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Shintani–Barnes zeta and gamma functions

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Abstract

We show that Shintani's work on multiple zeta and gamma functions can be simplified and extended by exploiting difference equations. We re-prove many of Shintani's formulas and prove several new ones. Among the latter is a generalization to the Shintani–Barnes gamma functions of Raabe's 1843 formula $\int_0^1 \log \Gamma(x)\, dx = \log \sqrt{2\pi}$, and a further generalization to the Shintani zeta functions. These explicit formulas can be interpreted as "vanishing period integral" side conditions for the ladder of difference equations obeyed by the multiple gamma and zeta functions. We also relate Barnes' triple gamma function to the elliptic gamma function appearing in connection with certain integrable systems.
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1. Introduction

Motivated by some problems in number theory, Shintani [13–18] introduced in the mid-1970s a multi-dimensional zeta function $\zeta(s, \mathscr{M}, x)$, with $\mathscr{M}$ an $N \times n$ matrix

$$\mathscr{M} := \{a_{ij}\}, \quad i = 1, \ldots, N, \; j = 1, \ldots, n, \tag{1.1}$$

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with positive entries, $x \in (0, \infty)^N$, and

$$\zeta(s, \mathcal{M}, x) := \sum_{m_1, \ldots, m_N = 0}^{\infty} \prod_{j=1}^{n} \left(\sum_{i=1}^{N} (x_i + m_i) a_{ij} \right)^{-s}, \quad \mathrm{Re}(s) > N/n. \tag{1.2}$$

For $n = 1$ this zeta function amounts to Barnes' multiple zeta function [3].

Shintani showed that his zeta function admits a meromorphic s-continuation with the same pole locations as the function

$$p_{N,n}(s) := \Gamma(ns - N)/\Gamma(s), \tag{1.3}$$

and obtained various explicit formulas. In particular, he expressed the s-value $\zeta(0, \mathcal{M}, x)$ and the s-derivative $\partial_s \zeta(s, \mathcal{M}, x)|_{s=0}$ in terms of Bernoulli polynomials and Barnes' multiple zeta and gamma functions. A certain multi-dimensional contour integral yielding the s-continuation played a pivotal role in Shintani's reasoning. Unfortunately, his impressive calculations rarely gave any insights as to why his formulas should hold or how he had discovered them.

A principal purpose of this paper is to present a simpler approach to Shintani's work. Along the way we also obtain with little extra effort various new results, including the formula

$$\int_{I^N} \zeta(s, \mathcal{M}, x)\, dx = 0, \quad I^N := (0,1)^N, \; \mathrm{Re}(s) < N/n. \tag{1.4}$$

To our knowledge, except when $N = n = 1$ [4,7], this result is new in the Barnes case, too.

A crucial point in our approach is that we mostly work with a zeta function that is somewhat more general than Shintani's $\zeta(s, \mathcal{M}, x)$, namely,

$$\zeta_{N,n}(s, w | a_1, \ldots, a_N) := \sum_{m_1, \ldots, m_N = 0}^{\infty} \prod_{j=1}^{n} \left(w_j + \sum_{i=1}^{N} m_i a_{ij} \right)^{-s}, \quad \mathrm{Re}(s) > N/n. \tag{1.5}$$

Here the a_i and w are elements of $\mathbb{C}^n$ whose coordinates a_{ij} and w_j have positive real parts. (In Section 6 we slightly relax this restriction.) Thus we have

$$\zeta(s, \mathcal{M}, x) = \zeta_{N,n}(s, W(x) | a_1, \ldots, a_N), \quad W(x) = W(x | a_1, \ldots, a_N) := \sum_{i=1}^{N} x_i a_i. \tag{1.6}$$

Clearly, for $N \geqslant n$ the two zeta functions are substantially equivalent, as the a_i generically span $\mathbb{C}^n$. For $N < n$, however, the function $\zeta_{N,n}(s, w)$ is more general than $\zeta(s, \mathcal{M}, x)$. The key advantage of working with the functions $\zeta_{N,n}(s, w)$ is that they satisfy

$$\zeta_{N,n}(s, w + a_N | a_1, \ldots, a_N) - \zeta_{N,n}(s, w | a_1, \ldots, a_N) = -\zeta_{N-1,n}(s, w | a_1, \ldots, a_{N-1}) \tag{1.7}$$

(with $\zeta_{0,n}(s,w) := \prod_{j=1}^{n} w_j^{-s}$). This recurrence relation is nearly immediate from the Dirichlet series definition, and has no analog for $\zeta(s,\mathcal{M},x)$. Viewing (1.7) as a ladder of analytic difference equations, we are able to guess and prove various results, including many of Shintani's formulas. (In the Barnes case $n=1$, the difference equation perspective was exploited before in [11], cf. also [12].)

A related point we emphasize is to work directly with the Dirichlet series (1.5) inside its domain of convergence, even though our interest lies mostly in the points $s=(N-m)/n$ with m a non-negative integer, especially $s=0$. The crux is that formulas obtained via the Dirichlet series for $\mathrm{Re}(s)$ large can be analytically continued. Shintani worked in the reverse order, first analytically continuing $\zeta(s,\mathcal{M},x)$ and then manipulating expressions valid at $s=0$, but these are more complicated than the series.

The simplicity of our ideas might easily remain hidden under the extensive bookkeeping needed to handle the general case. Therefore, we illustrate them in this introduction via the simplest non-trivial case, which is the well-known Hurwitz zeta function (cf. [1, Sections 1.2–1.3])

$$H(s,w) := \sum_{n=0}^{\infty} \frac{1}{(n+w)^s} = \zeta_{1,1}(s,w|1), \quad \mathrm{Re}(s)>1, \ \mathrm{Re}(w)>0. \tag{1.8}$$

This also serves to explain the organization and main results of Sections 2–5.

Our starting point is Euler's formula

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt. \tag{1.9}$$

It entails that we may rewrite $\Gamma(s)H(s,w)$ for $\mathrm{Re}(s)>1$ as

$$\sum_{m=0}^{\infty} \int_0^\infty e^{-r(w+m)} r^{s-1}\, dr = \int_0^\infty \frac{e^{-rw}}{r} \phi(r) r^{s-1} dr, \quad \phi(r) := \frac{r}{1-e^{-r}} = \sum_{l=0}^{\infty} \frac{\alpha_l r^l}{l!},$$

with the $\phi(r)$ power series converging for $|r|<2\pi$. Obviously, we have

$$\phi(r) - \sum_{l=0}^{M-1} \frac{\alpha_l r^l}{l!} = O(r^M), \ \ r\to 0, \quad \phi(r) = O(r), \ \ r\to\infty. \tag{1.10}$$

Using (1.9) again, we now obtain

$$\Gamma(s)H(s,w) = \sum_{l=0}^{M-1} \frac{\alpha_l}{l!} \frac{\Gamma(-1+l+s)}{w^{-1+l+s}} + \int_0^\infty \frac{e^{-rw}}{r} \left(\phi(r) - \sum_{l=0}^{M-1} \frac{\alpha_l r^l}{l!} \right) r^{s-1}\, dr. \tag{1.11}$$

Due to bounds (1.10), the integral yields a function that is analytic for $\mathrm{Re}(s) > -M+1$. From this we easily deduce that $H(s,w)$ extends to a meromorphic function of s, its only singularity being a simple pole at $s=1$.

Of course, this is just one of several ways to obtain these well-known facts. But the present approach admits a straightforward extension to the Barnes case (cf. [11]), and in Section 2 we generalize it to $\zeta_{N,n}$. The principal result of Section 2 is that $\zeta_{N,n}(s,w)/p_{N,n}(s)$ extends to a function that is entire in s and analytic in the w_j and a_{ij} in the domains

$$D_n := \{w \in \mathbb{C}^n | \mathrm{Re}(w_j) > 0,\ 1 \leqslant j \leqslant n\}, \tag{1.12}$$

and

$$\mathscr{D}_{N,n} := \{a_{ij} \in \mathbb{C} | \mathrm{Re}(a_{ij}) > 0,\ 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant n\}. \tag{1.13}$$

The associated Shintani–Barnes gamma function $\Gamma_{N,n}$ is defined by

$$\Gamma_{N,n}(w|a_1, \ldots, a_N) := \exp(\partial_s \zeta_{N,n}(s, w|a_1, \ldots, a_N)|_{s=0}), \tag{1.14}$$

where $\partial_s := \frac{\partial}{\partial s}$. Its analyticity features can be elucidated via the ladder of difference equations

$$\frac{\Gamma_{N,n}(w + a_N|a_1, \ldots, a_N)}{\Gamma_{N,n}(w|\, a_1, \ldots, a_N)} = \frac{1}{\Gamma_{N-1,n}(w|a_1, \ldots, a_{N-1})}, \quad \Gamma_{0,n}(w) = \prod_{j=1}^{n} w_j^{-1}. \tag{1.15}$$

In particular, $1/\Gamma_{N,n}(w)$ extends to an entire function on $\mathbb{C}^n$.

We now return to our account of further properties of the Hurwitz zeta function, obtained along the lines of Sections 3–5. Accordingly, we will only make use of $\zeta_{1,1}(s,w)/p_{1,1}(s)$ being entire in s, but not of formula (1.11) yielding the s-continuation. The entireness property amounts to $(s-1)H(s,w)$ being entire, and this is all we need to know to obtain the desired results via the Dirichlet series (1.8) and the pertinent difference equation, namely,

$$H(s, w+1) - H(s, w) = -w^{-s}. \tag{1.16}$$

As a first step, we take the w-derivative of the Dirichlet series, yielding a function

$$\partial_w H(s,w) = -s \sum_{n=0}^{\infty} \frac{1}{(n+w)^{s+1}}$$

that is clearly analytic for $\mathrm{Re}(s) > 0$. From this we see that the pole of $H(s,w)$ at $s=1$ has a constant residue. Next, we observe that the formula

$$\partial_w^2 H(s,w) = s(s+1) \sum_{n=0}^{\infty} \frac{1}{(n+w)^{s+2}}$$

yields a representation of $\partial_w^2 H(s,w)$ valid for $\mathrm{Re}(s) > -1$, whence it is clear that $\partial_w^2 H$ vanishes for $s=0$. Therefore, $H(0,w)$ is a polynomial of degree at most 1. Taking further derivatives, it follows more generally that $H(-k,w)$ with $k \in \mathbb{N}$

is a polynomial of degree at most $k+1$. Since the difference equation (1.16) specializes to

$$H(-k, w+1) - H(-k, w) = -w^k, \quad k \in \mathbb{N}, \tag{1.17}$$

we deduce that the degree of $H(-k, w)$ is in fact equal to $k+1$. (Here and below, we denote the set of non-negative integers by $\mathbb{N}$ and the set of positive integers by $\mathbb{N}_+$.)

In Section 3 we extend this reasoning to the general case, obtaining the polynomial property for several quantities of interest, together with an upper bound on the degree. By additional arguments we then show that the upper bound is optimal. Specifically, we prove that $\zeta_{N,n}(-k, w)$ with $k \in \mathbb{N}$ is a polynomial of degree $N + kn$, and that the eventual simple poles at the s-locations $(N-l)/n$ with $l \in \mathbb{N}$ do occur if $l \neq N + kn$ $(k \in \mathbb{N})$, the residues being polynomials of degree l. Finally, we demonstrate that the difference

$$\log \Gamma_{N,n}(w|a_1, \ldots, a_N) - \sum_{j=1}^{n} \log \Gamma_{N,1}(w_j|a_{1j}, \ldots, a_{Nj}), \quad N \geqslant 1, \ n \geqslant 2, \tag{1.18}$$

equals a degree-N polynomial, and determine this polynomial explicitly for $N = 1$. Except for the treatment of residues, these results are all due to Shintani [13, Proposition 1 and its Corollary] [16, Proposition 1], albeit for $\zeta(s, \mathscr{M}, x)$.

The main result of Section 4 is the integral formula (1.4) and its corollary

$$\int_{I^N} \log \Gamma_{N,n}(W(x))\, dx = 0. \tag{1.19}$$

For the Hurwitz case (1.4) specializes to

$$\int_0^1 H(s, w)\, dw = 0, \quad \mathrm{Re}(s) < 1. \tag{1.20}$$

We continue by demonstrating (1.20) as a template for the proof of (1.4).

