

FIVE LECTURES ON SOLITON EQUATIONS

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INTRODUCTION

In these lectures we review a new approach to soliton equations of KdV type developed by the author together with B. Feigin and B. Enriquez [11, 12, 8].

The KdV equation is the partial differential equation

$$(0.1) \quad v_\tau = 6vv_z + v_{zzz}.$$

Introduced by Korteweg and de Vries in 1895, it has a long and intriguing history, see, e.g., [26]. We will view this equation as a flow, parametrized by the variable τ , on the space of functions $v(z)$ in the variable z . Perhaps, the most remarkable aspect of the KdV equation is its *complete integrability*, i.e., the existence of infinitely many other flows defined by equations of the type

$$(0.2) \quad \partial_{\tau_n} = P_n[v, v_z, v_{zz}, \dots], \quad n = 3, 4, \dots,$$

which commute with the flow defined by (0.1) (we set $\tau_1 = z$ and $\tau_2 = \tau$). The important fact is that the right hand sides P_n of equations (0.2) are polynomials in v and its derivatives, i.e., *differential polynomials in v* .

This fact allows us to treat the KdV hierarchy (0.2) in the following way. Consider the ring $R = \mathbb{C}[v^{(n)}]_{n \geq 0}$ of polynomials in the variables $v^{(n)}$ (we shall write v for $v^{(0)}$). Let ∂_z be the derivation of R defined by the formula $\partial_z \cdot v^{(n)} = v^{(n+1)}$ (so that $v^{(n)} = \partial_z^n v$). Note that any derivation on R is uniquely determined by its values on $v^{(n)}$'s via the Leibnitz rule.

Any equation of the form $\partial_\tau v = P[v, v_z, \dots]$ gives rise to an evolutionary derivation D of R (i.e., such that commutes with ∂_z) defined by the formula $D \cdot v = P \in R$. The condition of commutativity with ∂_z implies that $D \cdot v^{(n)} = \partial_z^n \cdot P$, so that D can be written as

$$D = \sum_{n \geq 0} (\partial_z^n \cdot P) \frac{\partial}{\partial v^{(n)}}.$$

From this point of view, KdV hierarchy is an infinite set of mutually commuting evolutionary derivations of R . This is the way we will think of the KdV hierarchy and other, more general, hierarchies throughout these lectures.

We will not discuss the analytic issues related to the KdV equation, but will focus on its universal aspects, such as the origin of its integrability. This approach to soliton equations, initiated by I.M. Gelfand and L.A. Dickey [16], can be summarized as follows. A particular choice of functional space F , in which we want to look for solutions (we

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assume that it is an algebra closed under the action of the derivative ∂_z) gives us a specific realization of the hierarchy. On the other hand, R is nothing but the ring of functions on the space of ∞ -jets of functions at a point. Hence a solution $\mathbf{v}(z, \tau)$ in the realization F gives us a family of homomorphisms $r(t) : R \rightarrow F$, which send $v^{(n)}$ to $\partial_z^n \mathbf{v}(z, t)$. Thus, R can be considered as a scheme-like object associated with the KdV hierarchy, while various solutions correspond to various “points” of this “scheme”. This way R captures the universal, realization independent properties of the KdV equations.

In these notes we will try to answer the question: where do the infinitely many commuting derivations of KdV hierarchy (and other hierarchies of KdV type) come from? From the outset, they look rather mysterious. But in fact there is a very simple explanation. Very briefly, the answer is as follows:

- (i) we can identify the vector space with coordinates $v^{(n)}$, $n \geq 0$, with a homogeneous space of an infinite-dimensional unipotent algebraic group (a subgroup of a loop group);
- (ii) this homogeneous space has an obvious action of an infinite-dimensional abelian Lie algebra by vector fields;
- (iii) rewritten in terms of the coordinates $v^{(n)}$, these vector fields become the KdV flows (0.2).

Thus we will show that the KdV hierarchy is encoded in the differential geometry of a loop group.

For technical reasons, this program is easier to realize for a close relative of the KdV hierarchy – the so-called modified KdV (or mKdV) hierarchy; such a hierarchy is associated to an arbitrary affine Kac-Moody algebra. In Lecture 1 we explain (i)–(iii) for the generalized mKdV hierarchies. The reader who is only interested in the main ideas may read just this lecture.

