty$ (the $L^\infty(\mathbb{R}^n)$-norm of $|Df|$) is the Lipschitz constant of f. Let

$$\mathrm{Lip}_\alpha(\mathbb{R}^n) = \{f \in \mathrm{Lip}(\mathbb{R}^n) \mid e^f \in L^\alpha(\mathbb{R}^n)\}, \quad \alpha > 0, \tag{1.4}$$

where $\mathrm{Lip}(E)$ is the set of all Lipschitz continuous functions on E for $E = \mathbb{R}^n$ or $E = \mathbb{R}^n \times [0, T)$ $(T > 0)$.

Our goal of this paper is to derive both an optimal inequality including $\|Df\|_\infty$ for $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ $(\alpha > 0)$ and its two applications. First, we explain this optimal inequality including $\|Df\|_\infty$ for $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. This optimal inequality is a limit case of inequality (1.2) as $p \to \infty$. For $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, it is given by

$$\mathrm{Ent}(e^{\beta f}) \leqslant n \int e^{\beta f} \, dx \log\left(\frac{k_n \beta \|Df\|_\infty}{e}\right), \quad \beta > \alpha. \tag{1.5}$$

Here, the constant k_n is given by

$$k_n = \left(\frac{1}{\omega_{n-1}(n-1)!}\right)^{1/n} \tag{1.6}$$

and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit ball of $\mathbb{R}^n$. Inequality (1.5) is optimal in the sense that equality holds when $f(x) = C - \ell|x-a|$ in $\mathbb{R}^n$ for some constants $C \in \mathbb{R}$, $\ell > 0$ and $a \in \mathbb{R}^n$.

In this paper, we prove inequality (1.5) as follows: First, for $p > 1$ and $T > 0$, we use a hypercontractivity of the Cauchy problem of the Hamilton–Jacobi equation $u_t(x,t) + \frac{1}{p}|Du(x,t)|^p = 0$ in $\mathbb{R}^n \times (0,T)$. This hypercontractivity is equivalent to (1.2) by Gentil [7]. Then, by using a subsolution of this Cauchy problem and letting the parameter p tend to ∞, we show that if $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for some $\alpha > 0$, then $f \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for all $\beta > \alpha$, and the following inequality holds for $0 < \gamma < \delta < \infty$:

$$\|e^f\|_\delta \big(k_n \delta \|Df\|_\infty\big)^{n/\delta} \leqslant \|e^f\|_\gamma \big(k_n \gamma \|Df\|_\infty\big)^{n/\gamma}. \tag{1.7}$$

Here, for a function $f \in L^\alpha(\mathbb{R}^n)$ ($\alpha > 0$), we use the notation

$$\|f\|_\alpha = \left(\int \big|f(x)\big|^\alpha\, dx\right)^{1/\alpha}.$$

Then, (1.7) implies that

$$\frac{d}{d\beta}\big[\|e^f\|_\beta \big(k_n \beta \|Df\|_\infty\big)^{n/\beta}\big] \leqslant 0, \quad \beta > \alpha, \tag{1.8}$$

and from this inequality, we obtain (1.5) easily. Therefore, inequality (1.5) is a limit case of inequality (1.2) as $p \to \infty$.

On the other hand, the referee of this paper presented the following proof of inequality (1.5): In (1.2), replacing f by $e^{\beta f/p}$ and letting p tend to ∞, we surely obtain (1.5), and the extreme function $f(x) = C - \ell|x-a|$ is also obtained by letting p tend to ∞ in an extreme function of (1.2). This also shows that (1.5) is a limit case of (1.2) as $p \to \infty$. We provide this proof below.

Next, as an application of inequality (1.5), we show that the Lipschitz constant $\|Df\|_\infty$ of $f \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ can be expressed by using the entropy of f. Indeed, from (1.5), we have

$$\frac{e}{k_n \beta} \exp\left(\frac{1}{n} \frac{\mathrm{Ent}(e^{\beta f})}{\int e^{\beta f}\, dx}\right) \leqslant \|Df\|_\infty. \tag{1.9}$$

Then, we show that if $f \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ fulfills some additional condition, then

$$\|Df\|_\infty = \lim_{\beta \to \infty} \frac{e}{k_n \beta} \exp\left(\frac{1}{n} \frac{\mathrm{Ent}(e^{\beta f})}{\int e^{\beta f}\, dx}\right). \tag{1.10}$$

This is a link between the Lipschitz constant and the limit of the functional given by the left-hand side of (1.9) as $\beta \to \infty$. This link is similar to the one between $\|g\|_{\infty,\Omega}$ and the limit of $\|g\|_{p,\Omega}$ as $p \to \infty$, where $\|g\|_{q,\Omega}$ is the $L^q(\Omega)$-norm on a subset Ω of $\mathbb{R}^n$ for $1 \leqslant q \leqslant \infty$.