We begin by noting that $H(s, w+1)$ is analytic for $s \neq 1$ and $\mathrm{Re}(w) > -1$, so that it is integrable in w over (0,1). Specifically, we obtain from the Dirichlet series (1.8)

$$\int_0^1 H(s, w+1)\, dw = \int_1^{\infty} x^{-s}\, dx = \frac{1}{s-1}, \quad \mathrm{Re}(s) > 1. \tag{1.21}$$

By analytic continuation, the integral on the left-hand side equals $1/(s-1)$ for $\mathrm{Re}(s) < 1$ as well. Now for $\mathrm{Re}(s) < 1$, the function w^{-s} is also integrable over $(0, 1)$, the result being $1/(1-s)$. Using the difference equation (1.16), we therefore obtain (1.20).

At this point we would like to mention that the integral (1.20) was recently obtained by Broughan [4]. Likewise, for $\mathrm{Re}(s) < 0$, (1.20) occurs (among many other new integrals) in a recent paper by Espinosa and Moll [7].

Next, we deduce from the Cauchy integral formula that we may interchange the s-derivative of the left-hand side of (1.20) with the integration, obtaining in particular

$$\int_0^1 H'(0,w)\,dw = 0, \quad H'(s,w) := \partial_s H(s,w). \tag{1.22}$$

Recalling the well-known relation (Lerch formula)

$$H'(0,w) = \log\Big(\Gamma(w)/\sqrt{2\pi}\Big), \tag{1.23}$$

(cf. for example [1, p. 17]), we can rewrite (1.22) as the integral

$$\int_0^1 \log \Gamma(x)\,dx = \log\sqrt{2\pi}. \tag{1.24}$$

This integral is known as Raabe's formula [9, p. 89]. Accordingly, we may view (1.19) as a generalized Raabe formula. In the case of Barnes' multiple gamma function, (1.19) can be explicitly written out as

$$0 = \int_{I^N} \log \Gamma_N(a_1x_1 + a_2x_2 + \cdots + a_Nx_n|a_1, \ldots, a_N)\,dx. \tag{1.25}$$

(Here and from now on, we write $\Gamma_{N,1}$ as Γ_N; likewise, $\zeta_{N,1}$ will be written ζ_N.) Although Barnes [3, Section 53] proved a Raabe-type formula for Γ_N, it is complicated and involves a one-dimensional integral. (Note also that Barnes used a different normalization for his multiple gamma function.) Even in the Barnes case, formula (1.25) seems to be new.

In order to give an interpretation to the integral

$$\int_{I^N} \zeta_{N,n}(s, W(x))\,dx = 0, \quad \mathrm{Re}(s) < N/n \tag{1.26}$$

(which amounts to (1.4), cf. (1.6)), we observe that as x varies over I^N, the function $W(x)$ ranges over the "period parallelogram" $P \subset \mathbb{C}^n$, defined as the convex span of the a_i. For $N \leqslant n$ the a_i are (generically) linearly independent, so (1.26) can be restated as $\int_P \zeta_{N,n}(s,w)\,dw = 0$, where dw is N-dimensional Lebesgue measure on the subspace of $\mathbb{C}^n$ spanned by the a_i. This can be regarded as a "vanishing period integral" normalization, which fixes the constant left undetermined by the difference equation (1.7). Likewise, integral (1.19) fixes the constant in the difference equation (1.15).

Elaborating slightly, we note that the ambiguity in the solutions of the first-order partial analytic difference equations (1.7) and (1.15) (with the right-hand sides viewed as given functions) is not just a constant. Indeed, we can clearly add to solutions of (1.7) any meromorphic function $\alpha(w)$ having period a_N; likewise, we can multiply solutions of (1.15) by meromorphic functions $\mu(w)$ with period a_N. In the Barnes case $n = 1$, the multiple zeta and gamma functions can be singled out by "minimality": the singularities of ζ_N and Γ_N are enforced by the difference equation, and their asymptotics in a suitable strip is "best possible" [10–12]. In the Shintani case $n>1$, however, we are dealing with partial difference equations, for which no theory of minimal solutions is known to date.

In Section 4 we obtain not only integrals (1.26) and(1.19), but also generalizations to x-derivatives. Specifically, if $J=(J_1,J_2,\dots,J_N)$ is a multi-index of weight $|J|=\sum_{k=1}^N J_k$ and ∂_x^J denotes the differential operator

$$\partial_x^J := \frac{\partial^{|J|}}{\partial x_1^{J_1}\cdots\partial x_N^{J_N}},$$

then we show

$$\int_{I^N} \partial_x^J \zeta_{N,n}(s, W(x))\,dx = 0, \quad \mathrm{Re}(s)<(N-|J|)/n, \tag{1.27}$$

and

$$\int_{I^N} \partial_x^J \log \Gamma_{N,n}(W(x))\,dx = 0, \quad |J|<N. \tag{1.28}$$

In Section 5 we show that the latter integrals easily lead to Shintani's [13, p. 396] result stating that for $k\in\mathbb{N}$, the polynomial $\zeta_{N,n}(-k, W(x))$ is a sum of products of Bernoulli polynomials. We show that this also holds for the residues at each of the poles of $\zeta_{N,n}(s, W(x))$.

It is expedient to summarize next the salient features of the Bernoulli polynomials $B_l(t)$. We first recall that they can be defined via the generating function

$$\frac{ue^{tu}}{e^u-1} = \sum_{l=0}^{\infty} \frac{B_l(t)}{l!} u^l. \tag{1.29}$$

A more instructive definition is that they are the polynomials uniquely determined by the difference equation

$$B_l(t+1) - B_l(t) = lt^{l-1}, \tag{1.30}$$

together with the side conditions

$$B_0(t) = 1, \quad \int_0^1 B_l(t)\,dt = 0, \quad l>0. \tag{1.31}$$

(Indeed, these relations are easily derived from (1.29).) Another important feature, namely

$$B_l'(t) = lB_{l-1}(t), \tag{1.32}$$

is also clear from (1.29).

Specializing Section 5 to the Hurwitz case, we need only consider $H(-k,w)$ for $k\in\mathbb{N}$. We have already established that $H(-k,w)$ is a polynomial. Comparing the difference equations (1.17) and (1.30), we see that $H(-k,w)$ coincides with $-B_{k+1}(w)/(k+1)$ up to a constant. Comparing next (1.31) and (1.20) for $s=-k$, we deduce

$$H(-k,w) = -B_{k+1}(w)/(k+1), \quad k\in\mathbb{N}. \tag{1.33}$$

Once again, this is only one of various ways to obtain this relation, which has been known for a long time. We have spelled it out, since it illustrates our approach to the general case in Section 5.

In Section 5 we also prove three identities arising for $s=0$, namely

$$\zeta_{N,n}(0,w|a_1,\dots,a_N) = \frac{1}{n}\sum_{j=1}^{n} \zeta_N(0,w_j|a_{1j},\dots,a_{Nj}),$$

$$\zeta_{N,n}(0,w|a_1,\dots,a_N) = (-1)^N \zeta_{N,n}(0,A-w|a_1,\dots,a_N), \quad (A := a_1+\cdots+a_N)$$

$$\Gamma_{N,n}(w)(\Gamma_{N,n}(A-w))^{(-1)^{N+1}} = \prod_{j=1}^{n} \Gamma_N(w_j)(\Gamma_N(A_j-w_j))^{(-1)^{N+1}},$$

where

$$\Gamma_{N,n}(w) = \Gamma_{N,n}(w|a_1,\dots,a_N), \quad \Gamma_N(w_j) = \Gamma_N(w_j|a_{1j},\dots,a_{Nj}).$$

Furthermore, we determine the polynomial $\zeta_{N,n}(0,w)$ explicitly. In essence, all of these $s=0$ results were obtained first by Shintani [16, pp. 206, 210].

We begin Section 6 by detailing a slight generalization of our assumptions, for which all of the previous results still hold. Specifically, we allow the numbers $a_{1j},\dots,a_{Nj}$ to lie in any half-plane obtained by rotating the right half-plane over an angle less than $\pi/2$. (This angle restriction prevents multi-valuedness.) It is clear from the Dirichlet series (1.5) that we can do this when we choose w_j in the same half-plane, but a complete account of the pertinent analytic continuation involves a little more effort.

This generalization—already present in Barnes' and Shintani's work—enables us to relate Barnes' Γ_N to certain infinite products. The integral formulas (1.26) and (1.19) (with $n=1$) allow us to make this relation completely explicit. The pertinent result (Proposition 6.1) reduces to the reflection equation for Γ_1, whereas for Γ_2 it amounts to a result that can be found in Barnes' and Shintani's papers. For Γ_N with $N>2$, Proposition 6.1 seems to be new. As a corollary, we find an explicit relation between Γ_3 and the elliptic gamma function introduced in [10].

Since we are promoting in this paper a simplified approach to multiple zeta and gamma functions, we do not assume any familiarity with them. Although the theory of minimal solutions of first-order analytic difference equations [10] provided an important motivation for our work, we make no further appeal to this theory. Likewise, we avoid any reference to number theory, although this was the main motivation for Shintani's work. Lastly, we would like to mention that other approaches to special values of $\zeta_{N,n}(s,w)$ can be found in [5,6].

2. Analytic continuation of $\zeta_{N,n}$

The meromorphic continuation of $\zeta(s, \mathscr{M}, x)$ was proved by Shintani [13], who was interested in using it to calculate special values. Had he not been interested in this, he could have deduced the meromorphic continuation of $\zeta(s, \mathscr{M}, x)$ from an old result of Mahler's [8, Section 19].

We shall actually consider a generalization $Z_{N,n}(S, w, \mathscr{M})$ of $\zeta_{N,n}(s, w)$ (with $\mathscr{M}$ defined by (1.1)), replacing s by n complex variables $S = (S_1, \dots, S_n)$ in the half-space $\sum_{j=1}^{n} \mathrm{Re}(S_j) > N$. Namely,

$$Z_{N,n}(S, w, \mathscr{M}) \coloneqq \sum_{m_1,\dots,m_N=0}^{\infty} \prod_{j=1}^{n} (w_j + m_1 a_{1j} + \cdots + m_N a_{Nj})^{-S_j}. \tag{2.1}$$

Since we are assuming $\mathrm{Re}(a_{ij}) > 0$ and $\mathrm{Re}(w_j) > 0$, we may and will choose the principal branch of the logarithm to define the complex powers in (2.1).

To determine the region of absolute convergence of (2.1), let

$$c \coloneqq \min_{i,j}\{\mathrm{Re}(a_{ij})\} > 0, \quad C \coloneqq 1 + \max_{i,j}\ \{|a_{ij}|\}. \tag{2.2}$$

Then, for $m_i \geqslant 0$,

$$\begin{aligned} CN(||w|| + ||m||) &\geqslant |w_j + m_1 a_{1j} + \cdots + m_N a_{Nj}| \\ &\geqslant \mathrm{Re}(w_j + m_1 a_{1j} + \cdots + m_N a_{Nj}) > c \sum_{i=1}^{N} m_i \geqslant c\, ||m||, \end{aligned} \tag{2.3}$$

where $||m||^2 \coloneqq \sum_{i=1}^{N} |m_i|^2$, $||w||^2 \coloneqq \sum_{j=1}^{n} |w_j|^2$ for $m \in \mathbb{R}^N$, $w \in \mathbb{C}^n$. Thus (2.1) converges absolutely if and only if $\sum_{j=1}^{n} \mathrm{Re}(S_j) > N$. From this it readily follows that it defines an analytic function for $(S, w, \mathscr{M})$ in the subset of $\mathbb{C}^n \times D_n \times \mathscr{D}_{N,n}$ given by $\sum_{j=1}^{n} \mathrm{Re}(S_j) > N$, with D_n and $\mathscr{D}_{N,n}$ given by (1.12) and (1.13).

The zeta function $Z_{N,n}(S, w, \mathscr{M})$ reduces to $\zeta_{N,n}(s, w|a_1, \dots, a_N)$ when all the S_j are equal to s and $\mathrm{Re}(s) > N/n$. Furthermore, for $\mathrm{Re}(s) > N$,

$$Z_{N,n}((s, 0, 0 \dots, 0), w, \mathscr{M}) = \zeta_N(s, w_1|a_{11}, \dots, a_{N1}). \tag{2.4}$$

Similarly, by restriction of the S-variable, $Z_{N,n}$ yields any $\zeta_{N,n'}$ with $n' \leqslant n$. However, $Z_{N,n}$ has the drawback of being singular at $S = 0$, as we shall see in Section 3.

