ON HANKEL OPERATOR RANGES, MEROMORPHIC PSEUDO-CONTINUATIONS AND FACTORIZATION OF OPERATOR-VALUED ANALYTIC FUNCTIONS†

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ABSTRACT

Necessary and sufficient conditions for the existence of meromorphic pseudo-continuations of bounded type for bounded operator-valued analytic functions in the unit disc are given in terms of the ranges of the induced Hankel operators. Related factorizations of the functions are obtained.

In [1] Douglas and Helton raised the problem of studying the class of operator-valued analytic functions in the unit disc D that have meromorphic pseudo-continuations to $D_e = \{z \mid 1 < |z| \leqslant \infty\}$ which are of bounded type.

In the scalar case a function in H^2 has a meromorphic pseudo-continuation of bounded type in D_e if and only if it is a non-cyclic vector of the left shift in H^2. This has been proved by Douglas, Shapiro and Shields in [2] as well as by Kriete in [5].

Our generalization connects the meromorphic pseudo-continuation of a bounded operator-valued function in D with the range of the Hankel operator induced by it. The main result, stated below, uses concepts defined in the sequel.

THEOREM 1. *Let N, M be two separable Hilbert spaces and let A be a $B(N, M)$-valued bounded analytic function in the unit disc. Then A has a meromorphic pseudo-continuation of bounded type in D_e if and only if the Hankel operator, H_A, induced by A is associated with an inner function having a scalar multiple.*

For a separable Hilbert space N we let $L^2(N)$ be the space of all (equivalence classes) of weakly measurable N-valued functions on the unit circle which have square integrable norms. Functions in $L^2(N)$ have natural Fourier expansions [4] and we denote by $H^2(N)$ the subspace of $L^2(N)$ of all functions whose Fourier coefficients vanish for negative indices. $B(N, M)$ will denote the space of all bounded linear operators from N to another Hilbert space M. For separable Hilbert spaces N and M we let $H^\infty(N, M)$ be the space of all bounded $B(N, M)$-valued analytic functions in the unit disc. Elements of $H^\infty(N, M)$ have strong radial limits a.e. on the unit circle [7]. Elements of $H^2(N)$ have analytic extensions into the unit disc from which they can be recaptured as a.e. radial limits.

A $B(N, M)$-valued function H is meromorphic of bounded type in a domain V if it can be represented as $H = G/g$, where G is a bounded $B(N, M)$-valued analytic function in V and g is a scalar bounded analytic function in V. If H is $B(N, M)$-valued meromorphic of bounded type in D_e then the radial limits $\lim_{R \to 1^+} H(Re^{it})$ exist a.e. in the strong sense.

A function Q in $H^\infty(N, N)$ is called *rigid* if the boundary values $Q(e^{it})$ are partially isometric a.e. with a fixed initial space and *inner* if a.e. $Q(e^{it})$ is unitary.

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An inner function Q in $H^\infty(N, N)$ has a scalar multiple q if there exists a function Q_1 in $H^\infty(N, N)$ such that $Q(z)\,Q_1(z) = Q_1(z)\,Q(z) = q(z)\,I_N$, for all z in the open unit disc.

We define the right shift S in $H^2(N)$ letting $(Sf)(z) = zf(z)$ for all f in $H^2(N)$. The adjoint S^*, called the left shift, is given by $(S^*f)(z) = (f(z)-f(0))/z$. A subspace of $H^2(N)$ will be called right invariant or left invariant according as it is invariant under the right or left shift respectively. Right invariant subspaces will be also referred to simply as invariant subspaces.

Let R be the unitary map in $L^2(N)$ given by $(Rf)(e^{it}) = f(e^{-it})$ for all $f \in L^2(N)$ and let $P_{H^2(M)}$ be the orthogonal projection of $L^2(M)$ onto $H^2(M)$. Given A in $H^\infty(N, M)$, we define H_A as the bounded operator from $H^2(N)$ into $H^2(M)$ given by $H_A f = P_{H^2(M)}\,A(Rf)$ for all f in $H^2(N)$. Now it is easy to check that $\overline{\text{Range } H_A}$ is a left invariant subspace of $H^2(M)$, hence $H^2(M) \ominus \overline{\text{Range } H_A}$ is an invariant subspace of $H^2(M)$ and thus is represented, via the Beurling–Lax theorem, as $QH^2(M)$ for some rigid function Q. We say that S is associated with the Hankel operator H_A. In the case that Q is inner, any other inner function Q_1 associated with H_A differs from Q by a constant unitary factor from the right.