The second application is concerned with a hypercontractivity of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = \phi & \text{on } \mathbb{R}^n. \end{cases} \tag{1.11}$$

We show the hypercontractivity such that whenever $t > 0$, $e^{u(\cdot,t)}$ gets more integrability than e^{ϕ} with respect to the Lebesgue measure. An important factor to induce this hypercontractivity is the Lipschitz regularizing effect for (1.11).

The contents of this paper are as follows: In Section 2, we prove logarithmic Sobolev inequality (1.5) and inequality (1.7). We also provide the proof of (1.5) presented by the referee. In Section 3, we show (1.10). In Section 4, we consider a hypercontractivity for heat equation (1.11).

A special case of Theorem 2.2 below was announced in [5].

2. A logarithmic Sobolev inequality

In this section, we prove logarithmic Sobolev inequality (1.5) and inequality (1.7). In the following lemma, we use the concept of viscosity subsolutions. About it, refer to the books [1,4].

Lemma 2.1. *Let $p > 1$ and $T > 0$. For $f \in \mathrm{Lip}(\mathbb{R}^n)$, let $u \in \mathrm{C}(\mathbb{R}^n \times [0,T))$ be a viscosity subsolution of the Cauchy problem of the Hamilton–Jacobi equation*

$$\begin{cases} u_t(x,t) + \dfrac{1}{p}\left|Du(x,t)\right|^p = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot,0) = f & \text{on } \mathbb{R}^n. \end{cases} \tag{2.1}$$

If there is a constant $\alpha > 0$ such that $e^f \in L^{\alpha}(\mathbb{R}^n)$, then $e^{u(\cdot,t)} \in L^{\beta}(\mathbb{R}^n)$ for any $\beta \in (\alpha,\infty)$ and $t \in (0,T)$ and we have

$$\left\|e^{u(\cdot,t)}\right\|_{\beta} \leqslant \left\|e^f\right\|_{\alpha} \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p t}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}(\frac{\alpha}{p}+\frac{\beta}{q})}}{\beta^{\frac{n}{\alpha\beta}(\frac{\beta}{p}+\frac{\alpha}{q})}}, \tag{2.2}$$

where $q > 1$ is the exponent conjugate of p, i.e., $(1/p) + (1/q) = 1$, and L_p is the constant of (1.3).

Proof. Let

$$w(x,t) = \begin{cases} \inf_{y\in\mathbb{R}^n}[f(y) + \frac{t}{q}|\frac{x-y}{t}|^q] & \text{in } \mathbb{R}^n \times (0,T), \\ f(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

As is well known (cf. [4, Theorem 4 and Lemma 2 of Chap. 3.3], [9, Theorem 2.1]), $w \in \mathrm{Lip}(\mathbb{R}^n \times [0,T))$ and w is a viscosity solution of (2.1). By [7, Theorem 1.2], we have

$$\left\|e^{w(\cdot,t)}\right\|_{\beta} \leqslant \left\|e^f\right\|_{\alpha} \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p t}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}(\frac{\alpha}{p}+\frac{\beta}{q})}}{\beta^{\frac{n}{\alpha\beta}(\frac{\beta}{p}+\frac{\alpha}{q})}}, \quad t \in (0,T).$$

Since $u \leqslant w$ on $\mathbb{R}^n \times [0,T)$ by [9, Proposition A.2], we conclude the theorem. □

Now, we state our first result of this paper.