To keep this paper as self-contained as possible, rather than rely on Mahler's paper, we now give a detailed proof of the meromorphic continuation of $\zeta_{N,n}$. However, in later sections we shall need no formulas affording it. We will only use holomorphy of the function $\zeta_{N,n}(s, w|a_1, \dots, a_N)/p_{N,n}(s)$ in the domain $\mathbb{C} \times D_n \times \mathscr{D}_{N,n}$, as already discussed in the Introduction. Readers who are willing to take this analyticity for granted can safely pass to Proposition 2.2.

To establish the analytic continuation of $Z_{N,n}(S,w,\mathscr{M})$, we first use Euler's formula (1.9) to obtain the integral representation

$$Z_{N,n}(S,w,\mathscr{M})\prod_{j=1}^{n}\Gamma(S_j)=\int_{t\in\mathbb{R}_+^n}h(t)\prod_{j=1}^{n}t_j^{S_j-1}\,dt,\quad \mathrm{Re}(S_j)>0,\ \sum_{j=1}^{n}\mathrm{Re}(S_j)>N, \tag{2.5}$$

where $\mathbb{R}_+^n:=(0,\infty)^n\subset\mathbb{R}^n$, dt is Lebesgue measure on $\mathbb{R}^n$, and

$$\begin{aligned}h(t)&:=\sum_{m_1,\dots,m_N=0}^{\infty}\prod_{j=1}^{n}\exp\big(-t_j(w_j+m_1a_{1j}+\cdots m_Na_{Nj})\big)\\&=\frac{\prod_{j=1}^{n}\mathrm{e}^{-w_jt_j}}{\prod_{i=1}^{N}\left(\sum_{j=1}^{n}a_{ij}t_j\right)}f(t),\end{aligned}$$

with

$$f(t):=\prod\nolimits_{i=1}^{N}\varphi\left(\sum_{j=1}^{n}a_{ij}t_j\right),\quad \varphi(r):=\frac{r}{1-\mathrm{e}^{-r}}. \tag{2.6}$$

Following Shintani [13, Section 1], we write

$$\mathbb{R}_+^n=A_1\cup A_2\cup\cdots\cup A_n,\quad A_j:=\left\{t\in\mathbb{R}_+^n\,\middle|\,t_j=\max_{1\leqslant k\leqslant n}\{t_k\}\right\}, \tag{2.7}$$

where the union is disjoint up to sets of measure 0. Then (2.5) becomes

$$Z_{N,n}(S,w,\mathscr{M})\prod_{j=1}^{n}\Gamma(S_j)=\sum\nolimits_{j=1}^{n}\int_{A_j}h(t)\prod_{k=1}^{n}t_k^{S_k-1}\,dt=:\sum_{j=1}^{n}I_j(S,w,\mathscr{M}). \tag{2.8}$$

For $t\in A_j$, we switch to new coordinates (ρ,σ), where $\rho:=t_j$ and $\sigma_k:=t_k/t_j$ $(1\leqslant k\leqslant n,\ k\neq j)$. The Jacobian determinant is ρ^{n-1}, and the new coordinates range over $\sigma_k\in(0,1)$, $\rho\in(0,\infty)$. For convenience, on A_j we define $\sigma_j:=1,\sigma=(\sigma_1,\dots,\sigma_n)$.

We now change to the new coordinates. The piece of the integral (2.8) corresponding to $j=1$ becomes

$$I_1(S,w,\mathscr{M})=\int_{\rho=0}^{\infty}\rho^{-N-1+\sum_{k=1}^{n}S_k}\mathrm{e}^{-\rho w_1}\int_{\sigma_2=0}^{1}\cdots\int_{\sigma_n=0}^{1}g(\rho,\sigma)\prod_{k=2}^{n}\sigma_k^{S_k-1}\,d\sigma_k\,d\rho, \tag{2.9}$$

where

$$g(\rho,\sigma):=\frac{f(\rho,\rho\sigma_2,\dots,\rho\sigma_n)\prod_{k=2}^{n}\mathrm{e}^{-\rho\sigma_kw_k}}{\prod_{i=1}^{N}\left(a_{i1}+\sum_{k=2}^{n}a_{ik}\sigma_k\right)}.$$

For the innermost integral in (2.9), we integrate by parts to get

$$\begin{aligned}&\int_{\sigma_n=0}^{1} \sigma_n^{S_n-1} g(\rho,\sigma)\, d\sigma_n \\ &\quad = \frac{-1}{S_n}\int_{\sigma_n=0}^{1} \sigma_n^{S_n}\left(\frac{\partial g}{\partial \sigma_n}(\rho,\sigma) - (S_n+1)g(\rho,\sigma_2,\dots,\sigma_{n-1},1)\right) d\sigma_n \\ &\quad = \frac{1}{S_n}\int_{\sigma_n=0}^{1} \sigma_n^{S_n} g_0(S,\rho,\sigma)\, d\sigma_n, \qquad (2.10)\end{aligned}$$

with the obvious definition of g_0.

We can repeat the integration by parts M times in (2.10) to get

$$\int_{\sigma_n=0}^{1} \sigma_n^{S_n-1} g(\rho,\sigma)\, d\sigma_n = \left(\prod_{p=0}^{M}\frac{1}{S_n+p}\right)\int_{\sigma_n=0}^{1} \sigma_n^{S_n+M} g_M(S,\rho,\sigma)\, d\sigma_n,$$

where g_M is a sum of σ_n-derivatives of g (and some specializations of them at $\sigma_n = 1$) with coefficients which are monomials in S_n. We have thus replaced the exponent $\sigma_n^{S_n-1}$ in (2.7) by $\sigma_n^{S_n+M}$. The same procedure, applied to the remaining σ_k in (2.9), yields

$$I_1(S,w,\mathscr{M}) = T_{M,1}(S)\int_{\rho=0}^{\infty} \rho^{\sum_{k=1}^{n} S_k-N-1} \mathrm{e}^{-\rho w_1}\int_{\sigma} \tilde{g}(S,\rho,\sigma)\prod_{k=2}^{n}\sigma_k^{S_k+M} d\sigma_k\, d\rho, \qquad (2.11)$$

where

$$T_{M,j}(S) := \prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\prod_{p=0}^{M}\frac{1}{S_k+p}, \qquad \int_{\sigma} := \int_{\sigma_2=0}^{1}\cdots\int_{\sigma_n=0}^{1}, \qquad (2.12)$$

and where

$$\tilde{g}(S,\rho,\sigma) = \sum_r h_r(S) f_r(\rho,\sigma_2,\dots,\sigma_n)$$

is again essentially a finite sum of σ_k-derivatives f_r of g, with coefficients h_r which are monomials in the S_k.

We now expand each f_r in powers of ρ to obtain

$$f_r(\rho,\sigma) = \rho^M h_{r,M}(\rho,\sigma) + \sum_{l=0}^{M-1} b_{l,r}(\sigma)\rho^l,$$

with $|h_{r,M}(\rho,\sigma)|$ bounded above by a polynomial in ρ for $\rho\geqslant 0$.

Returning to our integral (2.11), we find

$$I_1 = T_{M,1}(S)\left(\sum_r h_r(S)\sum_{l=0}^{M-1}\frac{\Gamma(-N+l+\sum_{k=1}^n S_k)}{w_1^{-N+l+\sum_{k=1}^n S_k}}\int_\sigma b_{l,r}(\sigma)\prod_{k=2}^n \sigma_k^{S_k+M}d\sigma_k\right.$$
$$\left.+\sum_r h_r(S)\int_{\rho=0}^\infty\int_\sigma \mathrm{e}^{-\rho w_1}\rho^{M-N-1+\sum_{k=1}^n S_k}h_{r,M}(\rho,\sigma)\prod_{k=2}^n \sigma_k^{S_k+M}d\sigma_k\,d\rho\right). \tag{2.13}$$

We have proved most of

Proposition 2.1. *The function* $Z_{N,n}(S,w,\mathscr{M})/\Gamma(-N+\sum_{j=1}^n S_j)$ *extends to a holomorphic function on* $\mathbb{C}^n\times D_n\times\mathscr{D}_{N,n}$. *The function*

$$\zeta_{N,n}(s,w|a_1,\dots,a_N)\Gamma(s)/\Gamma(ns-N)$$

extends to a holomorphic function on $\mathbb{C}\times D_n\times\mathscr{D}_{N,n}$. *In particular, for a fixed* $(w,\mathscr{M})\in D_n\times\mathscr{D}_{N,n}$, *the function* $s\mapsto\zeta_{N,n}(s,w|a_1,\dots,a_N)$ *is meromorphic; it has at most simple poles for* $s=(N-l)/n$ *with* $l\in\mathbb{N}$, *and has no poles for* $s=-k$ *with* $k\in\mathbb{N}$.

As remarked earlier, the meromorphic continuation of $\zeta_{N,n}(s,\mathscr{M},x)$ was found by Shintani [13, Section 1]. He did not explicitly locate the poles, but they are easily deduced from his formulas.

Proof. Definition (2.12) of $T_{M,j}(S)$ entails that the functions

$$\frac{T_{M,j}(S)}{\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\Gamma(S_k)},\quad 1\leqslant j\leqslant n, \tag{2.14}$$

are entire. Hence we deduce from (2.13) and its analogs for $I_2,\dots,I_n$ that the functions

$$\frac{I_j(S,w,\mathscr{M})}{\Gamma(-N+\sum_{k=1}^n S_k)\prod_{k\neq j}\Gamma(S_k)},\quad j=1,\dots,n,$$

extend holomorphically to the domains given by

$$\mathrm{Re}(S_k)>-M-1,\ k\neq j,\ \sum_{k=1}^n\mathrm{Re}(S_k)>-M+N,\ (w,\mathscr{M})\in D_n\times\mathscr{D}_{N,n}.$$

As M is arbitrary, the first assertion of the proposition now readily follows from (2.8). Taking $S_j=s$ for all j, we obtain the second one. □

Since $s=0$ is a regular point of $\zeta_{N,n}(s,w|a_1,\dots,a_N)$, we can define a multiple gamma function by (1.14). Then the gamma recurrence (1.15) easily follows from the zeta recurrence (1.7). From its definition we see that $\Gamma_{N,n}(w|a_1,\dots,a_N)$ is holomorphic on $D_n \times \mathscr{D}_{N,n}$. We now use (1.15) to show that $\Gamma_{N,n}(w|a_1,\dots,a_N)$ continues meromorphically to $\mathbb{C}^n \times \mathscr{D}_{N,n}$.

Proposition 2.2. *The function $1/\Gamma_{N,n}(w|\, a_1,\dots,a_N)$ extends to a holomorphic function on $\mathbb{C}^n \times \mathscr{D}_{N,n}$. Its zero locus consists of the hyperplanes $\mathscr{H}_j := \{w\in\mathbb{C}^n|\, w_j = 0\}$ and their translates $\mathscr{H}_j - (m_1a_1 + \cdots + m_Na_N)$, with $j = 1,\dots,n$ and $m_1,\dots,m_N \in \mathbb{N}$.*

Although Shintani did not explicitly consider the nature of $\Gamma_{N,n}(w)$, Proposition 2.2 for $w = W(x)$ (cf. (1.6)) is a direct consequence of Barnes' study of Γ_N and of Shintani's formula [16, p. 204] relating $\Gamma_{N,n}$ to Γ_N (see Proposition 3.2 below).

Proof. Taking $N = 1$, we iterate the difference equation (1.15), obtaining

$$\frac{1}{\Gamma_{1,n}(w|\, a_1)} = \left(\prod_{m=0}^{l-1}\prod_{j=1}^{n}(w_j + ma_{1j})\right)\frac{1}{\Gamma_{1,n}(w + la_1|\, a_1)}, \quad l\in\mathbb{N}_+.$$

From this we deduce that $1/\Gamma_{1,n}(w|\, a_1)$ extends holomorphically to $(D_n - la_1) \times \mathscr{D}_{1,n}$. Since $\mathrm{Re}(a_{1j})>0$ for $j = 1,\dots,n$, and l is arbitrary, it follows that $1/\Gamma_{1,n}(w|a_1)$ extends to a holomorphic function on $\mathbb{C}^n \times \mathscr{D}_{1,n}$, whose zero locus equals $\mathscr{H}_j - ma_1$ with $j = 1,\dots,n$ and $m\in\mathbb{N}$. Using induction on N, we now obtain the proposition. □

3. Degree-m polynomials at $s = (N - m)/n$

The ladder of difference equations (1.7) begins with $\zeta_{0,n}(s,w) = w_1^{-s}w_2^{-s}\cdots w_n^{-s}$, which is evidently a polynomial of degree kn when $-s$ equals an integer $k\in\mathbb{N}$. Although we know of no theory of minimal solutions to first-order partial difference equations in $\mathbb{C}^n$, it is natural to surmise that the Nth level of the ladder $\zeta_{N,n}(-k,w)$ is a polynomial of degree N more than that of the base level.