In Lecture 2 we study the mKdV hierarchy in more detail, and write the equations of the hierarchy explicitly in the so-called zero-curvature form. In Lecture 3 we compare our approach to mKdV with the earlier approaches of Drinfeld-Sokolov and Wilson. We remark that in Sects. 2.1-2.2 and Sects. 3.4-3.6 we follow closely [8].

In Lecture 4 we consider the generalized KdV hierarchies from the point of view of our approach and that of the Drinfeld-Sokolov reduction [7]. Some of the results of this lecture (e.g., Theorem 4.1) are new. Finally, in Lecture 5 we discuss the Toda field theories and their connection to the mKdV and KdV hierarchies, following [12]. In particular, we show that the KdV and mKdV hierarchy are hamiltonian, and their hamiltonians are the integrals of motion of the corresponding affine Toda field theory.

The original motivation for the formalism explained in these lectures came from the study of deformations of conformal field theories, where one needs to show that the integrals of motion of affine Toda field theories can be quantized (see [32, 10]). It turns out that, in contrast to other approaches, our formalism is well suited for tackling the quantization problem. In particular, using the results of Lecture 5 and certain results on quantum groups, we were able to prove that the Toda integrals of motion can be quantized [10, 11]. We briefly discuss this at the end of Lecture 5. Recently these quantum integrals found some spectacular applications in two-dimensional quantum field theory [1]. We hope that quantization of our formalism will help elucidate further

the connections between differential geometry of integrable systems and quantum field theory.

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I am indebted to B. Kostant for kindly supplying a proof of Proposition 4.1, which allowed me to simplify the proof of Theorem 4.1 on the structure of the KdV jet space. I also thank L. Feher for pointing out an inaccuracy in [8], which is corrected here.

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LECTURE 1

1.1. Background material on affine algebras. Let $\tilde{\mathfrak{g}}$ be an affine Kac-Moody algebra. It has generators $e_i, f_i, \alpha_i^\vee, i = 0, \dots, \ell$, and d , which satisfy the standard relations, see [22]. The Lie algebra $\tilde{\mathfrak{g}}$ carries a non-degenerate invariant inner product (\cdot, \cdot) . One associates to $\tilde{\mathfrak{g}}$ the labels $a_i, a_i^\vee, i = 0, \dots, \ell$, the *exponents* $d_i, i = 1, \dots, \ell$, and the *Coxeter number* h , see [23, 22]. We denote by I the set of all positive integers, which are congruent to the exponents of $\tilde{\mathfrak{g}}$ modulo h (with multiplicities).

The exponents and the Coxeter numbers are given in a table below. Note that for all affine algebras except $D_{2n}^{(1)}$, each exponent occurs exactly once. In the case of $D_{2n}^{(1)}$, the exponent $2n - 1$ has multiplicity 2.

| Type | Coxeter number | Exponents |
|------------------|----------------|---|
| $A_n^{(1)}$ | $n + 1$ | $1, 2, \dots, n$ |
| $A_{2n}^{(2)}$ | $4n + 2$ | $1, 3, 5, \dots, 2n - 1, 2n + 3, \dots, 4n + 1$ |
| $A_{2n-1}^{(2)}$ | $4n - 2$ | $1, 3, 5, \dots, 4n - 3$ |
| $B_n^{(1)}$ | $2n$ | $1, 3, 5, \dots, 2n - 1$ |
| $C_n^{(1)}$ | $2n$ | $1, 3, 5, \dots, 2n - 1$ |
| $D_n^{(1)}$ | $2n - 2$ | $1, 3, 5, \dots, 2n - 3, n - 1$ |
| $D_{n+1}^{(2)}$ | $2n + 2$ | $1, 3, 5, \dots, 2n + 1$ |
| $D_4^{(3)}$ | 12 | 1, 5, 7, 11 |
| $E_6^{(1)}$ | 12 | 1, 4, 5, 7, 8, 11 |
| $E_6^{(2)}$ | 18 | 1, 5, 7, 11, 13, 17 |
| $E_7^{(1)}$ | 18 | 1, 5, 7, 9, 11, 13, 17 |
| $E_8^{(1)}$ | 30 | 1, 7, 11, 13, 17, 19, 23, 29 |
| $F_4^{(1)}$ | 12 | 1, 5, 7, 11 |
| $G_2^{(1)}$ | 6 | 1, 5 |

The elements $e_i, i = 0, \dots, \ell$, and $f_i, i = 0, \dots, \ell$, generate the *nilpotent subalgebras* \mathfrak{n}_+ and \mathfrak{n}_- of $\tilde{\mathfrak{g}}$, respectively. The elements $\alpha_i^\vee, i = 0, \dots, \ell$, and d generate the *Cartan subalgebra* $\tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$. We have: $\tilde{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{b}_-$, where $\mathfrak{b}_- = \mathfrak{n}_- \oplus \tilde{\mathfrak{h}}$.