We shall need the following lemma, from [**4**; p. 75]. $K^2(N)$ denotes the subspace of $L^2(N)$ of all functions whose non-negative indexed Fourier coefficients vanish.

LEMMA 1. *Let $Q \in H^\infty(N, N)$ be an inner function. Then $f \in H^2(N) \ominus QH^2(N)$ if and only if $Q^*f \in K^2(N) = L^2(N) \ominus H^2(N)$. Here $(Q^*f)(e^{it}) = Q(e^{it})^*f(e^{it})$.*

The next two lemmas are due to Douglas and Helton [**1**]. For completeness we shall include proofs.

LEMMA 2. *Let A be a $B(N, M)$-valued weakly measurable essentially bounded function on $\mathbb{T}$. Suppose there exists a function $\phi \neq 0$ in H^∞ such that ϕA belongs to $H^\infty(N, M)$; then there exists an inner function q in H^∞ for which qA is in $H^\infty(N, M)$.*

Proof. Let $J = \{\psi \in H^\infty \mid \psi A \in H^\infty(N, M)\}$. It is simple to check that J is a w^*-closed invariant subspace of H^∞. By a theorem of Srinivasan [**4**], $J = qH^\infty$ for an inner function q in H^∞.

It follows easily from Lemma 2 that in a representation $H = G/g$ of meromorphic functions of bounded type in D_e the denominator can always be taken to be an inner function in D_e.

LEMMA 3. *Let $Q \in H^\infty(N, N)$ be an inner function; then it has a meromorphic pseudo-continuation of bounded type if and only if it has a scalar multiple.*

Proof. Assume Q has a scalar multiple; then it has a scalar inner multiple q. Thus there exists an Ω in $H^\infty(N, N)$ such that $Q(z)\,\Omega(z) = \Omega(z)\,Q(z) = q(z)\,I_N$, for all z in the open unit disc. In fact necessarily Ω is also an inner function. For $z \in D_e$ we define $\hat{Q}$ by $\hat{Q}(z) = Q(1/\bar{z})^{*-1}$. Thus $\hat{Q}(z)$ is analytic except at most at the points for which $q(1/\bar{z}) = 0$. Now obviously $\Omega(\bar{z}^{-1})^*$ and $\overline{q(\bar{z}^{-1})}$ are both bounded and analytic in D_e and $\hat{Q}(z) = \Omega(1/\bar{z})^*/\overline{q(1/\bar{z})}$ and thus $\hat{Q}$ is meromorphic of bounded type. Clearly from the definition $\hat{Q}$ it follows that

$$\lim_{R\to 1+} \hat{Q}(Re^{it}) = \lim_{R\to 1+} Q\left(\frac{1}{R}e^{it}\right)^{*-1} = \lim_{r\to 1-} Q(re^{it})^{*-1} = Q(e^{it})^{*-1}$$

a.e., the limits taken in the strong sense. But a.e. $Q(e^{it})^{*-1} = Q(e^{it})$ as Q is inner and hence $\hat{Q}$ is the required pseudo-continuation of Q.

Conversely, assume Q has a meromorphic pseudo-continuation of bounded type in D_e. From the remark following Lemma 2, $\hat{Q}$ has a representation

$$\hat{Q}(e^{it}) = \Omega(e^{it})/q(e^{it})$$

for some inner function q in D_e. Clearly $\overline{q(e^{it})}$, which is inner in D, is a scalar multiple of Q.