As a first step, we prove the polynomial property by showing that the pertinent w-derivatives of $\zeta_{N,n}(s,w)$ vanish identically at these s-values. Specifically, for a multi-index $J = (J_1,\dots,J_n)$ of weight $|J| = \sum_{j=1}^n J_j$, denote by ∂_w^J the differential operator $\frac{\partial^{|J|}}{\partial w_1^{J_1}\cdots\partial w_n^{J_n}}$. In the region of absolute convergence of the Dirichlet series, direct differentiation yields

$$\begin{aligned}\partial_w^J\zeta_{N,n}(s,w) &= (-1)^{|J|}\left(\prod_{j=1}^{n}\prod_{p=0}^{J_j-1}(s+p)\right)\sum_{m\in\mathbb{N}^N}\prod_{j=1}^{n}(w_j + m_1a_{1j} + \cdots + m_Na_{Nj})^{-s-J_j}\\ &= (-1)^{|J|}Z_{N,n}((s+J_1,\dots,s+J_n),w,\mathscr{M})\prod_{j=1}^{n}\prod_{p=0}^{J_j-1}(s+p), \qquad (3.1)\end{aligned}$$

where $Z_{N,n}$ was defined in (2.1). The above series converges for $N < \sum_{j=1}^{n} \mathrm{Re}(s + J_j) = |J| + n\,\mathrm{Re}(s)$. In particular, by analytic continuation in s, series (3.1) represents $\partial_w^J \zeta_{N,n}(-k, w)$ for $|J| > N + kn$.

Just as in the example of the Hurwitz zeta function in Section 1, the analyticity of $\partial_w^J \zeta_{N,n}(s, w)$ for $\mathrm{Re}(s) > (N - |J|)/n$ entails that the residues at the poles in this s-region have been differentiated away. Therefore, the residues at the poles $(N - m)/n$ with $m \in \mathbb{N}$ are polynomials of degree at most m. For m of the form $N + kn$ with $k \in \mathbb{N}$, we showed in Proposition 2.1 that there are no poles. Since the product term in (3.1) vanishes for $s = -k$ and $|J| > N + kn$, we infer that $\zeta_{N,n}(-k, w)$ is a polynomial of degree at most $N + kn$. We now extend these results.

Proposition 3.1. *The functions*

$$P_{kn+N,N,n}(w) := \zeta_{N,n}(-k, w), \quad k \in \mathbb{N}, \tag{3.2}$$

are polynomials of degree $kn + N$. For $N \geqslant 1$, $\zeta_{N,n}(s, w)$ has simple poles at $s = s_{l,N,n}$, where

$$s_{l,N,n} := (N - l)/n, \ \ l \in \mathbb{N} \backslash (N + n\mathbb{N}), \tag{3.3}$$

with residues $P_{l,N,n}(w)$ that are polynomials of degree l, except possibly for non-generic $\mathscr{M} \in \mathscr{D}_{N,n}$. More precisely, whenever $s_{l,N,n} \leqslant 1/n$, the degree equals l on all of $\mathscr{D}_{N,n}$; in particular, the degree of the polynomials $P_{m,1,n}(w|\, a)$ equals m for all $m \in \mathbb{N}$ and $a \in \mathscr{D}_{1,n}$. For $N > 1$ and $s_{l,N,n} > 1/n$, the degree equals l on the polysector $\mathscr{S}_{N,n}(\frac{\pi}{2(N-l)})$, where

$$\mathscr{S}_{N,n}(\phi) := \{\mathscr{M} \in \mathscr{D}_{N,n} |\, |\mathrm{Arg}(a_{ij})| < \phi\}, \quad \phi \in (0, \pi/2). \tag{3.4}$$

All the above statements concerning the regular values were proved by Shintani [13, Section 1] in the case $w = W(x)$ (cf. (1.6)).

Proof. For the regular values $s = -k$, it remains to show that the degree of the polynomials (3.2) equals the upper bound $kn + N$ already established above. We prove this via the difference equations (1.7), as follows.

Let us assume that the polynomial (3.2) has degree $L < kn + N$. Now consider the monomials of highest degree L occurring in the two polynomials on the lhs of

$$\begin{aligned} &P_{kn+N,N,n}(w + a_N | a_1, \ldots, a_N) - P_{kn+N,N,n}(w | a_1, \ldots, a_N) \\ &\quad = -P_{kn+N-1,N-1,n}(w | a_1, \ldots, a_{N-1}). \end{aligned} \tag{3.5}$$

Clearly, their differences yield terms whose degree is at most $L - 1$. Thus the degree of the polynomial on the right-hand side is at most $L - 1$, too. Repeating this argument, we deduce that $P_{kn,0,n}(w)$ has degree at most $kn - 1$, contradicting $\zeta_{0,n}(-k, w) = w_1^k \cdots w_n^k$.

Passing to the pole and residue assertions, we first study the case that the numbers w_j and a_{ij} are positive. Consider the behavior of series (1.5) for real s near $s_{0,N,n}$. It follows from the paragraph containing estimate (2.3) that it diverges as $s \downarrow s_{0,N,n}$. Thus there must be a pole at $s = s_{0,N,n}$, yielding a constant non-zero residue. Turning to $s_{1,N,n}$ (for $N > 1$), we inspect (3.1) with $|J| = 1$. As before, we get a divergence for $s \downarrow s_{1,N,n}$, so $s_{1,N,n}$ is a pole with residue a degree-1 polynomial. Clearly, this reasoning can be repeated for $s = s_{l,N,n}$, so for positive a_{ij} the degree is always l. By analyticity in $\mathscr{D}_{N,n}$, the degree is therefore generically equal to l on $\mathscr{D}_{N,n}$.

To obtain the stronger assertions concerning the degree, we first take $N = 1$. Now we reconsider series (1.5), fixing $w \in (0, \infty)^n$ and $a \in \mathscr{D}_{1,n}$. Since a is fixed, it belongs to a sector $\mathscr{S}_{1,n}(\phi)$ for some $\phi < \pi/2$. As we let $s \downarrow s_{0,N,n} = 1/n$, all of the terms in the series belong to the sector $\mathscr{S}_{1,n}(\phi_0)$ with $\phi_0 \in (\phi, \pi/2)$ for s sufficiently close to $s_{0,N,n}$. Then the real parts of the terms in the series are bounded below by $\cos(\phi_0)$ times their modulus, so divergence as $s \downarrow s_{0,N,n}$ follows as before from (2.3). For $s_{l,N,n}$ we apply this argument to series (3.1) with $|J| = l$, obtaining once more divergence as $s \downarrow s_{l,N,n}$. We have therefore proved the degree assertion for $N = 1$.

Letting now $N > 1$ and $s_{l,N,n} \leqslant 1/n$ (so that $l \geqslant N - 1$), we can use the difference equations (1.7) in the same way as before to obtain the degree l assertion. To be quite specific, we can multiply (1.7) by $(s - s_{l,N,n})$ and take s to $s_{l,N,n}$ to get

$$P_{l,N,n}(w + a_N | a_1, \dots, a_N) - P_{l,N,n}(w | a_1, \dots, a_N) = -P_{l-1,N-1,n}(w | a_1, \dots, a_{N-1}). \quad (3.6)$$

Iterating downward, we can relate $P_{l,N,n}$ to the degree-$(l - N + 1)$ polynomial $P_{l-N+1,1,n}$.

It remains to prove the last assertion. Taking first $l = 0$, we need only inspect the argument variation of the terms in the pertinent series to obtain the desired divergence for $\mathscr{M}$ in the specified sector. For $l > 0$ we can use (3.6) once more, this time to relate $P_{l,N,n}$ to the non-zero constant $P_{0,N-l,n}$. □

In fact, we surmise that the non-generic subsets of $\mathscr{D}_{N,n}$ where the degree is lower than l are empty.

We proceed by pointing out that (3.1) yields in particular

$$\frac{\partial^n}{\partial w_1 \partial w_2 \cdots \partial w_n} \zeta_{N,n}(s, w) = (-s)^n \zeta_{N,n}(s + 1, w). \quad (3.7)$$

For $s = -k$ with $k \in \mathbb{N}_+$, this may be viewed as a generalization of the Bernoulli property (1.32). Specializing to the Barnes case $n = 1$, it entails that the above non-generic subsets of $\mathscr{D}_{N,1}$ are indeed empty. To explain this, we recall that $\zeta_N(0, w)$ has degree N, so (3.7) implies that the residue at $s = 1$ has degree $N - 1$, etc. (Alternatively, the explicit residue formulas in terms of Barnes' multiple Bernoulli polynomials can be invoked, cf. Eq. (3.9) in [11].) Unfortunately, for $n > 1$ the partial differential operator occurring in (3.7) can lower the degree of polynomials by more than n, so that it cannot be used to rule out non-generic degree lowering.

We can use Proposition 3.1 to verify that $Z_{N,n}(S, w, \mathcal{M})$ is singular at $S = 0$ for $N > 0$ and $n > 1$, as mentioned in Section 2. Indeed, if the origin were regular, it would follow from (2.4) that $Z_{N,n}(0, w, \mathcal{M})$ equals $\zeta_N(0, w_1|a_{11}, \ldots, a_{N1})$ (by taking $s \to 0$). Likewise, $Z_{N,n}(0, w, \mathcal{M})$ would be equal to $\zeta_N(0, w_j|a_{1j}, \ldots, a_{Nj})$ for $1 < j \leqslant n$. This would imply that $\zeta_N(0, w_1|a_{11}, \ldots, a_{N1})$ is constant as a function of w_1, contradicting Proposition 3.1.

Turning to the multiple gamma function $\Gamma_{N,n}(w)$ (1.14), we notice that at the base level $N = 0$ of the ladder (1.15) we have

$$\log \Gamma_{0,n}(w) = \sum_{j=1}^{n} \log \Gamma_{0,1}(w_j).$$

As before, we expect the same to hold for arbitrary N, up to a polynomial of degree at most N. To study this, we note that for $|J| \geqslant N + 1$, the series in (3.1) converges for $\mathrm{Re}(s) > -1/n$. We can differentiate it with respect to s and set $s = 0$ to obtain

$$\partial_s \partial_w^J \zeta_{N,n}(s, w|a_1, \ldots, a_N)|_{s=0}$$
$$= \begin{cases} (-1)^{|J|}\Gamma(|J|)\zeta_N(|J|, w_j|a_{1j}, \ldots, a_{Nj}), & \text{if } |J| = J_j \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the above to $\zeta_{N,n}$ and to $\zeta_{N,1} = \zeta_N$, we obtain for all $|J| \geqslant N + 1$,

$$0 = \partial_w^J \left(\partial_s \zeta_{N,n}(s, w|a_1, \ldots, a_N)|_{s=0} - \sum_{j=1}^{n} \partial_s \zeta_N(s, w_j|a_{1j}, \ldots, a_{Nj})|_{s=0} \right).$$

Recalling definition (1.14), we see that $\Gamma_{N,n}$ reduces to a sum of Barnes' Γ_N functions, up to a polynomial of degree at most N.

We now render this result more precise. To this end we define for $a \in D_n$ a coefficient vector $c(a) \in \mathbb{C}^n$ by

$$c(a)_j := \frac{1}{n a_j} \sum_{k=1}^{n} (\log(a_k) - \log(a_j)), \quad j = 1, \ldots, n, \tag{3.8}$$

the logarithm branch being the principal one.

Proposition 3.2. *Let $N \geqslant 1$ and $n \geqslant 2$. Then we have*

$$\log \Gamma_{N,n}(w|a_1, \ldots, a_N) = \Pi_{N,n}(w|a_1, \ldots, a_N) + \sum_{j=1}^{n} \log \Gamma_N(w_j|a_{1j}, \ldots, a_{Nj}), \tag{3.9}$$

with $\Pi_{N,n}(w)$ a polynomial of degree at most N. Moreover, $\Pi_{1,n}(w)$ is given by

$$\Pi_{1,n}(w|a) = \sum_{j=1}^{n} w_j c(a)_j, \tag{3.10}$$

and the degree of $\Pi_{N,n}(w|a_1,\dots,a_N)$ *equals* N *whenever at least one of* $a_1,\dots,a_N$ *satisfies* $c(a_i)\neq 0$.