The element

$$C = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$$

of $\tilde{\mathfrak{h}}$ is a *central element* of $\tilde{\mathfrak{g}}$. Let \mathfrak{g} be the quotient of $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ by $\mathbb{C}C$. We can identify $\tilde{\mathfrak{g}}$ with the direct sum $\mathfrak{g} \oplus \mathbb{C}C \oplus \mathbb{C}d$. The Lie algebra \mathfrak{g} has a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where \mathfrak{h} is spanned by $\alpha_i^\vee, i = 1, \dots, \ell$. Each $x \in \mathfrak{g}$ can be uniquely written as $x_+ + x_-$, where $x_+ \in \mathfrak{n}_+$ and $x_- \in \mathfrak{b}_-$.

There exists a unique element $\rho^\vee \in \tilde{\mathfrak{h}}$, such that $(\alpha_i, \rho^\vee) = 1, \forall i = 0, \dots, \ell$, and $(d, \rho^\vee) = 0$. But $\tilde{\mathfrak{h}}^*$ is isomorphic to $\tilde{\mathfrak{h}}$ via the non-degenerate inner product (\cdot, \cdot) . We shall use the same notation for the image of ρ^\vee in $\tilde{\mathfrak{h}}$ under this isomorphism. Then ρ^\vee satisfies: $[\rho^\vee, e_i] = e_i, [\rho^\vee, h_i] = 0, [\rho^\vee, f_i] = -f_i, i = 0, \dots, \ell$. The adjoint action of ρ^\vee on \mathfrak{g} defines the *principal \mathbb{Z} -gradation* on \mathfrak{g} .

Denote by $\bar{\mathfrak{g}}$ the simple Lie algebra of rank ℓ , generated by $e_i, f_i, \alpha_i^\vee, i = 1, \dots, \ell$ (the Dynkin diagram of $\bar{\mathfrak{g}}$ is obtained from that of $\tilde{\mathfrak{g}}$ by removing the 0th node). It has the Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}_-$, where \mathfrak{h} is the Cartan subalgebra of $\bar{\mathfrak{g}}$, spanned by $\alpha_i^\vee, i = 1, \dots, \ell$.

Remark 1.1. Affine algebra $\tilde{\mathfrak{g}}$ can be identified with the universal central extension of the loop algebra $\mathfrak{g}^0[t, t^{-1}]$, where \mathfrak{g}^0 is a simple Lie algebra, or its subalgebra [22]. More precisely, if $\tilde{\mathfrak{g}}$ is non-twisted, then $\mathfrak{g}^0 = \bar{\mathfrak{g}}$ and $\mathfrak{g} = \bar{\mathfrak{g}}[t, t^{-1}]$, so that $\tilde{\mathfrak{g}}$ is the universal central extension of $\bar{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}d$:

$$0 \rightarrow \mathbb{C}K \rightarrow \tilde{\mathfrak{g}} \rightarrow \bar{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}d \rightarrow 0.$$

A twisted affine algebra $\tilde{\mathfrak{g}}$ can be identified with a subalgebra in the universal central extension of $\mathfrak{g}^0[t, t^{-1}] \oplus \mathbb{C}d$. \square

1.2. The principal abelian subalgebra. Set

$$p_{-1} = \sum_{i=0}^{\ell} \frac{(\alpha_i, \alpha_i)}{2} f_i,$$

where α_i 's are the simple roots of \mathfrak{g} , considered as elements of \mathfrak{h} using the inner product. Let \mathfrak{a} be the centralizer of p_{-1} in \mathfrak{g} . Recall that we have a principal gradation on $\tilde{\mathfrak{g}}$, such that $\deg e_i = -\deg f_i = 1, \deg \alpha_i^\vee = \deg d = 0$. Since p_{-1} is a homogeneous element with respect to this gradation, the Lie algebra \mathfrak{a} is a \mathbb{Z} -graded subalgebra of \mathfrak{g} . In particular, it can be decomposed as $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, where $\mathfrak{a}_+ = \mathfrak{a} \cap \mathfrak{n}_+$, and $\mathfrak{a}_- = \mathfrak{a} \cap \mathfrak{b}_-$.