Theorem 2.2. *Let $\alpha > 0$ and $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. Then, $f \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for all $\beta > \alpha$, and inequality* (1.7) *holds for $0 < \gamma < \delta$, where k_n is the constant of* (1.6). *Inequality* (1.7) *is optimal in the sense that equality holds when $f(x) = C - \ell|x - a|$ for some constants $C \in \mathbb{R}$, $\ell > 0$ and $a \in \mathbb{R}^n$.*

Proof. Let $\alpha > 0$ and $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. We set $\theta = \|Df\|_\infty$. Then, the function $v(x,t) = f(x) - (\theta^p t/p)$ is a viscosity subsolution of (2.1) (cf. [1, Proposition 5.1 of Chap. 2]). By Lemma 2.1, we have, for any $\beta \in (\alpha, \infty)$ and $t \in (0, \infty)$,

$$\|e^f\|_\beta \leqslant \|e^f\|_\alpha e^{\theta^p t/p} t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \times \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}(\frac{\alpha}{p}+\frac{\beta}{q})}}{\beta^{\frac{n}{\alpha\beta}(\frac{\beta}{p}+\frac{\alpha}{q})}}, \tag{2.3}$$

where $q > 1$ is the exponent conjugate of p, i.e., $(1/p) + (1/q) = 1$, and L_p is the constant of (1.3). By minimizing the right-hand side of (2.3) with respect to the t-variable, we have

$$\begin{aligned}\|e^f\|_\beta &\leqslant \|e^f\|_\alpha \left(\frac{\theta^p e}{n\frac{\beta-\alpha}{\alpha\beta}}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \times \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}(\frac{\alpha}{p}+\frac{\beta}{q})}}{\beta^{\frac{n}{\alpha\beta}(\frac{\beta}{p}+\frac{\alpha}{q})}} \\ &= \|e^f\|_\alpha \left(\frac{\theta e (L_p)^{1/p}}{p}\right)^{\frac{n}{\alpha}-\frac{n}{\beta}} \times \alpha^{\frac{n}{\alpha}}\beta^{-\frac{n}{\beta}}.\end{aligned}$$

Hence, we obtain

$$\|e^f\|_\beta \big(k_p^{(n)}\beta\theta\big)^{n/\beta} \leqslant \|e^f\|_\alpha \big(k_p^{(n)}\alpha\theta\big)^{n/\alpha}, \tag{2.4}$$

where

$$k_p^{(n)} = \frac{e(L_p)^{1/p}}{p} = \left(\frac{n}{eq}\right)^{1/q}\left[\Gamma\left(\frac{n}{q}+1\right)\right]^{-1/n} \frac{e}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{1/n}. \tag{2.5}$$

Now, letting p tend to ∞ in (2.5), i.e., letting q tend to 1 in (2.5), we conclude that

$$\begin{aligned}\lim_{p\to\infty} k_p^{(n)} &= \lim_{q\to 1}\left(\frac{n}{eq}\right)^{1/q}\left[\Gamma\left(\frac{n}{q}+1\right)\right]^{-1/n} \frac{e}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{1/n} \\ &= \frac{1}{\sqrt{\pi}}\frac{1}{(n!)^{1/n}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{1/n} = \left(\frac{1}{\omega_{n-1}(n-1)!}\right)^{1/n} = k_n.\end{aligned} \tag{2.6}$$

Thus, we have obtained $\|e^f\|_\beta (k_n\beta\theta)^{n/\beta} \leqslant \|e^f\|_\alpha (k_n\alpha\theta)^{n/\alpha}$ for $\alpha \leqslant \beta < \infty$.

Next, let $0 < \gamma < \delta$. When $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$, inequality (1.7) easily follows from the arguments above. When $f \notin \mathrm{Lip}_\gamma(\mathbb{R}^n)$, we have $f \in \mathrm{Lip}(\mathbb{R}^n)$ but $f \notin L^\gamma(\mathbb{R}^n)$, so that inequality (1.7) is trivial. Hence, inequality (1.7) holds for $0 < \gamma < \delta$.

When $f(x) = C - \ell|x - a|$ for some constants $C \in \mathbb{R}$, $\ell > 0$ and $a \in \mathbb{R}^n$, we see that $\ell = \|Df\|_\infty$ and inequality (1.7) is reduced to equality. □

Remark 2.3. (1) In general, the condition $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for some $\alpha > 0$ does not imply that $f \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for $0 < \beta < \alpha$. Indeed, let $\alpha > 0$ and $f(x) = -\theta \log(1 + |x|)$, where $\theta > n/\alpha$. Then, $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ but $f \notin \mathrm{Lip}_\beta(\mathbb{R}^n)$ for $\beta \leqslant n/\theta$ $(< \alpha)$.