For $w=W(x)$, the polynomial in (3.9) was made quite explicit by Shintani [16, pp. 204, 206]. We return to his formula below (5.10).

Proof. Thanks to the difference equations

$$\Pi_{N,n}(w+a_N|a_1,\dots,a_N)-\Pi_{N,n}(w|a_1,\dots,a_N)=-\Pi_{N-1,n}(w|a_1,\dots,a_{N-1}), \quad (3.11)$$

we need only show (3.10). Indeed, we have already proved that $\Pi_{N,n}(w)$ is a polynomial of degree at most N, and (3.10) shows that $\Pi_{1,n}(w|a)$ has degree 1, provided $c(a)\neq 0$. Hence the degree assertion follows from (3.11) by the argument in the paragraph containing (3.5).

By analyticity in w, it suffices to prove (3.10) for $w\in(0,\infty)^n$, which we require from now on. We begin by observing that for $\mathrm{Re}(s)>1/n$ we can invoke the Dirichlet series representation to obtain

$$\zeta_{1,n}(s,w|a)=\left(\prod_{j=1}^{n}a_j^{-s}\right)\zeta_{1,n}(s,(w_1/a_1,\dots,w_n/a_n)|\,\mathbf{1}), \quad \mathbf{1}_j:=1,\ j=1,\dots,n. \quad (3.12)$$

Likewise, for $\mathrm{Re}(s)>1$ we have

$$\zeta_{1,1}(s,w_j|\,a_j)=a_j^{-s}H(s,w_j/a_j), \quad j=1,\dots,n, \quad (3.13)$$

with $H(s,w)$ the Hurwitz zeta function (1.8). Using meromorphic continuation in s and regularity at $s=0$, this yields

$$\begin{aligned}\Pi_{1,n}(w|a)=&-\sum_{j=1}^{n}\log(a_j)[\zeta_{1,n}(0,(w_1/a_1,\dots,w_n/a_n)|\,\mathbf{1})-H(0,w_j/a_j)]\\&+\partial_s\zeta_{1,n}(s,(w_1/a_1,\dots,w_n/a_n)|\,\mathbf{1})|_{s=0}-\sum_{j=1}^{n}H'(0,w_j/a_j).\end{aligned} \quad (3.14)$$

Next, we study the difference function

$$\zeta_{1,n}(s,x|\,\mathbf{1})-\frac{1}{n}\sum_{j=1}^{n}H(ns,x_j)=:d_n(s,x), \quad x\in[1,\infty)^N. \quad (3.15)$$

From the series representation we have

$$d_n(s,x)=\sum_{m=0}^{\infty}I_n(m,s,x), \quad \mathrm{Re}(s)>1/n, \quad (3.16)$$

where

$$I_n(m,s,x) := \prod_{j=1}^{n} (x_j+m)^{-s} - \frac{1}{n}\sum_{j=1}^{n}(x_j+m)^{-ns}. \tag{3.17}$$

Here, $x \in [1, \infty)^N$ is viewed as fixed, whereas m varies over $\mathbb{N}$ and s over $\mathbb{C}$, with the principal branch of the logarithm understood for the complex powers. Clearly, this function is entire in s and obeys the bound

$$I_n(m,s,x) = O(m^{-ns-2}), \quad m \to \infty. \tag{3.18}$$

Hence we infer that the series on the rhs of (3.16) converges for $\mathrm{Re}(s) > -1/n$. By analytic continuation, it follows that (3.16) actually holds for $\mathrm{Re}(s) > -1/n$.

Since $I_n(m,s,x)$ vanishes at $s=0$, we obtain from (3.15)

$$\zeta_{1,n}(0,x|\,\mathbf{1}) = \frac{1}{n}\sum_{j=1}^{n} H(0,x_j) = \frac{1}{2} - \frac{1}{n}\sum_{j=1}^{n} x_j, \tag{3.19}$$

where we used (1.33). By analyticity in s, we may interchange the s-derivative of (3.16) with the summation, so we also have

$$\partial_s d_n(s,x) = \sum_{m=0}^{\infty} \partial_s I_n(m,s,x), \quad \mathrm{Re}(s) > -1/n. \tag{3.20}$$

Now the s-derivative of I_n vanishes at $s=0$ as well, so we obtain from (3.15)

$$\partial_s \zeta_{1,n}(s,x|\,\mathbf{1})|_{s=0} = \sum_{j=1}^{n} H'(0,x_j). \tag{3.21}$$

We continue by noting that the functions appearing in (3.19) and (3.21) are analytic for $x \in D_n$, so that we may substitute $x_j \to w_j/a_j$. Substitution in (3.14) then yields

$$\Pi_{1,n}(w|a) = -\sum_{j=1}^{n} \log(a_j)\left(\left(\frac{1}{2} - \frac{1}{n}\sum_{k=1}^{n}\frac{w_k}{a_k}\right) - \left(\frac{1}{2} - \frac{w_j}{a_j}\right)\right).$$

This can be rewritten as (3.10), so our proof is complete. □

We have gone to some lengths to obtain the explicit formula (3.10), since it has two illuminating features (apart from implying the degree-N property). First, consider the difference equation (3.11) with $N=1$. Obviously, the right-hand side vanishes, so if we were dealing with an ordinary difference equation, it would follow that $\Pi_{1,n}(w)$ could be at most a non-zero constant. That $\Pi_{1,n}(w)$ can have degree 1 for the partial difference equation at issue is due to the existence in $\mathbb{C}^n$, $n>1$, of a coefficient vector $c(a)$ (see (3.8)) that is orthogonal to a in the sense that

$$\sum_{j=1}^{n} a_j c(a)_j = 0. \tag{3.22}$$

Secondly, when we specialize $\Pi_{1,n}(w|a)$ to the Shintani case, where w is replaced by $W(x) = xa$, the polynomial vanishes identically by (3.22).

Another interesting corollary of our calculation leading to (3.10) is an explicit formula for the residue of $\zeta_{1,n}(s, w|a)$ at its simple pole $s = 1/n$, namely,

$$\lim_{s\to 1/n}\left(s-\frac{1}{n}\right)\zeta_{1,n}(s,w|a) = \frac{1}{n}\prod_{j=1}^{n} a_j^{-1/n}. \tag{3.23}$$

Indeed, the residue of $H(s, w)$ at $s = 1$ equals 1 (as is for example plain from (1.11) with $M = 1$), so from (3.15) and analyticity of $d_n(s, x)$ for $\mathrm{Re}(s) > -1/n$ we obtain (3.23) for $a = \mathbf{1}$. The general case is then clear from (3.12).

4. Vanishing zeta and gamma integrals

In this section we obtain the "vanishing period integrals" (1.26)–(1.27) and their corollary (1.28).

Proposition 4.1. *Let* $N \in \mathbb{N}_+$, $\mathrm{Re}(s) < N/n$ *and* s *not a pole of* $\zeta_{N,n}(s, w)$. *Then the function* $x \mapsto \zeta_{N,n}(s, W(x))$ *is integrable with respect to Lebesgue measure* dx *on the unit* N*-dimensional cube* $I^N := (0,1)^N$, *and satisfies*

$$\int_{I^N} \zeta_{N,n}(s, W(x))\, dx = 0, \quad \mathrm{Re}(s) < N/n. \tag{4.1}$$

More generally, $x \mapsto \partial_x^J \zeta_{N,n}(s, W(x))$ *is integrable on* I^N *for* $\mathrm{Re}(s) < (N - |J|)/n$, *and satisfies*

$$\int_{I^N} \partial_x^J \zeta_{N,n}(s, W(x))\, dx = 0, \quad \mathrm{Re}(s) < (N - |J|)/n. \tag{4.2}$$

Finally, *we have the integrals*

$$\int_{I^N} \partial_x^J P_{m,N,n}(W(x))\, dx = 0, \quad 0 \leqslant |J| < m, \ m \in \mathbb{N}_+, \tag{4.3}$$

where $P_{m,N,n}(w)$ *are the polynomials from Proposition* 3.1.

Corollary 4.2. *The function* $\log \Gamma_{N,n}(W(x))$ *is integrable on* I^N, *and satisfies*

$$\int_{I^N} \log \Gamma_{N,n}(W(x))\, dx = 0. \tag{4.4}$$

More generally, *for* $N > 1$ *and* $|J| < N$ *the function* $\partial_x^J \log \Gamma_{N,n}(W(x))$ *is integrable on* I^N, *and satisfies*

$$\int_{I^N} \partial_x^J \log \Gamma_{N,n}(W(x))\, dx = 0, \quad 0 \leqslant |J| < N. \tag{4.5}$$

Proof of Proposition 4.1. Just as in the Hurwitz case treated in Section 1, we subtract the $m = 0$ term from the Dirichlet series (1.5). This yields a function

$$\widehat{\zeta}_{N,n}(s,w) := \zeta_{N,n}(s,w) - \prod_{j=1}^{n} w_j^{-s}, \tag{4.6}$$

whose analyticity domain in the w_j is larger than D_n, containing in particular the origin, where the $m = 0$ term is singular. Specifically, an inspection of the series shows that for $\mathrm{Re}(s) > N/n$ we have holomorphy in

$$V_n := \{w \in \mathbb{C}^n \mid \mathrm{Re}(w_j) > -c, \ 1 \leqslant j \leqslant n\},$$

with $c > 0$ given by (2.2) (say).

To see that this actually holds true whenever s is not a pole of $\zeta_{N,n}(s,w)$, we note that the difference equation (1.7) yields

$$\widehat{\zeta}_{N,n}(s,w) = \zeta_{N,n}(s, w + a_N) + \widehat{\zeta}_{N-1,n}(s,w), \quad \widehat{\zeta}_{0,n}(s,w) = 0.$$

By induction on N, this entails that $\widehat{\zeta}_{N,n}(s,w)$ is indeed holomorphic in V_n.

Thus, for any value of s outside the poles, $\widehat{\zeta}_{N,n}(s, W(x))$ is infinitely differentiable in x on an open neighborhood of $[0,1]^N$. The map $x \mapsto \widehat{\zeta}_{N,n}(s, W(x))$, as well as any of its x-derivatives, is therefore integrable on $I^N = (0,1)^N$, so the function

$$\widehat{Q}^J(s) := \int_{I^N} \partial_x^J \widehat{\zeta}_{N,n}(s, W(x))\, dx \tag{4.7}$$

is well defined and analytic in s away from the poles of $\zeta_{N,n}(s,w)$.

Now from (4.6) we have for $x \in I^N$

$$\zeta_{N,n}(s, W(x)) = \widehat{\zeta}_{N,n}(s, W(x)) + W(x)^{-S(s)}, \tag{4.8}$$

where

$$w^S := \prod_{j=1}^{n} w_j^{S_j}, \quad S(s) := (s, s, \ldots, s) \in \mathbb{C}^n, \quad w \in D_n.$$

It follows that $\partial_x^J \zeta_{N,n}(s, W(x))$ is integrable on I^N if and only if $\partial_x^J W(x)^{-S(s)}$ is. (We shall see shortly that this happens if $\mathrm{Re}(s) < (N - |J|)/n$.)

We proceed by computing $\widehat{Q}^J(s)$, assuming first that s belongs to the region $\mathrm{Re}(s) > N/n$, where the Dirichlet series and all series obtained by taking term-wise derivatives converge absolutely. Then we have from (4.6) and (4.7)

$$\begin{aligned} \widehat{Q}^J(s) &= \int_{I^N} \partial_x^J \sum_m{}' \prod_{j=1}^{n} (W(x)_j + m_1 a_{1j} + \cdots + m_N a_{Nj})^{-s}\, dx \\ &= \int_{I^N} \partial_x^J \sum_m{}' (W(x+m))^{-S(s)}\, dx = \int_{\mathbb{R}_+^N - I^N} \partial_x^J W(x)^{-S(s)}\, dx, \end{aligned} \tag{4.9}$$

where the sums are over all $m=(m_1,\dots,m_N)$, save for the term $m=0$. By induction on $|J|$, we see that the integrand is of the form

$$\begin{aligned}\partial_x^J W(x)^{-S(s)} &= \sum_I c_I^J(s) \prod_{j=1}^n (x_1 a_{1j} + \cdots + x_N a_{Nj})^{-s-I_j} \\ &= \sum_I c_I^J(s)\, W(x)^{-S(s)-I}, \end{aligned} \tag{4.10}$$

where $I=(I_1,\dots,I_n)$ ranges over multi-indices satisfying $|I|=|J|$, and $c_I^J(s)$ is a polynomial in s whose coefficients depend on J and on the a_{ij}, but not on x.