Proposition 1.1 ([20], Prop. 3.8). (1) *The Lie algebra \mathfrak{a} is abelian.*

(2) *The homogeneous component of \mathfrak{a}_\pm of degree i with respect to the principal gradation has dimension equal to the number of occurrences of i in the set $\pm I$.*

We call \mathfrak{a} the principal abelian subalgebra of \mathfrak{g} . Its pull-back to $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ is called the principal Heisenberg subalgebra.

Proposition 1.1 means that for all $\tilde{\mathfrak{g}}$ except $D_{2n}^{(1)}$, each homogeneous component of \mathfrak{a}_{\pm} is either zero-dimensional or one-dimensional, and it is one-dimensional if and only if $i \in I$. In the latter case we choose a generator p_i of this component. In the case $\tilde{\mathfrak{g}} = D_{2n}^{(1)}$, the homogeneous component of degree i congruent to $2n - 1$ modulo the Coxeter number $4n - 2$ is two-dimensional. We choose two generators of \mathfrak{a} , p_i^1 and p_i^2 , that span this component.

In particular, we choose

$$p_1 = \sum_{i=0}^l a_i e_i.$$

The properties of \mathfrak{a} that are most important to us are described in the following fundamental proposition.

Proposition 1.2 ([20], Prop. 3.8). (1) $\text{Ker}(\text{ad } p_{-1}) = \mathfrak{a}$;

(2) The Lie algebra \mathfrak{g} has a decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \text{Im}(\text{ad } p_{-1});$$

(3) With respect to the principal gradation, $\text{Im}(\text{ad } p_{-1}) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where $\dim \mathfrak{g}_j = \ell$, and the map $\text{ad } p_{-1} : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j-1}$ is an isomorphism.

Propositions 1.1 and 1.2 can be derived from B. Kostant's results about cyclic elements in simple Lie algebras (see Sect. 6 of [23]).

Remark 1.2. L. Feher has pointed out to us that Prop. 5 of [8] was stated incorrectly: it is true only if n is relatively prime to h (see, e.g., [6]). However, the case $n = 1$ (given above) is sufficient for the purposes of [8], as well as for us here. \square

Examples. Consider first the case of

$$L\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, a(t) + d(t) = 0 \right\}.$$

Then

$$\mathfrak{a} = \text{span}\{p_i\}_{i \text{ odd}},$$

where

$$p_{2j+1} = \begin{pmatrix} 0 & t^j \\ t^{j+1} & 0 \end{pmatrix}.$$

More generally, for $\mathfrak{g} = L\mathfrak{sl}_N$,

$$\mathfrak{a} = \text{span}\{p_i\}_{i \not\equiv 0 \pmod N},$$

where

$$p_i = (p_1)^i, \quad p_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ t & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Remark 1.3. It is known that in the non-twisted case maximal abelian subalgebras of \mathfrak{g} are parametrized by the conjugacy classes of the Weyl group of $\bar{\mathfrak{g}}$, see [21]. In particular, $\mathfrak{h}[t, t^{-1}]$ corresponds to the identity class, and \mathfrak{a} corresponds to the class of the *Coxeter element*. \square

1.3. The unipotent subgroup. Let $\widehat{\mathfrak{n}}_+$ be the completion of the Lie subalgebra \mathfrak{n}_+ of \mathfrak{g} defined as the inverse limit of finite-dimensional nilpotent Lie algebras $\mathfrak{n}_+^{(m)}$, $m > 0$, where $\mathfrak{n}_+^{(m)} = \mathfrak{n}_+ / (\mathfrak{g}^0 \otimes t^m \mathbb{C}[[t]] \cap \mathfrak{n}_+)$. In the non-twisted case, $\mathfrak{n}_+ = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{g}} \otimes t\mathbb{C}[t]$, and so $\widehat{\mathfrak{n}}_+ = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{g}} \otimes t\mathbb{C}((t))$. In general, $\widehat{\mathfrak{n}}_+$ is a Lie subalgebra of $\mathfrak{g}^0((t))$.