LEMMA 4. *Let $A \in H^\infty(N, M)$; then* $\{\text{Range } H_A\}^\perp \supset QH^2(M)$ *for some inner function Q in $H^\infty(M, M)$ if and only if $A = QB$ for some bounded $B(N, M)$-valued analytic function B in D_e vanishing at infinity. The equality* $\{\text{Range } H_A\}^\perp = QH^2(M)$ *occurs if and only if Q and C, with $C(e^{it}) = e^{-it}B(e^{it})^*$, have no common non-trivial right inner factor. In this case Q is uniquely determined up to a constant unitary factor W on the right and W^* on the left for B.*

Proof. Assume A has such a representation. For $\xi \in N$,

$$H_A(\xi z^n) = P_{H^2(M)}AR(\xi z^n) = P_{H^2(M)}\bar{z}^n QB\xi.$$

Since $z^m Q^* QB\xi$ is in $K^2(M)$ and $Q^* K^2(M) \subset K^2(M)$, we have by Lemma 1 $H_A(\xi z^n) \in H^2(M) \ominus QH^2(M)$ for all $\xi \in N$ and $n \geqslant 0$. Since the functions ξz^n span $H^2(N)$ it follows that Range $H_A \subset H^2(M) \ominus QH^2(M)$.

Conversely, suppose A is in $H^\infty(N, M)$ and Range $H_A \subset H^2(M) \ominus QH^2(M)$ for some inner function Q. Let $\xi \in N$; then $H_A\xi = A\xi$ and, by Lemma 1, $Q^*A\xi$ is in $K^2(M)$. Since $Q^*A\bar{z}^n\xi = \bar{z}^n Q^*A\xi$, it follows that Q^*A can be extended to a bounded operator from $zK^2(N)$ into $K^2(M)$ which intertwines the corresponding multiplication by $\bar{z}$ operators. Thus, by a theorem of Lax [6], it is given by multiplication by a bounded $B(N, M)$-valued analytic function B in D_e. So $B = Q^*A$ or $A = QB$. Since $BzK^2(N) \subset K^2(M)$ we have $B(\infty) = 0$.

Let $K = H^2(M) \ominus QH^2(M)$ and let $\tilde{K} = H^2(M) \ominus \tilde{Q}H^2(M)$ where $\tilde{Q}(z) = Q(\bar{z})^*$. Define the map $\tau: K \to \tilde{K}$ by $(\tau f)(e^{it}) = e^{-it}\tilde{Q}(e^{it})f(e^{-it})$. It is known [3] that τ is unitary and satisfies $\tau P_K = P_{\tilde{K}}\tau$ where P_K and $P_{\tilde{K}}$ are the orthogonal projections of $H^2(M)$ on K and $\tilde{K}$ respectively. Hence Range H_A is dense in K if and only if Range τH_A is dense in $\tilde{K}$. Since we assume Range $H_A \subset H^2(M) \ominus QH^2(M)$ it follows that $H_A f = P_{H^2(M)}A(Rf) = P_K A(Rf)$ and hence $\tau H_A f = \tau P_K A(Rf) = P_{\tilde{K}}\tau A(Rf)$. Now $A = QB$; hence $(\tau A(Rf))(e^{it}) = e^{-it}\tilde{Q}(e^{it})Q(e^{-it})B(e^{-it})f(e^{it}) = D(e^{it})f(e^{it})$ where $D(e^{it}) = e^{-it}B(e^{-it})$. Thus $D \in H^\infty(N, M)$ and the problem reduces to the density of the range of the operator $\Gamma_D: H^2(N) \to \tilde{K}$ defined by $\Gamma_D f = P_{\tilde{K}}Df$. Now the range of Γ_D is not dense in $\tilde{K}$ if and only if, for some non-zero F in $\tilde{K}$, $(F, P_{\tilde{K}}Dz^n\xi) = (F, Dx^n\xi) = 0$ for all $n \geqslant 0$ and ξ in N. Since $(F, \tilde{Q}z^n\xi) = 0$ for all $n \geqslant 0$ and ξ in M, F is orthogonal to span of the ranges of $\tilde{Q}$ and D which is given by $SH^2(N)$ for the greatest common non-trivial left inner factor S of D and $\tilde{Q}$. Thus

Γ_D has dense range if and only if D, $\tilde{Q}$ have no non-trivial common left inner factor, or $\tilde{D}$, Q have no non-trivial common right inner factor. But

$$\tilde{D}(e^{it}) = e^{-it} B(e^{it})^* = C(e^{it}).$$

Since two inner functions corresponding to the same invariant subspace can differ at most by a constant unitary factor on the right the uniqueness part follows.