(2) Let $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ for some $\gamma > 0$. Letting δ tend to ∞ in (1.7), we have

$$\|e^f\|_\infty \leqslant \|e^f\|_\gamma \big(k_n \gamma \|Df\|_\infty\big)^{n/\gamma}.$$

This gives an upper bound of $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ by using $\|e^f\|_\gamma$ and $\|Df\|_\infty$.

Next, we show inequality (1.5).

Theorem 2.4. *Let $\alpha > 0$. For $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, we have inequality* (1.5)*. Inequality* (1.5) *is optimal in the sense that equality holds when $f(x) = C - \ell|x - a|$ for some constants $C \in \mathbb{R}$, $\ell > 0$ and $a \in \mathbb{R}^n$.*

Proof. Let $\alpha > 0$ and $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. Set $\theta = \|Df\|_\infty$. Since

$$\frac{d}{d\beta}\|e^f\|_\beta = \frac{1}{\beta^2}\|e^f\|_\beta^{1-\beta}\,\mathrm{Ent}\big(e^{\beta f}\big),$$

$$\frac{d}{d\beta}(k_n\theta\beta)^{n/\beta} = (k_n\theta\beta)^{n/\beta}\frac{n}{\beta^2}\big[1 - \log(k_n\theta\beta)\big],$$

we have easily (1.5) by (1.8). When $f(x) = C - \ell|x - a|$ for some constants $C \in \mathbb{R}$, $\ell > 0$ and $a \in \mathbb{R}^n$, we have $\|Df\|_\infty = \ell$ and inequality (1.7) is reduced to equality. Thus, equality holds in (1.8), so that equality holds in (1.5). □

Remark 2.5. (1) By the proof of Theorem 2.4, we see that inequality (1.7) is equivalent to logarithmic Sobolev inequality (1.5).

(2) By Gentil [7], a hypercontractivity for (2.1) is equivalent to (1.2). Recall that the key to derive (1.7) is to let the parameter p tend to ∞ in this hypercontractivity. This implies that (1.5) is a limit case of (1.2) as $p \to \infty$.

The following proof of Theorem 2.4 was presented by the referee.

Proof. In (1.2), replace f by $e^{\beta f/p}$. Then, we obtain

$$\mathrm{Ent}\big(e^{\beta f}\big) \leqslant n \int e^{\beta f}\,dx \log\left(\frac{(L_p)^{1/p}\beta}{p}\frac{(\int |Df|^p e^{\beta f}\,dx)^{1/p}}{(\int e^{\beta f}\,dx)^{1/p}}\right). \tag{2.7}$$

It is easy to see that

$$\limsup_{p\to\infty}\frac{(\int |Df|^p e^{\beta f}\,dx)^{1/p}}{(\int e^{\beta f}\,dx)^{1/p}} \leqslant \|Df\|_\infty.$$

On the other hand, for any $\lambda < \|Df\|_\infty$ which is close to $\|Df\|_\infty$, we see that the set $E_\lambda := \{x \in \mathbb{R}^n \mid |Df(x)| > \lambda\}$ has a positive Lebesgue measure. Hence,

$$\liminf_{p\to\infty} \frac{(\int |Df|^p e^{\beta f}\,dx)^{1/p}}{(\int e^{\beta f}\,dx)^{1/p}} \geqslant \lambda \liminf_{p\to\infty} \frac{(\int_{E_\lambda} e^{\beta f}\,dx)^{1/p}}{(\int e^{\beta f}\,dx)^{1/p}} = \lambda.$$

Letting λ tend to $\|Df\|_\infty$, we see that

$$\lim_{p\to\infty} \frac{(\int |Df|^p e^{\beta f}\,dx)^{1/p}}{(\int e^{\beta f}\,dx)^{1/p}} = \|Df\|_\infty.$$

Letting p tend to ∞ in (2.7), we have inequality (1.5) by (2.5) and (2.6).