Let $N_+^{(m)}$ be the unipotent algebraic group over \mathbb{C} corresponding to $\mathfrak{n}_+^{(m)}$. Clearly, the exponential map $\mathfrak{n}_+^{(m)} \rightarrow N_+^{(m)}$ is an isomorphism. We define the pronipotent proalgebraic group N_+ as the inverse limit of $N_+^{(m)}$, $m > 0$. We have the exponential map $\widehat{\mathfrak{n}}_+ \rightarrow N_+$, which is an isomorphism of proalgebraic varieties. Thus, the ring $\mathbb{C}[N_+]$ of regular functions on N_+ , is isomorphic (non-canonically) to the ring of polynomials in infinitely many variables. Below we will construct such an isomorphism explicitly, i.e., we will construct explicitly a natural coordinate system on N_+ .

1.4. The action of \mathfrak{g} . The group N_+ acts on itself from the left and from the right:

$$n \cdot_L g = n^{-1}g, \quad n \cdot_R g = gn.$$

In addition, it has a remarkable structure, which is the key in soliton theory: N_+ is equipped with an infinitesimal action of the Lie algebra \mathfrak{g} by vector fields. Let \bar{G} be the simply-connected simple algebraic group over \mathbb{C} with Lie algebra $\bar{\mathfrak{g}}$. In the non-twisted case, there exists an ind-group G , whose set of \mathbb{C} -points is $\bar{G}((t))$. Thus, G can be viewed as the Lie group of \mathfrak{g} . In the twisted case G is defined in a similar way. The group G has subgroups N_+ and B_- , where the latter is the ind-group corresponding to the Lie subalgebra \mathfrak{b}_- . We have an open embedding $N_+ \rightarrow B_- \backslash G$. Hence the obvious infinitesimal action of \mathfrak{g} on $B_- \backslash G$ from the right can be restricted to N_+ .

It is easy to write down the resulting action of \mathfrak{g} on N_+ explicitly. We can faithfully represent N_+ in the Lie algebra of matrices, whose entries are Taylor power series; for instance, N_+ is faithfully represented on $\mathfrak{g}^0[[t]]$. Each Fourier coefficient of such a series defines an algebraic function on N_+ , and the ring $\mathbb{C}[N_+]$ is generated by these functions. Hence any derivation of $\mathbb{C}[N_+]$ is uniquely determined by its action on these functions. We can write the latter concisely as follows: $\nu \cdot x = y$, where x is a ‘‘test’’ matrix representing an element of N_+ , i.e. its (i, j) th entry is considered as a regular function f_{ij} on N_+ , and y is another matrix, whose (i, j) th entry is $\nu \cdot f_{ij} \in \mathbb{C}[N_+]$.

Remark 1.4. One can also take a slightly different point of view: all formulas below make sense for an arbitrary representation of N_+ . Hence instead of picking a particular faithful representation of N_+ , one can consider all representations simultaneously. The corresponding matrix elements generate $\mathbb{C}[N_+]$. \square

For $a \in \mathfrak{g}$, let us denote by a^R the derivation of $\mathbb{C}[N_+]$ corresponding to the right infinitesimal action of a . Also, for $b \in \mathfrak{n}_+$ we denote by b^L the derivation of $\mathbb{C}[N_+]$ corresponding to the left infinitesimal action of b .

In order to simplify notation, we will write below axa^{-1} instead of $\text{Ad}_a(x)$.

Lemma 1.1.

$$(1.3) \quad a^R \cdot x = (xax^{-1})_+x, \quad \forall a \in \bar{\mathfrak{g}},$$

$$(1.4) \quad b^L \cdot x = -bx, \quad \forall b \in \mathfrak{n}_+.$$

Proof. Consider a one-parameter subgroup $a(\epsilon)$ of G , such that $a(\epsilon) = 1 + \epsilon a + o(\epsilon)$. We have: $x \cdot a(\epsilon) = x + \epsilon xa + o(\epsilon)$. For infinitesimally small ϵ we can factor $x \cdot a(\epsilon)$ into a product $y_- y_+$, where $y_+ = x + \epsilon y_+^{(1)} + o(\epsilon) \in N_+$ and $y_- = 1 + \epsilon y_-^{(1)} \in B_-$. We then find that $y_-^{(1)} x + y_+^{(1)} = xa$, from which we conclude that $y_+^{(1)} = (xax^{-1})_+x$. This proves formula (1.3). Formula (1.4) is obvious. \square

1.5. The main homogeneous space. Now we introduce our main object. Denote by $\hat{\mathfrak{a}}_+$ the completion of \mathfrak{a}_+ in $\hat{\mathfrak{n}}_+$. Let A_+ be the abelian subgroup of N_+ , which is the image of $\hat{\mathfrak{a}}_+$ in N_+ under the exponential map. Consider the homogeneous space N_+/A_+ . Since \mathfrak{g} acts on N_+ infinitesimally from the right, the normalizer of $\hat{\mathfrak{a}}_+$ in \mathfrak{g} acts on N_+/A_+ infinitesimally from the right. In particular, \mathfrak{a}_- acts on N_+/A_+ , and each $p_{-n}, n \in I$, gives rise to a derivation of $\mathbb{C}[N_+/A_+]$, which we denote by p_{-n}^R .

