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ADVANCES IN Applied Mathematics

Advances in Applied Mathematics 31 (2003) 527–531

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q-identities from Lagrange and Newton interpolation

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Received 20 November 2002; accepted 22 November 2002

Abstract

Combining Newton and Lagrange interpolation, we give q-identities which generalize results of Van Hamme, Uchimura, Dilcher, and Prodinger.

1. Introduction

Van Hamme [7] gave the following identity involving Gauss polynomials, see also Andrews [1]:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^i} = \sum_{i=1}^{n} \frac{q^i}{1-q^i}, \tag{1.1}$$

where $\begin{bmatrix} n \\ i \end{bmatrix}$ is the Gauss polynomials defined by $\begin{bmatrix} n \\ i \end{bmatrix} = (q;q)_n \left((q;q)_i (q;q)_{n-i}\right)^{-1}$ with $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Uchimura [6] generalize (1.1) as following:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^{i+m}} = \sum_{i=1}^{n} \frac{q^i}{1-q^i} \Big/ \begin{bmatrix} i+m \\ i \end{bmatrix}, \quad m \geqslant 0, \tag{1.2}$$

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0196-8858/$ – see front matter
doi:10.1016/S0196-8858(03)00024-1

and by Dilcher [2]:

$$\sum_{1\leqslant i\leqslant n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^{i-1}\frac{q^{\binom{i}{2}+mi}}{(1-q^i)^m}=\sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_m\leqslant n}\frac{q^{i_1}}{1-q^{i_1}}\cdots\frac{q^{i_m}}{1-q^{i_m}}. \tag{1.3}$$

Prodinger [5] mentioned the following identity as a q-analogue of Kirchenhofer's [3] formula,

$$\sum_{i=0,\, i\neq M}\begin{bmatrix}n\\i\end{bmatrix}\frac{(-1)^{i-1}q^{\binom{i+1}{2}}}{1-q^{i-M}}=(-1)^M q^{\binom{M+1}{2}}\begin{bmatrix}n\\M\end{bmatrix}\sum_{i=0,\, i\neq M}\frac{q^{i-M}}{1-q^{i-M}} \tag{1.4}$$

and explained how to obtain all these formulas by using Cauchy residues.

We shall show that in fact, all the above formulas are a direct consequence of Newton and Lagrange interpolation.

Given two finite sets of variables $\mathbb{A}$ and $\mathbb{B}$, we denote by $R(\mathbb{A},\mathbb{B})$ the product $\prod_{a\in\mathbb{A},b\in\mathbb{B}}(a-b)$, and by $\mathbb{A}\setminus\mathbb{B}$ the set difference of $\mathbb{A}$ and $\mathbb{B}$.

Let $\mathbb{A}=\{x_1,x_2,\ldots\}$, $\mathbb{A}_n=\{x_1,x_2,\ldots,x_n\}$, for any $n\geqslant 0$. Lagrange wrote the following summation:

$$f(x)=\sum_{i=1}^{n}f(x_i)\frac{R(x,\mathbb{A}\setminus x_i)}{R(x_i,\mathbb{A}\setminus x_i)}\quad \operatorname{mod} R(x,\mathbb{A}_n). \tag{1.5}$$

On the other hand, Newton's development is:

$$f(x)=f(x_1)+f\partial_1 R(x,\mathbb{A}_1)+f\partial_1\partial_2 R(x,\mathbb{A}_2)+\cdots, \tag{1.6}$$

where ∂_i, $i\geqslant 1$, operating on its left, is defined by

$$f(x_1,x_2,\ldots,x_i,x_{i+1},\ldots)\partial_i=\frac{f(\ldots,x_i,x_{i+1},\ldots)-f(\ldots,x_{i+1},x_i,\ldots)}{x_i-x_{i+1}}.$$

Taking $x_{n+1}=x$, Newton's summation terminates:

$$f(x)=f(x_1)+f\partial_1 R(x,\mathbb{A}_1)+f\partial_1\partial_2 R(x,\mathbb{A}_2)+\cdots+f\partial_1\cdots\partial_n R(x,\mathbb{A}_n).$$

Newton's and Lagrange's expressions imply the same remainder $f\partial_1\cdots\partial_n R(x,\mathbb{A}_n)$, and the polynomial $g_n(x)=f(x)-f\partial_1\cdots\partial_n R(x,\mathbb{A}_n)$ is the only polynomial of degree $\leqslant n-1$ such that $f(x_i)=g(x_i)$, $1\leqslant i\leqslant n$.

Taking now $f(x)=(y-x)^{-1}$, since

$$\frac{1}{y-x_1}\partial_1\cdots\partial_{n-1}=\frac{1}{(y-x_1)\cdots(y-x_n)},$$

one gets, by comparing Newton's and Lagrange's expressions, the identity:

$$\sum_{i=0}^{n-1} \frac{R(x, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=1}^{n} \frac{f(x_i) R(x, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)} = \frac{1}{y-x} - \frac{R(x, \mathbb{A}_n)}{R(y, \mathbb{A}_n)(y-x)}. \tag{1.7}$$

Letting $x = 1$, we derive:

$$\sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=1}^{n} \frac{f(x_i) R(1, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)}. \tag{1.8}$$

Expanding $(y-x)^{-1} = \sum_i y^{-i-1} x^i$ and taking the coefficient of y^{-m-1}, we have:

$$x_1^m \partial_1 \cdots \partial_{n-1} = \sum_{i=1}^{n} \frac{x_i^m}{\prod_{j \neq i} (x_i - x_j)} = h_{m-n+1}(x_1, x_2, \ldots, x_n). \tag{1.9}$$

Recall that complete functions [4] h_k are defined by

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

2. General identities

The identities that we present just correspond to taking

$$\mathbb{A} = \left\{ \frac{a - bq}{c - zq}, \frac{a - bq^2}{c - zq^2}, \ldots \right\}$$

in Lagrange or Newton interpolation. In that case, the products $R(x_i, \mathbb{A} \setminus x_i)$ are immediate to write.