By Gentil [7, Theorem 1.1], a function f for which (2.7) is reduced to equality is given by

$$e^{\beta f(x)} = k \exp\bigl(-m|x-a|^q\bigr)$$

for some constant $k, m > 0$ and $a \in \mathbb{R}^n$, where $(1/p) + (1/q) = 1$. Letting p tend to ∞, i.e., letting q tend to 1, we conclude that an extreme function for (1.5) is $(\log k - m|x-a|)/\beta$. □

3. An expression of Lipschitz constants

In this section, we consider (1.10). We denote by $\mathrm{Lip}_{\log}(\mathbb{R}^n)$ the set of functions $f \in \mathrm{Lip}(\mathbb{R}^n)$ such that there exists $a \in \mathbb{R}^n$ satisfying

$$-\theta \log\bigl(1 + |x-a|\bigr) \geqslant f(x) - f(a), \quad x \in \mathbb{R}^n \ \bigl(\theta := \|Df\|_\infty > 0\bigr).$$

It is easy to see that if $f \in \mathrm{Lip}_{\log}(\mathbb{R}^n)$, then $f \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for $\beta > n/\|Df\|_\infty$. As an example of functions of $\mathrm{Lip}_{\log}(\mathbb{R}^n)$, we have

$$f(x) = f(a) - \int_0^{|x-a|} g(t)\,dt, \quad x \in \mathbb{R}^n,$$

where $a \in \mathbb{R}^n$ and $g : (0, \infty) \to \mathbb{R}$ is a Lebesgue measurable function such that $A/(1+t) \leqslant g(t) \leqslant A$ a.e. on $(0, \infty)$ for some constant $A > 0$.

Now, we state our result of this section.

Theorem 3.1. *For $f \in \mathrm{Lip}_{\log}(\mathbb{R}^n)$, we have* (1.10).

Proof. 1. By the definition of $\mathrm{Ent}(e^{\beta f})$, we have

$$\frac{e}{k_n \beta} \exp\biggl(\frac{1}{n} \frac{\mathrm{Ent}(e^{\beta f})}{\int e^{\beta f}\,dx}\biggr) = \frac{e}{k_n \beta} \biggl(\int e^{\beta f}\,dx\biggr)^{-1/n} \exp\biggl(\frac{\beta}{n} \frac{\int f e^{\beta f}\,dx}{\int e^{\beta f}\,dx}\biggr). \tag{3.1}$$

We note that, for $\theta > 0$ and $a \in \mathbb{R}^n$,

$$\int \big(1+|x-a|\big)^{-\beta\theta}\,dx = \omega_{n-1}(n-1)!\frac{\Gamma(\beta\theta-n)}{\Gamma(\beta\theta)} \quad (\beta > n/\theta), \tag{3.2}$$

$$\int |x-a|\big(1+|x-a|\big)^{-\beta\theta}\,dx = \omega_{n-1}n!\frac{\Gamma(\beta\theta-n-1)}{\Gamma(\beta\theta)} \quad \left(\beta > \frac{n+1}{\theta}\right), \tag{3.3}$$

$$\int e^{-\beta\theta|x-a|}\,dx = \omega_{n-1}\frac{(n-1)!}{(\beta\theta)^n} \quad (\beta > 0). \tag{3.4}$$

2. Let $f \in \mathrm{Lip}_{\log}(\mathbb{R}^n)$ and set $\theta = \|Df\|_\infty$. Since

$$-\theta\log\big(1+|x-a|\big) \geqslant f(x)-f(a) \geqslant -\theta|x-a|, \quad x \in \mathbb{R}^n,$$

we have, by (3.2) and (3.4),

$$e^{\beta f(a)}\omega_{n-1}(n-1)!\frac{\Gamma(\beta\theta-n)}{\Gamma(\beta\theta)} \geqslant \int e^{\beta f(x)}\,dx \geqslant e^{\beta f(a)}\omega_{n-1}\frac{(n-1)!}{(\beta\theta)^n}, \tag{3.5}$$

whenever $\beta > n/\theta$. On the other hand, since

$$0 \leqslant \theta\log\big(1+|x-a|\big) \leqslant f(a)-f(x) \leqslant \theta|x-a|, \quad x \in \mathbb{R}^n,$$

we have

$$0 \leqslant \big(f(a)-f(x)\big)e^{\beta f(x)} \leqslant \theta|x-a|\exp\big[\beta\big(f(a)-\theta\log\big(1+|x-a|\big)\big)\big], \quad x \in \mathbb{R}^n,$$

so that

$$f(a)e^{\beta f(x)} - \theta e^{\beta f(a)}|x-a|\big(1+|x-a|\big)^{-\beta\theta} \leqslant f(x)e^{\beta f(x)}, \quad x \in \mathbb{R}^n.$$