We now switch to cubical coordinates (ρ,σ) on $\mathbb{R}^N$, as we did in Section 2 (between formulas (2.8) and (2.9)) without giving them this name. Namely, we adopt coordinates with respect to the unit "sphere"

$$C^{N-1} := \{\sigma\in\mathbb{R}^N \mid ||\sigma||_\infty = 1\}, \quad ||x||_\infty := \max_{1\leqslant i\leqslant N}\{|x_i|\}.$$

Thus, $\rho=\rho(x):=||x||_\infty$ and $\sigma=\sigma(x):=\rho^{-1}x\in C^{N-1}$. The new volume element is $\rho^{N-1}\,d\rho\,d\sigma$, where $d\sigma$ is the $(N-1)$-dimensional Lebesgue measure on C^{N-1}. We let $C_+^{N-1}:=C^{N-1}\cap\mathbb{R}_+^N$.

Then we have

$$\int_{\mathbb{R}_+^N - I^N} W(x)^{-S(s)-I}\,dx = \int_{\sigma\in C_+^{N-1}} W(\sigma)^{-S(s)-I} \int_{\rho=1}^{\infty} \rho^{N-1-ns-|J|}\,d\rho\,d\sigma. \tag{4.11}$$

The crux is now that the ρ-integral in (4.11) is elementary, so that from (4.9) and (4.10) we get

$$\widehat{Q}^J(s) = -\frac{1}{N-|J|-ns} \sum_I c_I^J(s) \int_{\sigma\in C_+^{N-1}} W(\sigma)^{-S(s)-I}\,d\sigma. \tag{4.12}$$

The integrals over C_+^{N-1} in (4.12) yield entire functions of s, and the coefficients $c_I^J(s)$ are polynomials in s. As a consequence, (4.12) holds for $s\in\mathbb{C}$. (In particular, the only eventual pole of $\widehat{Q}^J(s)$ occurs for $s=(N-|J|)/n$.)

Next, we use (4.10) and the above change of variables to verify that the function

$$P^J(s) := \int_{I^N} \partial_x^J W(x)^{-S(s)}\,dx \tag{4.13}$$

is well defined and analytic for $\mathrm{Re}(s)<(N-|J|)/n$. Indeed, we have

$$P^J(s) = \sum_I c_I^J(s) \int_{\sigma\in C_+^{N-1}} W(\sigma)^{-S(s)-I} \int_{\rho=0}^{1} \rho^{N-1-ns-|J|}\,d\rho\,d\sigma, \tag{4.14}$$

where the σ-integrand is bounded and the ρ-integral is again elementary. Comparing the result to (4.12), we obtain

$$P^J(s) = -\widehat{Q}^J(s), \quad \mathrm{Re}(s)<(N-|J|)/n. \tag{4.15}$$

This is the key equality: combining it with (4.8) and definitions (4.7) and (4.13) of $\widehat{Q}^J(s)$ and $P^J(s)$, resp., we obtain integral (4.2).

Integrals (4.3) with $m = kn + N,\ k \in \mathbb{N}$, are immediate from (4.2) with $s = -k$, cf. (3.2). To obtain them for the residue polynomials, we need only divide all of the above quantities by $\Gamma(ns - N)$ to obtain entire functions of s. Letting s converge to the locations $s_{l,N,n}$ (3.3), the residue integrals (4.3) result. □

To appreciate the above proof in one fell swoop, it may help to reinspect the reasoning for the Hurwitz case, cf. the paragraph containing (1.21). Key equality (4.15) can be viewed as a higher-dimensional version of the equality of the integrals $\int_0^1 x^{-s}\,dx$ and $-\int_1^\infty x^{-s}\,dx$, in the sense of their analytic continuations to $\mathbb{C}\backslash\{1\}$ being equal.

Proof of Corollary 4.2. As we have seen in the above proof, the integrand in (4.7) is continuous in (s, x) on sets of the form $K \times [0, 1]^N$, where K is any s-compact not containing poles of $\zeta_{N,n}$. Thus we may interchange the s-derivative of the integral with the integration. Take $|J| < N$ and $\mathrm{Re}(s) < 1/n$ from now on. Using the Cauchy integral formula and dominated convergence, we deduce from Eq. (4.14) that the s-derivative of $P^J(s)$ exists and that we may interchange the s-derivative of the integral on the rhs of (4.13) with the integration.

Recalling (4.8), we now see that $\partial_s \partial_x^J \zeta_{N,n}(s, W(x))$ is integrable on I^N, and that we have

$$\frac{d}{ds} \int_{I^N} \partial_x^J \zeta_{N,n}(s, W(x))\,dx = \int_{I^N} \partial_s \partial_x^J \zeta_{N,n}(s, W(x))\,dx. \tag{4.16}$$

Since the lhs vanishes by (4.2), so does the rhs. Hence the assertions follow upon taking $s = 0$ (recall (1.14)). □

5. Applications of the vanishing period integrals to Shintani–Barnes functions

In this section we exploit the results of Section 4 to derive various explicit formulas, most of which were obtained before by Shintani. We shall need

Lemma 5.1. *Suppose $R(x) \in \mathbb{C}[x_1, \ldots, x_N]$ is a polynomial of degree at most m satisfying*

$$\int_{I^N} \partial_x^J R(x)\,dx = 0, \quad 0 \leqslant |J| < m. \tag{5.1}$$

Then we have the identities

$$R(\mathbf{1} - x) = (-1)^m R(x), \quad (\mathbf{1} - x)_i := 1 - x_i,\ 1 \leqslant i \leqslant N, \tag{5.2}$$

and

$$R(x) = \sum_{\substack{L \\ |L|=m}} c_L \frac{B_L(x)}{L!}, \tag{5.3}$$

where L ranges over all N-multi-indices of weight m,

$$c_L := \int_{I^N} \partial_x^L R(x)\, dx, \quad L! := \prod_{i=1}^N L_i!, \quad B_L(x) := \prod_{i=1}^N B_{L_i}(x_i), \tag{5.4}$$

with $B_l(t)$ *the Bernoulli polynomial defined in* (1.29). *Moreover*, *any polynomial of form* (5.3) *satisfies* (5.1).

Proof. Since $R(x)$ has degree at most m, the difference polynomial

$$D(x) := R(\mathbf{1} - x) - (-1)^m R(x)$$

has degree at most $m - 1$. Next, we observe that (5.1) and the change of variables $x \mapsto \mathbf{1} - x$ imply

$$\int_{I^N} \partial_x^J D(x)\, dx = 0, \quad 0 \leqslant |J| \leqslant m - 1.$$

Hence it recursively follows that the coefficients of the terms of degree $m - 1$, $m - 2, \dots, 0$ vanish, yielding (5.2).

To prove (5.3), we first use (1.31) and (1.32) to obtain

$$\int_{I^N} \partial_x^J B_L(x)\, dx = \begin{cases} L! & \text{if } L = J; \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Now let c_L be given by (5.4) and set

$$Q(x) := R(x) - \sum_{\substack{L \\ |L|=m}} c_L \frac{B_L(x)}{L!}.$$

Then (5.1) and (5.5) imply $\int_{I^N} \partial_x^J Q(x)\, dx = 0$ for all J, proving (5.3). The final statement in the lemma follows from (5.5). □

Specializing the argument yielding (5.2) to $N = 1$, we obtain the well-known identity $B_l(1 - t) = (-1)^l B_l(t)$. Note that the latter, combined with (5.3), yields an alternative proof of (5.2).

Combining Lemma 5.1 with our previous information on the polynomials $\Pi_{N,n}$ and $P_{m,N,n}$, we easily obtain the following formulas.

Corollary 5.2. *The above polynomials have the symmetries*

$$\Pi_{N,n}(W(\mathbf{1}-x)) = (-1)^N \Pi_{N,n}(W(x)), \tag{5.6}$$

$$P_{m,N,n}(W(\mathbf{1}-x)) = (-1)^m P_{m,N,n}(W(x)), \quad m \in \mathbb{N}, \tag{5.7}$$

and admit the "Bernoulli representations"

$$\Pi_{N,n}(W(x)) = \sum_{|L|=N} \gamma_{L,N,n} \frac{B_L(x)}{L!}, \tag{5.8}$$

$$P_{m,N,n}(W(x)) = \sum_{|L|=m} g_{L,m,N,n} \frac{B_L(x)}{L!}. \tag{5.9}$$

Here, the coefficients can be written

$$\gamma_{L,N,n} = \int_{I^N} \partial_x^L \Pi_{N,n}(W(x))\, dx, \quad g_{L,m,N,n} = \int_{I^N} \partial_x^L P_{m,N,n}(W(x))\, dx. \tag{5.10}$$

Proof. Using (3.9) and (4.5), we obtain

$$\int_{I^N} \partial_x^J \Pi_{N,n}(W(x))\, dx = 0, \quad 0 \leqslant |J| < N.$$

Since by Proposition 3.2, $\Pi_{N,n}(W(x))$ has degree at most N, Lemma 5.1 entails (5.8) and the integral formula (5.10) for $\gamma_{L,N,n}$. Likewise, (5.9) and the coefficient formula (5.10) follow from (4.3). Finally, the symmetries (5.6) and (5.7) are plain from (5.2). □

We recall that for $m = kn + N$ with $k \in \mathbb{N}$, the polynomial $P_{m,N,n}(W(x))$ equals $\zeta_{N,n}(-k, W(x))$. In this case, representation (5.9) was obtained first by Shintani [13, p. 398]. Moreover, Shintani not only obtained (5.8), but also stated an impressive formula for $\gamma_{L,N,n}$ [16, p. 206]. It is not clear how he obtained the latter, and we have not tried to supply a proof.

We sketch, however, a proof that if $N \leqslant n$, then $\Pi_{N,n}(w)$ is a sum of monomials of the kind $w_i^{b_i} w_j^{b_j}$, i.e., no products of three or more distinct w_j's appear in $\Pi_{N,n}(w)$. This fact is not clear from Shintani's formula. For the proof we may assume $N = n$, as the case $N < n$ follows from the difference equation. In this "equidimensional" case, the techniques in Sections 4 and 5 involving derivatives with respect to the x_i can all be replaced by derivatives with respect to the w_j. The integrals over I^N are now replaced by integrals over the convex span P of the a_i. Where we had $W(x)$ for x in I^N, we now have w in P. The advantage is that the w-partials $\partial_w^J w^{-S(s)}$ are very much simpler than (4.10). By studying these we find that there are no terms of degree

N involving three distinct w_j's. Using the vanishing of integrals in degree less than N, one can show that there are no terms in $\Pi_{N,n}(w)$ of any degree involving three or more distinct w_j's.

The results in the following proposition are due to Shintani [16, pp. 204, 206, 210] when $w = W(x)$.

Proposition 5.3. *Setting* $A := a_1 + a_2 + \cdots + a_N$, *we have the identity*

$$\Gamma_{N,n}(w)\,(\Gamma_{N,n}(A-w))^{(-1)^{N+1}} = \prod_{j=1}^{n} \Gamma_N(w_j)(\Gamma_N(A_j - w_j))^{(-1)^{N+1}}, \tag{5.11}$$

where $N \geqslant 1$, $w \in \mathbb{C}^n$, *and*

$$\Gamma_{N,n}(w) = \Gamma_{N,n}(w|a_1, \ldots, a_N), \quad \Gamma_N(w_j) = \Gamma_N(w_j|a_{1j}, \ldots, a_{Nj}).$$

Moreover, for $N \geqslant 1$, $w \in \mathbb{C}^n$ *and* $x \in \mathbb{C}^N$, *we have*

$$\zeta_{N,n}(0, w|a_1, \ldots, a_N) = (-1)^N \zeta_{N,n}(0, A - w|a_1, \ldots, a_N), \tag{5.12}$$

$$\zeta_{N,n}(0, w|a_1, \ldots, a_N) = \frac{1}{n} \sum_{j=1}^{n} \zeta_N(0, w_j|a_{1j}, \ldots, a_{Nj}), \tag{5.13}$$

$$\zeta_N(0, W(x)|a_1, \ldots, a_N) = \frac{(-1)^N}{a_1 a_2 \cdots a_N} \sum_{\substack{L \\ |L|=N}} \frac{a^L}{L!} B_L(x), \quad a^L := \prod_{i=1}^{N} a_i^{L_i}. \tag{5.14}$$

Corollary 5.4.

$$\zeta_N(0, w|a_1, \ldots, a_N) = \frac{(-1)^N}{a_1 a_2 \cdots a_N} \sum_{j=0}^{N} \frac{w^j}{j!} \sum_{\substack{J \\ |J|=N-j}} \frac{a^J}{J!} B_J, \quad B_J := B_J(0). \tag{5.15}$$

Proof of Proposition 5.3. From (3.9) and (5.6) we obtain (5.11) for $w = W(x)$, with $x \in \mathbb{C}^N$. (Note $A - W(x) = W(\mathbf{1} - x)$.) Now for $N \geqslant n$, the a_i generically span $\mathbb{C}^n$, so we can generically write any $w \in \mathbb{C}^n$ as $w = W(x)$ for some $x \in \mathbb{C}^N$. By analyticity in the a_i and w, (5.11) therefore holds whenever $N \geqslant n$. Then the case $N < n$ of (5.11) follows recursively from (1.15).