Our goal is to show that the derivations p_{-n}^R are “responsible” for the equations of the mKdV hierarchy.

1.6. The space of jets. Now we change the subject and consider the space \mathbf{U} of ∞ -jets of an \mathfrak{h} -valued smooth function $\mathbf{u}(z)$ at $z = 0$. Equivalently, this is the space of functions $\mathbf{u}(z) : \mathcal{D} \rightarrow \mathfrak{h}$, where \mathcal{D} is the formal disk, $\mathcal{D} = \text{Spec } \mathbb{C}[[z]]$.

We can view $\mathbf{u}(z)$ as a vector (u_1, \dots, u_ℓ) , where $u_i = (\alpha_i, \mathbf{u}(z))$. The space \mathbf{U} is therefore the inverse limit of the finite-dimensional vector spaces

$$\mathbf{U}_m = \text{span}\{u_i^{(n)}\}_{i=1, \dots, \ell; n=1, \dots, m},$$

where $u_i = (\alpha_i, \mathbf{u}(0))$, and $u_i^{(n)} = \partial_z^n u_i$ (so, the value of $u_i^{(n)}$ on $\mathbf{u}(z)$ is given by $\partial_z^n u_i(0)$). Thus, the ring $\mathbb{C}[\mathbf{U}]$ of regular functions on \mathbf{U} is $\mathbb{C}[u_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}$. The derivative ∂_z gives rise to a derivation ∂_z of $\mathbb{C}[\mathbf{U}]$, such that $\partial_z \cdot u_i^{(n)} = u_i^{(n+1)}$. We will write u_i for $u_i^{(0)}$.

Introduce a \mathbb{Z} -gradation on $\mathbb{C}[\mathbf{U}]$ by setting $\deg u_i^{(n)} = n + 1$.

1.7. Main result.

Theorem 1.1 ([12], Prop. 4). *There is an isomorphism of rings*

$$\mathbb{C}[N_+/A_+] \simeq \mathbb{C}[\mathbf{U}],$$

under which p_{-1}^R gets identified with ∂_z .

Proof. We will explicitly construct a homomorphism $\mathbb{C}[\mathbf{U}] \rightarrow \mathbb{C}[N_+/A_+]$ and then show that it is actually an isomorphism. In order to do that, we have to construct regular functions on N_+/A_+ corresponding to $u_i^{(n)} \in \mathbb{C}[\mathbf{U}], i = 1, \dots, \ell; n \geq 0$.

Define a function \tilde{u}_i on N_+ by the formula

$$(1.5) \quad \tilde{u}_i(K) = (\alpha_i, K p_{-1} K^{-1}), \quad K \in N_+.$$

This function is invariant under the right action of A_+ . Indeed, if $y \in A_+$, then $yp_{-1}y^{-1} = p_{-1}$, and so $\tilde{u}_i(Ky) = \tilde{u}_i(K)$. Hence \tilde{u}_i can be viewed as a regular function on N_+/A_+ . Next, we define the functions

$$\tilde{u}_i^{(n)} = (p_{-1}^R)^n \cdot \tilde{u}_i, \quad n > 0,$$

on N_+/A_+ . Consider the homomorphism $\mathbb{C}[\mathbf{U}] \rightarrow \mathbb{C}[N_+/A_+]$, which sends $u_i^{(n)}$ to $\tilde{u}_i^{(n)}$.