Proof of Theorem 1. Suppose first that with the Hankel operator H_A is associated an inner function Q having a scalar multiple. By Lemma 4 we have $A = QB$ with B bounded analytic in D_e and vanishing at ∞. Let $\hat{Q}$ be the peusdo-continuation of Q which is, by Lemma 3, meromorphic of bounded type in D_e. So if we let $\hat{A} = \hat{Q}B$ then $\hat{A}$ is meromorphic of bounded type in D_e. Moreover since Q and B have boundary values a.e. on $\mathbb{T}$ so has $\hat{A}$ and a.e.

$$\hat{A}(e^{it}) = \hat{Q}(e^{it}) B(e^{it}) = Q(e^{it}) B(e^{it}) = A(e^{it}).$$

Thus $\hat{A}$ is the required pseudo-continuation.

Conversely assume $A \in H^\infty(N, M)$ has a meromorphic pseudo-continuation of bounded type $\hat{A}$. Thus $\hat{A}(z) = B(z)/b(z)$ with B bounded $B(N, M)$-valued analytic in D_e. By Lemma 2 we may assume b is inner in D_e. Without loss of generality we may assume $B(\infty) = 0$. Since boundary values exist in the strong sense, we have, for each $\xi \in N$, $A(e^{it}) \xi = B(e^{it}) \xi/b(e^{it})$ a.e. Let $b_1(e^{it}) = \overline{b(e^{it})}$; then b_1 is inner in D and clearly, for all $n \geqslant 0$, $\overline{b_1(e^{it})} A(e^{it}) e^{-int} = e^{-int} B(e^{it}) \xi \in K^2(M)$. Hence it follows that Range $H_A \subset H^2(M) \ominus b_1 H^2(M)$ for the scalar inner function b_1. Since $b_1 H^2(M)$ is an invariant subspace of full range [**4**; p. 70] so is Range H_A and is given therefore by an inner function Q. Since $QH^2(M) \supset b_1 H^2(M)$, b_1 is a scalar multiple of Q. This completes the proof.

We wish to add that if N is finite-dimensional, any inner function Q in $H^\infty(N, N)$ has a scalar multiple [**4**].

COROLLARY 1. *Let $A \in H^\infty(N, M)$. Then A has a factorization $A = QB$ with Q inner in $H^\infty(M, M)$ having a scalar multiple and B analytic in D_e vanishing at infinity if and only if $A = B_1 Q_1$ with Q_1 inner in $H^\infty(N, N)$ having a scalar multiple and B_1 analytic in D_e and vanishing at infinity.*

Proof. Assume $A = QB$ with Q and B as above. By Theorem 1, A has a meromorphic psuedo-continuation of bounded type in D_e. Let $\tilde{A}(z) = A(\bar{z})^*$. It is clear that $\tilde{A} \in H^\infty(M, N)$ and has also a meromorphic pseudo-continuation of bounded type in D_e; in fact $\hat{\tilde{A}}(z) = \tilde{\hat{A}}(z)$. Thus, again by Theorem 1, $\tilde{A} = \tilde{Q}_1 \tilde{B}_1$ for some $\tilde{Q}_1$, $\tilde{B}_1$ as above. It follows that $A = B_1 Q_1$ as required.

References

1. R. G. Douglas and J. W. Helton, " Inner Dilations of Analytic Matrix Functions and Darlington Synthesis ", *Acta Sci. Math. (Szeged)*, 34 (1973), 61–67.
2. ———, H. S. Shapiro and A. L. Shields, " Cyclic Vectors and Invariant Subspaces for the Backward Shift, *Ann. Inst. Fourier, Grenoble* 20, 1 (1971), 37–76.
3. P. A. Fuhrmann, " On the Corona Theorem and its Application to Spectral Problems in Hilbert Space ", *Trans. Amer. Math. Soc.*, 132 (1968), 55–66.

4. H. Helson, *Lectures on Invariant Subspaces* (Academic Press, N.Y., 1964).
5. T. L. Kriete, "A Generalized Paley–Wiener Theorem", *J. Math. Anal. Appl.*, 36 (1971), 529–555.
6. P. D. Lax, "Translation Invariant Subspaces", *Proc. Inter. Symp. on Linear Spaces*, Jerusalem Academic Press, 1961.
7. B. Sz-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space* (North Holland, Amsterdam, 1970).

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