Thus, by (3.3), we obtain

$$f(a)\int e^{\beta f(x)}\,dx - \theta e^{\beta f(a)}\omega_{n-1}n!\frac{\Gamma(\beta\theta-n-1)}{\Gamma(\beta\theta)} \leqslant \int f(x)e^{\beta f(x)}\,dx, \tag{3.6}$$

whenever $\beta > (n+1)/\theta$. Dividing each term of (3.6) by $\int e^{\beta f(x)}\,dx$ and using (3.5), we have

$$\frac{\beta}{n}\frac{\int fe^{\beta f}\,dx}{\int e^{\beta f}\,dx} \geqslant \frac{\beta}{n}f(a) - (\beta\theta)^{n+1}\frac{\Gamma(\beta\theta-n-1)}{\Gamma(\beta\theta)}, \tag{3.7}$$

whenever $\beta > (n+1)/\theta$. Therefore, by (1.9), (3.5) and (3.7), we conclude that

$$\begin{aligned}
\theta &\geqslant \frac{e}{k_n\beta}\left(\int e^{\beta f}\,dx\right)^{-1/n}\exp\left(\frac{\beta}{n}\frac{\int fe^{\beta f}\,dx}{\int e^{\beta f}\,dx}\right)\\
&\geqslant \frac{e}{k_n\beta}\left(e^{\beta f(a)}\omega_{n-1}(n-1)!\frac{\Gamma(\beta\theta-n)}{\Gamma(\beta\theta)}\right)^{-1/n}\\
&\quad\times\exp\left[\frac{\beta}{n}f(a) - (\beta\theta)^{n+1}\frac{\Gamma(\beta\theta-n-1)}{\Gamma(\beta\theta)}\right]
\end{aligned}$$

$$= \frac{e}{\beta}\left(\frac{\Gamma(\beta\theta)}{\Gamma(\beta\theta-n)}\right)^{1/n}\exp\left[-(\beta\theta)^{n+1}\frac{\Gamma(\beta\theta-n-1)}{\Gamma(\beta\theta)}\right],$$

whenever $\beta > (n+1)/\theta$. Since

$$\frac{\Gamma(\beta\theta)}{\Gamma(\beta\theta-k)} = (\beta\theta-1)(\beta\theta-2)\cdots(\beta\theta-k), \quad k\in\mathbb{N},\ \beta\theta>k,$$

we have just obtained

$$\begin{aligned}\theta &= \lim_{\beta\to\infty}\frac{e}{\beta}\left(\frac{\Gamma(\beta\theta)}{\Gamma(\beta\theta-n)}\right)^{1/n}\exp\left[-(\beta\theta)^{n+1}\frac{\Gamma(\beta\theta-n-1)}{\Gamma(\beta\theta)}\right]\\ &\leqslant \liminf_{\beta\to\infty}\frac{e}{k_n\beta}\left(\int e^{\beta f}\,dx\right)^{-1/n}\exp\left(\frac{\beta}{n}\frac{\int fe^{\beta f}\,dx}{\int e^{\beta f}\,dx}\right)\\ &\leqslant \limsup_{\beta\to\infty}\frac{e}{k_n\beta}\left(\int e^{\beta f}\,dx\right)^{-1/n}\exp\left(\frac{\beta}{n}\frac{\int fe^{\beta f}\,dx}{\int e^{\beta f}\,dx}\right)\\ &\leqslant\theta,\end{aligned}$$

which implies (1.10) by (3.1). □

Remark 3.2. We do not know whether the functional of the right-hand side of (1.9) is non-decreasing or non-increasing in β.

4. Hypercontractivity of heat equation

In this section, as an application of (1.5), we consider a hypercontractivity of heat equation (1.11). Recall that, by Remark 2.5, (1.7) is equivalent to logarithmic Sobolev inequality (1.5).

We show a hypercontractivity in heat equation (1.11) such that whenever $e^{\phi}\in L^{\alpha}(\mathbb{R}^n)$ ($\alpha>0$), we have $e^{u(\cdot,t)}\in L^{\beta}(\mathbb{R}^n)$ for any $\beta>\alpha$ and $t>0$.