To prove that this homomorphism is injective, we have to show that the functions $\tilde{u}_i^{(n)}$ are algebraically independent. We will do that by showing that the values of their differentials at the identity coset $\bar{1} \in N_+/A_+$, $d\tilde{u}_i|_{\bar{1}}$, are linearly independent. Those are elements of the cotangent space to N_+/A_+ at $\bar{1}$, which is canonically isomorphic to $(\hat{\mathfrak{n}}_+/\hat{\mathfrak{a}}_+)^*$. Using Proposition 1.2 and the invariant inner product on $\tilde{\mathfrak{g}}$, we identify $(\hat{\mathfrak{n}}_+/\hat{\mathfrak{a}}_+)^*$ with $(\mathfrak{a}_+)^{\perp} \cap \mathfrak{n}_-$. By Proposition 1.2, with respect to the principal gradation, $(\mathfrak{a}_+)^{\perp} = \bigoplus_{j>0} (\mathfrak{a}_+)^{\perp}_{-j}$, where $\dim(\mathfrak{a}_+)^{\perp}_{-j} = \ell$.

Let us first show that the vectors $d\tilde{u}_i|_{\bar{1}}, i = 1, \dots, \ell$, form a linear basis in $(\mathfrak{a}_+)^{\perp}_{-1}$. Indeed, the tangent space to $\bar{1}$ is isomorphic to $\hat{\mathfrak{n}}_+/\hat{\mathfrak{a}}_+$, and it is clear that the projections of $e_i, i = 1, \dots, \ell$, onto $\hat{\mathfrak{n}}_+/\hat{\mathfrak{a}}_+$ are linearly independent. Hence it suffices to check that the matrix $[\langle e_i, d\tilde{u}_i|_{\bar{1}} \rangle]_{1 \leq i, j \leq \ell}$ is non-degenerate. But

$$\langle e_i, d\tilde{u}_i|_{\bar{1}} \rangle = (e_i^L \cdot \tilde{u}_j)(\bar{1}),$$

and we find

$$(1.6) \quad (e_i^L \cdot \tilde{u}_j)(K) = -(\alpha_j, [e_i, Kp_{-1}K^{-1}]) \\ = -(\alpha_j, [e_i, p_{-1}] + \dots)$$

$$(1.7) \quad = -(\alpha_j, \alpha_i)$$

(the dots above stand for the terms lying in $\hat{\mathfrak{n}}_+$, and so their inner product with α_j is 0). Thus, we see that the matrix $[\langle e_i, d\tilde{u}_i|_{\bar{1}} \rangle]_{1 \leq i, j \leq \ell}$ coincides with minus the symmetrized Cartan matrix of $\tilde{\mathfrak{g}}$, $-[(\alpha_i, \alpha_j)]_{1 \leq i, j \leq \ell}$, and hence is non-degenerate. Therefore the vectors $d\tilde{u}_i|_{\bar{1}}, i = 1, \dots, \ell$, are linearly independent.

Now, by definition,

$$d\tilde{u}_i^{(m+1)}|_{\bar{1}} = \text{ad } p_{-1} \cdot d\tilde{u}_i^{(m)}|_{\bar{1}}.$$

But according to Proposition 1.2, $\text{ad } p_{-1} : (\mathfrak{a}_+)^{\perp}_{-j} \rightarrow (\mathfrak{a}_+)^{\perp}_{-j-1}$ is an isomorphism for all $j > 0$. Hence the vectors $d\tilde{u}_i^{(m)}|_{\bar{1}}$ are all linearly independent. Therefore the functions $\tilde{u}_i^{(n)}, i = 1, \dots, \ell, n \geq 0$, are algebraically independent and our homomorphism $\mathbb{C}[\mathbf{U}] \rightarrow \mathbb{C}[N_+/A_+]$ is an embedding.

Now let us compare gradations. Consider the derivation $(\rho^{\vee})^R$ on N_+ . It is clear that for any $v \in \mathfrak{a}_+, [\rho^{\vee}, v] \in \mathfrak{a}_+$ (see Sect. 1.1). Hence $(\rho^{\vee})^R$ is a well-defined derivation on $\mathbb{C}[N_+/A_+]$. We take $-(\rho^{\vee})^R$ as the gradation operator on $\mathbb{C}[N_+/A_+]$ (to simplify notation, we will denote it by $-\rho^{\vee}$).

We have by formula (1.3),

$$\begin{aligned}
 (-\rho^\vee \cdot \tilde{u}_i)(K) &= -(\alpha_i, [(K\rho^\vee K^{-1})_+, Kp_{-1}K^{-1}]) \\
 &= -(\alpha_i, [K\rho^\vee K^{-1}, Kp_{-1}K^{-1}]) + (\alpha_i, [(K\rho^\vee K^{-1})_-, Kp_{-1}K^{-1}]) \\
 &= -(\alpha_i, K[\rho^\vee, p_{-1}]K^{-1}) + ([\alpha_i, (K\rho^\vee K^{-1})_-], Kp_{-1}K^{-1}) \\
 &= (\alpha_i, Kp_{-1}K^{-1}) = \tilde{u}_i(K).
 \end{aligned}$$

Hence the degree of \tilde{u}_i equals 1. Since $[-\rho^\vee, p_{-1}] = p_{-1}$, the degree of p_{-1}^R also equals 1, and so $\deg wt u_i^{(n)} = n + 1$.