Analogously, we obtain (5.12) from (5.7) with $m = N$, the zeta recurrence (1.7) playing the role of the gamma recurrence (1.15).

We now prove (5.13). For $x \in \mathbb{C}^N$, let

$$R_{N,n}(x) := \zeta_{N,n}(0, W(x)|a_1, \ldots, a_N) - \frac{1}{n} \sum_{j=1}^{n} \zeta_N(0, W(x)_j|a_{1j}, \ldots, a_{Nj}),$$

where

$$W(x)_j = W(x|a_1, \dots, a_N)_j = W(x|a_{1j}, \dots, a_{Nj}), \tag{5.16}$$

cf. (1.6). In view of (5.9) and (5.10) with $m = N$, we need only show that all of the integrals

$$\int_{I^N} \partial_x^L R_{N,n}(x)\, dx, \quad |L| = N, \tag{5.17}$$

vanish. For this purpose, we turn to the proof of Proposition 4.1, whose notation we retain. Recall from (4.8) that

$$\widehat{\zeta}_{N,n}(s, W(x)) = \zeta_{N,n}(s, W(x)) - W(x)^{-S(s)}, \quad x \in I^N.$$

Since $|L| = N \geqslant 1$, this entails

$$\partial_x^L \widehat{\zeta}_{N,n}(0, W(x)) = \partial_x^L \zeta_{N,n}(0, W(x)), \quad x \in \mathbb{C}^N.$$

Thus we have

$$\int_{I^N} \partial_x^L \zeta_{N,n}(0, W(x))\, dx = \int_{I^N} \partial_x^L \widehat{\zeta}_{N,n}(0, W(x))\, dx = \widehat{Q}_{N,n}^L(0),$$

where we have now added the subscript N, n to the notation defined in (4.7).

In order to compute $\widehat{Q}_{N,n}^L(s)$ when $s = 0$, we need to examine the terms $c_I^L(s)/(ns)$ in (4.12) as $s \to 0$. Recall from (4.10) that $c_I^L(s)$ is defined by

$$\partial_x^L W(x)^{-S(s)} = \sum_{\substack{I \\ |I|=N}} c_I^L(s) \prod_{j=1}^n (x_1 a_{1j} + \dots + x_N a_{Nj})^{-s-I_j}. \tag{5.18}$$

Since $c_I^L(s)/(ns)$ vanishes at $s = 0$ if $c_I^L(s)$ contains an s^2 factor, it follows from (5.18) that $c_I^L(s)/(ns)$ vanishes at $s = 0$ unless $N = |I| = I_j$ for some j. Applying this to N, n and to $N, 1$ we find

$$\widehat{Q}_{N,n}^L(0) = \frac{1}{n} \sum_{j=1}^N \widehat{Q}_{N,1}^L(0).$$

Therefore, (5.13) holds when $w = W(x)$. In the same way as for the previous formulas (5.11) and (5.12), this entails (5.13) for all $w \in D_n$ and $\mathcal{M} \in \mathcal{D}_{N,n}$.

It remains to prove (5.14). To this end, we compute the coefficient

$$g_{L,N,N,1} = \int_{I^N} \partial_x^L \zeta_N(0, W(x))\, dx.$$

Since $n = 1$, we have $a_{ij} = a_{i1} = a_i$ and $I = I_1 = N$, so (5.18) becomes

$$\partial_x^L W(x)^{-s} = (-1)^N (x_1 a_1 + \dots + x_N a_N)^{-s-N} \prod_{i=1}^N a_i^{L_i} (s + i - 1).$$

As a consequence, (4.12) yields

$$\int_{I^N} \partial_x^L \zeta_N(0, W(x))\, dx = d_N(-1)^N \prod_{i=1}^N a_i^{L_i},$$

where

$$d_N = (N-1)! \int_{C_+^{N-1}} (\sigma_1 a_1 + \cdots + \sigma_N a_N)^{-N}\, d\sigma.$$

Note that d_N depends on the a_i and N, but not on L.

We have thus far proved

$$\zeta_N(0, W(x)|a_1, \ldots, a_N) = d_N(-1)^N \sum_{|L|=N} \frac{\prod_{i=1}^N a_i^{L_i}}{L!} B_L(x). \tag{5.19}$$

Rather than compute d_N directly from the integral above, we shall use the difference equation (1.7) to relate d_N to d_{N-1}. Note that $d_1 = 1/a_1$, since $C_+^0 = \{1\}$. In (5.19) we let $x' := (x_1, \ldots, x_{N-1})$ be arbitrary, but take first $x_N = 1$, then $x_N = 0$, and subtract to get

$$\begin{aligned} &\zeta_N(0, W((x',1))|a_1, \ldots, a_N) - \zeta_N(0, W((x',0))|a_1, \ldots, a_N) \\ &\quad = \zeta_N(0, W((x',0)) + a_N) - \zeta_N(0, W((x',0))) \\ &\quad = -\zeta_{N-1}(0, W(x')|a_1, \ldots, a_{N-1}). \end{aligned} \tag{5.20}$$

Write an N-multi-index L of weight N as $(L', i_{L'})$, with L' an $(N-1)$-multi-index and $i_{L'} = N - |L'|$. Then, from (5.19) and (5.20),

$$-\zeta_{N-1}(0, W(x')) = d_N(-1)^N \sum_{|L'| \leqslant N} \frac{\prod_{l=1}^{N-1} a_l^{L'_l}}{L'!} B_{L'}(x') \frac{a_N^{i_{L'}}}{i_{L'}!}(B_{i_{L'}}(1) - B_{i_{L'}}(0)).$$

At face value, the latter substitution seems to complicate matters. The point is, however, that $B_l(1) - B_l(0) = 0$ unless $l = 1$, in which case $B_1(1) - B_1(0) = 1$. Before proving this assertion, we show that it entails (5.14). Indeed, it yields

$$-\zeta_{N-1}(0, W(x')|a_1, \ldots, a_{N-1}) = a_N d_N(-1)^N \sum_{|L'|=N-1} \frac{\prod_{l=1}^{N-1} a_l^{L'_l}}{L'!} B_{L'}(x').$$

Comparing this with (5.19) with N replaced by $N-1$, we obtain $a_N d_N = d_{N-1}$. It follows that in (5.19) we have $d_N = (a_1 a_2 \cdots a_N)^{-1}$, so that (5.14) results.

Finally, to prove the assertion, we use (1.29) to obtain

$$\sum_{l=0}^{\infty} \frac{u^l}{l!} [(B_l(1) - B_l(0)] = u,$$

whence its validity is clear. □

Proof of Corollary 5.4. In (5.14) let $x_i = 0$ for $i<N$ and $x_N = w/a_N$, so that $W(x)$ reduces to w. Then we find

$$\zeta_N(0,w) = \frac{(-1)^N}{a_1 a_2 \cdots a_N} \sum_{l=0}^{N} \frac{a_N^l}{l!} k_l \, B_l(w/a_N), \quad k_l := \sum_{\substack{L \\ |L|=N-l}} \frac{\prod_{i=1}^{N-1} a_i^{L_i}}{L!} B_L, \tag{5.21}$$

where L runs over $(N-1)$-multi-indices. Using

$$B_l(t) = \sum_{j=0}^{l} \frac{l!}{(l-j)!j!} B_{l-j} t^j,$$

formula (5.15) follows from (5.21) on reversing the order of sums over l and j. (Note that the well-known expression above for $B_l(t)$ is readily proved: the rhs has the differentiation property (1.32) and coincides with the lhs at $t = 0$.) □

Possibly, Barnes was aware of explicit formula (5.15). However, he only wrote $\zeta_N(0,w)$ as a multiple Bernoulli polynomial, cf. Eq. (3.10) in [11]. From (5.15) we obtain after some calculation

$$\zeta_1(0,w|a_1) = \frac{1}{2} - \frac{w}{a_1}, \tag{5.22}$$

$$\zeta_2(0,w|a_1,a_2) = \frac{1}{12a_1a_2}(6w^2 - 6(a_1+a_2)w + a_1^2 + a_2^2 + 3a_1a_2), \tag{5.23}$$

$$\begin{aligned} \zeta_3(0,w|a_1,a_2,a_3) = & \frac{1}{24a_1a_2a_3}(-4w^3 + 6(a_1+a_2+a_3)w^2 \\ & - (2a_1^2 + 2a_2^2 + 2a_3^2 + 6a_1a_2 + 6a_1a_3 + 6a_2a_3)w \\ & + a_1^2a_2 + a_1^2a_3 + a_2^2a_1 + a_2^2a_3 + a_3^2a_1 + a_3^2a_2 + 3a_1a_2a_3). \end{aligned} \tag{5.24}$$

6. Applications of the Raabe formula to certain infinite products

A glance at the Dirichlet series (1.5) defining $\zeta_{N,n}$ shows that restricting the variables w_j and a_{kj} to the right half-plane is somewhat artificial. Indeed, for

$\mathrm{Re}(s) > N/n$ the series also converges whenever $\mathrm{Re}(e^{i\vartheta_j} w_j), \mathrm{Re}(e^{i\vartheta_j} a_{kj}) > 0$, with ϑ_j any angle in $(-\pi, \pi]$ and $j = 1, \dots, n$. (Of course, this involves a fixed choice of the logarithm branch used to define the complex powers $(w_j + \sum_{k=1}^{N} m_k a_{kj})^{-s}$.) For $n = 1$, this was the setting chosen by Barnes [3] to define his multiple gamma function.

As will be clear from the following, this more general situation can be handled by analytic continuation in the vectors w and $a_1, \dots, a_N$. In order to steer clear of multi-valuedness, however, we restrict attention to vectors ϑ in $(-\pi/2, \pi/2)^n$ from now on.

First, we recall from Proposition 2.1 and (1.3) that the function

$$\kappa(s, w, a_1, \dots, a_N) \coloneqq \zeta_{N,n}(s, w | a_1, \dots, a_N)/p_{N,n}(s) \tag{6.1}$$

is holomorphic in $\mathbb{C} \times \mathcal{T}$, where $\mathcal{T}$ is the tube-like domain

$$\mathcal{T} \coloneqq \{(w, a_1, \dots, a_N) \in D_n^{N+1}\}, \quad D_n \coloneqq \{v \in \mathbb{C}^n \mid \mathrm{Re}(v_j) > 0\}.$$

For $v \in \mathbb{C}^n$ we now introduce

$$v(\vartheta) \coloneqq (e^{i\vartheta_1} v_1, \dots, e^{i\vartheta_n} v_n).$$

Next, we define domains

$$D_n(\vartheta) \coloneqq \{v \in \mathbb{C}^n \mid v(\vartheta) \in D_n\},$$

$$\mathcal{T}(\vartheta) \coloneqq \{(w, a_1, \dots, a_N) \in D_n(\vartheta)^{N+1}\},$$

$$\mathcal{T}_{\mathrm{ext}} \coloneqq \bigcup_{\vartheta \in (-\pi/2, \pi/2)^n} \mathcal{T}(\vartheta). \tag{6.2}$$

The ϑ-restriction ensures that none of the w_j, a_{kj} in the "extended tube" $\mathcal{T}_{\mathrm{ext}}$ belongs to $(-\infty, 0]$. Therefore, $\mathcal{T}_{\mathrm{ext}}$ is a simply connected domain, on which no multi-valuedness can occur.