If ϕ of (1.11) is a bounded and uniformly continuous function on $\mathbb{R}^n$, (1.11) has a unique solution $u\in C^{2,1}(\mathbb{R}^n\times(0,\infty))\cap C(\mathbb{R}^n\times[0,\infty))$ such that u is bounded on $\mathbb{R}^n\times[0,\infty)$. Then, for any $\alpha>0$, we observe that $e^{u(\cdot,t)}\notin L^{\alpha}(\mathbb{R}^n)$. To overcome this situation, we make the following assumption:

$$\begin{cases}\phi \text{ of (1.11) has the form such that } \phi(x) = -\epsilon|x|+\rho(x) \text{ in } \mathbb{R}^n,\\ \text{where } \epsilon>0 \text{ is a constant and } \rho \text{ is a bounded and}\\ \text{uniformly continuous function on } \mathbb{R}^n.\end{cases} \tag{4.1}$$

In this case, for any $\alpha>0$, we observe that $e^{u(\cdot,t)}\in L^{\alpha}(\mathbb{R}^n)$ whenever $t>0$, since $\epsilon>0$. Although the influence $\epsilon>0$ is strong, we will seek a hypercontractivity which is independent of the parameter ϵ and arises from an intrinsic character of (1.11). Such an intrinsic character of (1.11) considered here is the Lipschitz regularizing effect of (1.11) explained below. In the following, we set

$$C_n = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}. \tag{4.2}$$

Lemma 4.1. *Assume* (4.1). *Then, we have*

$$\left\| Du(\cdot,t)\right\|_\infty \leqslant \epsilon + \frac{C_n}{\sqrt{t}}\|\rho\|_\infty, \quad t>0. \tag{4.3}$$

In the following, we define the operator $\{P_t\}$ on $\mathrm{C}_p(\mathbb{R}^n)$ by

$$P_t\psi(x) = \int \Phi(x-y,t)\psi(y)\,dy, \quad x\in\mathbb{R}^n,\ t>0, \tag{4.4}$$

where $\mathrm{C}_p(\mathbb{R}^n)$ is the set of all continuous functions on $\mathbb{R}^n$ with polynomial growth order, and $\Phi(x,t) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}$. Note that, for a uniformly continuous function ϕ on $\mathbb{R}^n$, $P_t\phi$ is a solution of heat equation (1.11).

Proof. Let $\eta(x) = -\epsilon|x|$. It is easy to see that $|P_t\eta(x) - P_t\eta(y)| \leqslant \epsilon|x-y|$. Next, we have

$$D(P_t\rho)(x) = (4\pi t)^{-n/2}\int \rho(y)e^{-\frac{|x-y|^2}{4t}}\left(-\frac{x-y}{2t}\right)dy.$$

Since

$$(4\pi t)^{-n/2}\int |x-y|e^{-\frac{|x-y|^2}{4t}}\,dy = 2C_n\sqrt{t}, \tag{4.5}$$

we have

$$\left|D(P_t\rho)(x)\right| \leqslant (4\pi t)^{-n/2}\int \|\rho\|_\infty e^{-\frac{|x-y|^2}{4t}}\left|\frac{x-y}{2t}\right|dy = \frac{C_n}{\sqrt{t}}\|\rho\|_\infty.$$

The proof is complete. □

We prepare one more lemma.

Lemma 4.2. *Assume* (4.1). *Then, there exists a bounded and non-decreasing function ν on* $(0,\infty)$ *such that* $\nu(0+) = 0$ *and*

$$\left\|u(\cdot,t)-\phi\right\|_\infty \leqslant 2\epsilon C_n\sqrt{t} + \nu(t), \quad t>0. \tag{4.6}$$

Proof. Let $\eta(x) = -\epsilon|x|$. By (4.5), it is easy to see that $|P_t\eta(x) - \eta(x)| \leqslant 2\epsilon C_n\sqrt{t}$. Next, for any $\delta>0$, we have

$$\begin{aligned} P_t\rho(x)-\rho(x) = {} & \int_{B(x,\delta)} \Phi(x-y,t)\big[\rho(y)-\rho(x)\big]\,dy \\ & + \int_{\mathbb{R}^n\setminus B(x,\delta)} \Phi(x-y,t)\big[\rho(y)-\rho(x)\big]\,dy \\ =: {} & I + J, \end{aligned}$$

where $B(x,\delta) = \{y \in \mathbb{R}^n \mid |y-x| \leqslant \delta\}$. Since ρ is a uniformly continuous function on $\mathbb{R}^n$, we have