Proposition 2.1. *Let $m, n \in \mathbb{N}$, $\tau = m - n + 1$, a, b, z, q be variables, and*

$$\mathbb{A} = \left\{ \frac{a - bq}{c - zq}, \frac{a - bq^2}{c - zq^2}, \ldots, \frac{a - bq^n}{c - zq^n} \right\}.$$

We have:

$$\sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_\tau \leqslant n} \frac{a - bq^{i_1}}{c - zq^{i_1}} \frac{a - bq^{i_2}}{c - zq^{i_2}} \cdots \frac{a - bq^{i_\tau}}{c - zq^{i_\tau}}$$
$$= \frac{c^n (zq/c; q)_n}{(q; q)_n (az - bc)^{n-1}} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i)(a - bq^i)^m}{(c - zq^i)^{\tau+1}}. \tag{2.10}$$

In particular, for $m = n$:

$$\sum_{i=1}^{n} \frac{a - bq^i}{c - zq^i} = \frac{c^n (zq/c; q)_n}{(q; q)_n (az - bc)^{n-1}} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i)(a - bq^i)^n}{(c - zq^i)^2}. \tag{2.11}$$

Proof. For our choice of $\mathbb{A}$, then

$$R(x_i, \mathbb{A} \setminus x_i) = \prod_{j \neq i} \frac{(az - bc)(q^i - q^j)}{(c - zq^i)(c - zq^j)},$$

and

$$\sum_{i=1}^{n} \frac{((a - bq^i)/(c - zq^i))^m}{\prod_{j \neq i} (\frac{a - bq^i}{c - zq^i} - \frac{a - bq^j}{c - zq^j})}$$
$$= \frac{c^n (zq/c; q)_n}{(q; q)_n (az - bc)^{n-1}} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i)(a - bq^i)^m}{(c - zq^i)^{m-n+2}}. \qquad \square$$

Consider the case $a = 0$, $b = -1$, $c = 1$, and $z = 1$, i.e.,

$$\mathbb{A} = \left\{ \frac{q}{1 - q}, \frac{q^2}{1 - q^2}, \ldots, \frac{q^n}{1 - q^n} \right\}.$$

Equation (2.11) implies Van Hamme identity:

$$\sum_{i=1}^{n} \frac{q^i}{1 - q^i} = \frac{(q; q)_n}{(q; q)_n} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i) q^{ni}}{(1 - q^i)^2}$$
$$= \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - q^i}.$$

From (2.10), we have

$$\sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{m-n+1} \leqslant n} \frac{q^{i_1}}{1 - q^{i_1}} \frac{q^{i_2}}{1 - q^{i_2}} \cdots \frac{q^{i_{m-n+1}}}{1 - q^{i_{m-n+1}}} = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i+1} q^{\binom{i+1}{2} - ni} q^{mi}}{(1 - q^i)^{m-n+1}},$$

replacing $m - n + 1$ by m, we derive (1.3).

Take now $a = 1$, $b = 0$, $c = 0$, $z = -1$, then $\mathbb{A} = \{q^{-1}, q^{-2}, \ldots, q^{-n}\}$, and Eq. (1.8) implies:

$$\text{R.H.S.} = \sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=0}^{n-1} \frac{\prod_{j=1}^{i}(1-q^{-j})}{\prod_{j=1}^{i+1}(y-q^{-j})} = -\sum_{i=0}^{n-1} \frac{q^{i+1}(q;q)_i}{(yq;q)_{i+1}},$$

$$\text{L.H.S.} = \sum_{i=1}^{n} \frac{f(x_i)R(1, \mathbb{A}\setminus x_i)}{R(x_i, \mathbb{A}\setminus x_i)}$$

$$= \sum_{i=1}^{n} \frac{(y-q^{-i})^{-1}\prod_{j\neq i}(1-q^{-j})}{\prod_{j\neq i}(q^{-i}-q^{-j})} = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^i q^{\binom{i+1}{2}}}{1-yq^i}.$$

Clearly:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-yq^i} = \sum_{i=0}^{n-1} \frac{q^{i+1}(q;q)_i}{(yq;q)_{i+1}} = \sum_{i=1}^{n} \frac{q^i(q;q)_{i-1}}{(yq;q)_i}.$$

In the special case $y = q^m$, we obtain (1.2).

We take now $\mathbb{A} = \{q^M, q^{M-1}, q^{M-2}, \ldots, q, q^{-1}, \ldots, q^{M-n}\}$, with $M \in \mathbb{N}$, and $y = 1$. Equation (1.8) becomes:

$$\text{R.H.S.} = \sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(1, \mathbb{A}_{i+1})} = -\sum_{i=0, i\neq M}^{n} \frac{q^{i-M}}{1-q^{i-M}}$$

and

$$\text{L.H.S.} = \sum_{i=0, i\neq M}^{n} \frac{1}{1-q^{M-i}} \frac{\prod_{j\neq i,M}(1-q^{M-j})}{\prod_{j\neq i,M}(q^{M-i}-q^{M-j})}$$

$$= \sum_{i=0, i\neq M}^{n} \frac{(-1)^{M+i} q^{\binom{i+1}{2}-\binom{M+1}{2}}(q;q)_M(q;q)_{n-M}}{(q;q)_i(q;q)_{n-i}},$$

which proves (1.4).

As a final comment, we would like to stress that we have just used simple alphabets in Newton and Lagrange interpolation; it is easy to generalize the above formulas by taking more sophisticated alphabets.

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