We assert that κ in (6.1) has a holomorphic extension to $\mathbb{C} \times \mathcal{T}_{\mathrm{ext}}$. To show this, we fix $\vartheta \in (-\pi/2, \pi/2)^n$ and define a function κ_ϑ on the domain $\mathbb{C} \times \mathcal{T}(\vartheta)$ by

$$\kappa_\vartheta(s, w, a_1, \dots, a_N) \coloneqq e^{is\sum_{j=1}^{n} \vartheta_j} \kappa(s, w(\vartheta), a_1(\vartheta), \dots, a_N(\vartheta)). \tag{6.3}$$

Obviously, κ_ϑ is holomorphic in $\mathbb{C} \times \mathcal{T}(\vartheta)$. On the subdomain

$$\{\mathrm{Re}(s) > N/n\} \times (\mathcal{T} \cap \mathcal{T}(\vartheta)),$$

κ_ϑ coincides with κ, as is clear from the series representation (1.5). Thus our assertion readily follows.

Multiplying κ by $p_{N,n}(s)$, we obtain analyticity properties of $\zeta_{N,n}$ in $\mathbb{C} \times \mathcal{T}_{\mathrm{ext}}$ that are plain. They entail in particular that all of our previous results regarding $\zeta_{N,n}$ have generalizations to the extended tube $\mathscr{T}_{\mathrm{ext}}$ (6.2). Since $s=0$ is a regular value, the gamma functions are well defined and holomorphic on $\mathscr{T}_{\mathrm{ext}}$ as well, and Proposition 2.2 has an immediate generalization that need not be spelled out. Likewise, our previous results on $\Gamma_{N,n}$ can be analytically continued to all $w \in \mathbb{C}^n$ and $a_1, \ldots, a_N$ in $D_n(\vartheta)$ for any $\vartheta \in (-\pi/2, \pi/2)^n$; in particular, this is the case for the generalized Raabe formula (1.25), whose extended version we will have occasion to invoke shortly. Note that the analog of (6.3) reads

$$\Gamma_{N,n,\vartheta}(w|a_1, \ldots, a_N) = \mathrm{e}^{i\zeta_{N,n}(0,w|a_1,\ldots,a_N)\sum_{j=1}^{n}\vartheta_j}\,\Gamma_{N,n}(w(\vartheta)|a_1(\vartheta), \ldots, a_N(\vartheta)).$$

(The $s=0$ value of $\zeta_{N,n}$ does not depend on ϑ, cf. (6.3).)

We can now relate Barnes' multiple gamma function to certain infinite products.

Proposition 6.1. *Let $\alpha_1, \ldots, \alpha_N$ be N complex numbers in the upper half-plane, and set*

$$\alpha := (\alpha_1, \ldots, \alpha_N), \quad m\alpha := \sum_{k=1}^{N} m_k\alpha_k, \quad m = (m_1, \ldots, m_N) \in \mathbb{N}^N.$$

Then we have the following equality between meromorphic functions of w:

$$\Gamma_{N+1}(w|1,\alpha)\Gamma_{N+1}(1-w|1,-\alpha) = \mathrm{e}^{-\pi i\zeta_{N+1}(0,w|1,\alpha)} \prod_{m\in\mathbb{N}^N} (1-\mathrm{e}^{2\pi i(w+m\alpha)})^{-1}. \tag{6.4}$$

When $N=0$, the above formula amounts to

$$\Gamma_1(w|1)\Gamma_1(1-w|1) = \frac{\mathrm{e}^{-\pi i(\frac{1}{2}-w)}}{1-\mathrm{e}^{2\pi i w}} \quad \text{or} \quad \Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin(\pi w)},$$

as can be seen from Lerch's formula (1.23). The case $N=1$ of Proposition 6.1 was proved by Barnes [2, p. 376], and re-proved by Shintani [18, p. 196]. For $N>1$ we have not been able to find (6.4) in the literature. Since the ratio of the two sides is easily seen to have neither zeroes nor poles, the main point of Proposition 6.1 is the exact determination of the entire function appearing in the exponential.

From (5.12), (5.23) and (6.4) with $N=1$, one obtains a relation between Barnes' double gamma function and the modular functions appearing in Kronecker's second limit formula. Namely [18, Proposition 2],

$$\Gamma_2(w|1,\tau)\Gamma_2(1-w|1,-\tau)\Gamma_2(1+\tau-w|1,\tau)\Gamma_2(w-\tau|1,-\tau)$$
$$= \mathrm{e}^{\pi i T(w)} \prod_{j=0}^{\infty} (1-\mathrm{e}^{2\pi i w}\mathrm{e}^{2\pi i j\tau})^{-1}(1-\mathrm{e}^{-2\pi i w}\mathrm{e}^{2\pi i(j+1)\tau})^{-1},$$

where τ is in the upper half-plane, $w \in \mathbb{C}$, and the quadratic polynomial T is given by

$$T(w) = -2\zeta_2(0, w|1, \tau) = w - \frac{1}{2} - \frac{\tau}{6} - \frac{w^2 - w + \frac{1}{6}}{\tau}.$$

Proof of Proposition 6.1. The analyticity properties established above entail that we need only prove (6.4) for w in the half-strip $\mathrm{Re}(w) \in (0, 1)$, $\mathrm{Im}(w) \in [0, \infty)$, and $\alpha_1, \ldots, \alpha_N \in i(0, \infty)$. Assuming this from now on, we begin by observing that the second logarithmic derivative of the well-known $N = 0$ case of (6.4), combined with the relation of $\Gamma_1(w|\,1)$ to the Hurwitz zeta function $H(s, w)$, yields the identity

$$\sum_{k \in \mathbb{Z}} \frac{1}{(w + k)^2} = \frac{\pi^2}{\sin^2(\pi w)}, \quad \mathrm{Re}(w) \in (0, 1).$$

(Of course, this is another well-known identity.) Now we replace w above by $w + m\alpha$ and take the Nth w-derivative. Summing over $m \in \mathbb{N}^N$, we obtain an identity of convergent series, namely,

$$(-1)^N (N + 1)! \sum_{m \in \mathbb{N}^N} \sum_{k \in \mathbb{Z}} \frac{1}{(w + k + m\alpha)^{N+2}} = \frac{d^N}{dw^N} \sum_{m \in \mathbb{N}^N} \frac{\pi^2}{\sin^2 \pi(w + m\alpha)}. \tag{6.5}$$

Taking the s-derivative at $s = 0$ of (3.1) with $J = N + 2$ and $n = 1$, we recognize the lhs of (6.5) as

$$\frac{d^{N+2}}{dw^{N+2}} \log(\Gamma_{N+1}(w|1, \alpha)\Gamma_{N+1}(1 - w|1, -\alpha)).$$

The rhs is minus the $(N + 2)$th logarithmic derivative of the infinite product

$$f_{N+1}(w) := \prod_{m \in \mathbb{N}^N} (1 - \mathrm{e}^{2\pi i(w + m\alpha)}). \tag{6.6}$$

Thus, (6.4) is correct up to a factor $\mathrm{e}^{p(w)}$, where p is a polynomial of degree at most $N + 1$.

We can now prove (6.4) using the Raabe-type formula (1.25) and induction on N. To this end we introduce

$$-\pi i Q_{N+1}(w|\,\alpha) := \log \Gamma_{N+1}(w|1, \alpha) + \log \Gamma_{N+1}(1 - w|1, -\alpha) + \log f_{N+1}(w). \tag{6.7}$$

Here, the logarithms of the gamma functions are the ones fixed by the s-derivative of ζ_N at $s = 0$, whereas the logarithm branch for $f_{N+1}(w)$ is fixed by requiring that when we let w converge to $i\infty$ in the above half-strip, the limit vanishes. From the foregoing discussion, we already know that $Q_{N+1}(w)$ is a polynomial.

It remains to prove that $Q_{N+1}(w) = \zeta_{N+1}(0, w)$. As discussed above, this is true for $N = 0$. A short calculation, using (6.6), (6.7), (1.15) and the inductive hypothesis

$Q_N = \zeta_N$, shows

$$Q_{N+1}(w+\alpha_N) = Q_{N+1}(w) - Q_N(w|\,\alpha') = Q_{N+1}(w) - \zeta_N(0,w|\,1,\alpha'),$$

where $\alpha' = \{\alpha_1,\dots,\alpha_{N-1}\}$ and $N\geqslant 1$. Since $\zeta_{N+1}(0,w|1,\alpha)$ is another polynomial satisfying the same ordinary difference equation as Q_{N+1}, we have

$$Q_{N+1}(w) = \zeta_{N+1}(0,w|1,\alpha) + c$$

for some constant c independent of w. For $x = (x_0,x_1,\dots,x_N)$, we substitute in (6.7)

$$w = W(x) = W(x|1,\alpha) = x_0 + x_1\alpha_1 + x_2\alpha_2 + \cdots + x_N\alpha_N, \quad x\in I^{N+1}.$$

From (4.1) we find

$$c = \int_{I^{N+1}} Q_{N+1}(W(x)|\,\alpha)\,dx.$$

The vanishing of c now follows from (6.7) and the vanishing of the following three integrals:

$$\begin{aligned} K &:= \int_{I^{N+1}} \log\Gamma_{N+1}(W(x|1,\alpha)|1,\alpha)\,dx,\\ L &:= \int_{I^{N+1}} \log\Gamma_{N+1}((1-W(x|1,\alpha))|1,-\alpha)\,dx,\\ M &:= \int_{I^{N+1}} \log f_{N+1}(W(x|1,\alpha))\,dx. \end{aligned} \tag{6.8}$$

The vanishing of K is a direct application of the Raabe formula (1.25). Switching now to L, note that

$$1 - W(x|1,\alpha) = 1 - x_0 - x_1\alpha_1 - x_2\alpha_2 - \cdots - x_N\alpha_N = W(y|1,-\alpha),$$

$$\text{where } y := (y_0,y_1,\dots,y_N),\ \ y_0 := 1-x_0,\ \ y_k := x_k,\ \ 1\leqslant k\leqslant N.$$

Therefore, changing variables from x to y in definition (6.8) of L, we obtain

$$\int_{I^{N+1}} \log\Gamma_{N+1}(1-W(x|1,\alpha)|1,-\alpha)\,dx = \int_{I^{N+1}} \log\Gamma_{N+1}(W(y|1,-\alpha)|1,-\alpha)\,dy,$$

which again vanishes by the Raabe formula.

The vanishing of M can be seen as follows. Writing $x\in I^{N+1}$ as $x=(t,y)$ with $0<t<1$ and $y\in I^N$, we have

$$M = \int_{I^N}\int_{t=0}^{1} \log f_{N+1}(t+W(y|\,\alpha))\,dt\,dy.$$

Now the integrand has no singularities in the half-strip $\mathrm{Re}(t) \in (0, 1)$, $\mathrm{Im}(t) \in [0, \infty)$, and it has period 1 in t. Thus we may shift the contour of the t-integral to $i\infty$, and deduce it vanishes. $\square$

From Proposition 6.1 we can derive a formula for the elliptic gamma function given by

$$G(r, a, b; w) := \prod_{j,k=0}^{\infty} \frac{1 - e^{-2irw} e^{-ra(2j+1)} e^{-rb(2k+1)}}{1 - e^{2irw} e^{-ra(2j+1)} e^{-rb(2k+1)}}, \tag{6.9}$$

where $r > 0$ and $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ [10, p. 1104].

Corollary 6.2. *Setting* $\alpha := ia$, $\beta := ib$ *and* $\gamma := (\alpha + \beta)/2$, *we have*

$$G(\pi, a, b; w) = e^{\pi i R(w)} \frac{\Gamma_3(w + \gamma | 1, \alpha, \beta) \Gamma_3(1 - w - \gamma | 1, -\alpha, -\beta)}{\Gamma_3(-w + \gamma | 1, \alpha, \beta) \Gamma_3(1 + w - \gamma | 1, -\alpha, -\beta)},$$

where $R(w)$ *is the cubic polynomial*

$$R(w) = \zeta_3(0, w + \gamma | 1, \alpha, \beta) - \zeta_3(0, -w + \gamma | 1, \alpha, \beta) = \frac{w^3}{3ab} + \frac{a^2 + b^2 + 2}{12ab} w.$$

Proof. This can be verified from Proposition 6.1, definition (6.9) and Eq. (5.24) by a simple calculation. $\square$

Note that G is invariant under $r, a, b, w \to rt, a/t, b/t, w/t$, so fixing $r = \pi$ in Corollary 6.2 is no restriction.

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