$$|I| \leqslant \max\bigl\{\bigl|\rho(z) - \rho(w)\bigr|,\ z, w \in \mathbb{R}^n,\ |z-w| \leqslant \delta\bigr\} =: f(\delta)$$

and $f(\delta) \to 0$ as $\delta \to 0$. On the other hand, we have

$$\begin{aligned} |J| &\leqslant 2\|\rho\|_\infty \int\limits_{\mathbb{R}^n \setminus B(x,\delta)} \Phi(x-y,t)\,dy \\ &= \frac{2\|\rho\|_\infty}{(4\pi t)^{n/2}} \int\limits_{\delta}^{\infty} \omega_{n-1} r^{n-1} e^{-r^2/4t}\,dr \\ &= \frac{2\omega_{n-1}}{\pi^{n/2}} \|\rho\|_\infty \int\limits_{\frac{\delta}{2\sqrt{t}}}^{\infty} s^{n-1} e^{-s^2}\,ds =: g(\delta, t). \end{aligned}$$

Hence,

$$\bigl\|u(\cdot,t) - \phi\bigr\|_\infty \leqslant 2\epsilon C_n \sqrt{t} + f(\delta) + g(\delta,t), \quad \delta, t > 0.$$

Now, for example, let

$$\nu(t) := f\bigl(t^{1/4}\bigr) + g\bigl(t^{1/4}, t\bigr), \quad t > 0.$$

Then, it is easy to see that the function ν on $(0,\infty)$ is bounded and non-decreasing with $\nu(0+) = 0$ and (4.6) holds. □

Now, we state our result of this section.

Theorem 4.3. *Assume* (4.1). *Let* $\alpha > 0$. *Then, for a unique solution* $u(\cdot,t)$ *of* (1.11), *we have, for* $\beta > \alpha$, $t > 0$,

$$\bigl\|e^{u(\cdot,t)}\bigr\|_\beta \leqslant \bigl\|e^\phi\bigr\|_\alpha e^{2\epsilon C_n\sqrt{t}+\nu(t)} (k_n\alpha)^{n/\alpha} (k_n\beta)^{-n/\beta} \left(\frac{C_n}{\sqrt{t}}\|\rho\|_\infty + \epsilon\right)^{\frac{n}{\alpha}-\frac{n}{\beta}}, \tag{4.7}$$

where ν *is the function of Lemma* 4.2. *In particular, we have, for* $\beta > \alpha$, $t > 0$,

$$\limsup_{\epsilon \searrow 0} \frac{\|e^{u(\cdot,t)}\|_\beta}{\|e^\phi\|_\alpha} \leqslant e^{\nu(t)} (k_n\alpha)^{n/\alpha} (k_n\beta)^{-n/\beta} \left(\frac{C_n}{\sqrt{t}}\|\rho\|_\infty\right)^{\frac{n}{\alpha}-\frac{n}{\beta}}. \tag{4.8}$$

Proof. Since

$$u(x,t) \leqslant -\epsilon|x| + 2\epsilon C_n\sqrt{t} + \|\rho\|_\infty,$$

we have $u(\cdot,t) \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for $\alpha, t > 0$ by Lemma 4.1. By (1.7), we have

$$\|e^{u(\cdot,t)}\|_\beta \leqslant \|e^{u(\cdot,t)}\|_\alpha \big(k_n\|Du(\cdot,t)\|_\infty \alpha\big)^{n/\alpha}\big(k_n\|Du(\cdot,t)\|_\infty \beta\big)^{-n/\beta}.$$

Since $(n/\alpha) - (n/\beta) > 0$, we have, by Lemma 4.1,

$$\|Du(\cdot,t)\|_\infty^{\frac{n}{\alpha}-\frac{n}{\beta}} \leqslant \left(\epsilon + \frac{C_n}{\sqrt{t}}\|\rho\|_\infty\right)^{\frac{n}{\alpha}-\frac{n}{\beta}}, \quad t > 0.$$

By Lemma 4.2, we conclude (4.7). □

Remark 4.4. We note that the hypercontractivity of (4.8) is due to the Lipschitz regularizing effect of (1.11), since it is independent of the parameter ϵ. By the proof of Theorem 4.3, in another Cauchy problem, there exists a possibility such that the similar hypercontractivity occurs provided that the Lipschitz regularizing effect occurs in this Cauchy problem.

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