On the other hand, the function $u_i^{(n)}$ has degree $n + 1$ with respect to the \mathbb{Z} -gradation on $\mathbb{C}[\mathbf{U}]$. Hence our homomorphism $\mathbb{C}[\mathbf{U}] \rightarrow \mathbb{C}[N_+/A_+]$ is homogeneous of degree 0. To show its surjectivity, it suffices to prove that the characters of $\mathbb{C}[\mathbf{U}]$ and $\mathbb{C}[N_+/A_+]$ coincide. Here by character of a \mathbb{Z} -graded vector space V we understand the formal power series

$$\text{ch } V = \sum_{n \in \mathbb{Z}} \dim V_n q^n,$$

where V_n is the homogeneous subspace of V of degree n . Clearly,

$$\text{ch } \mathbb{C}[\mathbf{U}] = \prod_{n > 0} (1 - q^n)^{-\ell}.$$

Now, it is well-known that as a \mathfrak{g} -module, $\mathbb{C}[N_+]$ is isomorphic to the module M_0^* contragradient to the Verma module with highest weight 0 (these modules are defined in Lecture 4). Hence, as an \mathfrak{n}_+ module, it is isomorphic to the restricted dual $U(\mathfrak{n}_+)^*$ of $U(\mathfrak{n}_+)$. This implies that $\mathbb{C}[N_+/A_+]$ is isomorphic to the space of \mathfrak{a}_+ -invariants on $U(\mathfrak{n}_+)^*$.

By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{n}_+)^*$ has a basis dual to the basis of lexicographically ordered monomials in $U(\mathfrak{n}_+)$. We can choose this basis in such a way that each element is the product AB , where A is the product of $p_n, n \in I$ (a basis in \mathfrak{a}_+), and B is a product of elements of some basis in $\text{Im}(\text{ad } p_{-1}) \cap \mathfrak{n}_+$. Then the elements of $U(\mathfrak{n}_+)^*$ dual to the monomials of the second type form a basis of the space of \mathfrak{a}_+ -invariants on $U(\mathfrak{n}_+)^*$. Hence the character of $\mathbb{C}[N_+/A_+]$ equals that of a free polynomial algebra with ℓ generators of each positive degree. Thus its character coincides with the character of $\mathbb{C}[\mathbf{U}]$, and this completes the proof. \square

1.8. Definition of the mKdV hierarchy. Having identified $\mathbb{C}[\mathbf{U}]$ with $\mathbb{C}[N_+/A_+]$ in the theorem above, we can now view the derivation $p_{-n}^R, n \in I$ of the latter, as a derivation of $\mathbb{C}[\mathbf{U}]$, which we denote by ∂_n . Thus, we obtain an *infinite set of commuting derivations* of the ring of differential polynomials $\mathbb{C}[u_i^{(n)}]$, of degrees equal to the exponents of \mathfrak{g} modulo the Coxeter number.

We call this set the *mKdV hierarchy* associated to the affine algebra \mathfrak{g} .

According to Theorem 1.1, to each map $\mathcal{D} \rightarrow \mathfrak{h}$ we can attach a map $\mathcal{D} \rightarrow N_+/A_+$ (here \mathcal{D} is the formal disk). From analytic point of view, this means that we have a mapping that to a every smooth function $\mathbf{u}(z)$ from, say, \mathbb{R} to \mathfrak{h} , assigns a smooth function $K(z) : \mathbb{R} \rightarrow N_+/A_+$. Moreover, this mapping is *local* in the sense that $K(z)$ depends on \mathbf{u} only through the values of its derivatives at z (the jets of $\mathbf{u}(z)$ at z). Due

to this property, $\mathbf{u}(z)$ can just as well be a smooth function on a circle, or an analytic function on a domain in \mathbb{C} .

Next we say that on N_+/A_+ we have a family of commuting vector fields p_{-n}^R that come from the right action of the Lie algebra \mathfrak{a}_- . The flows of the mKdV hierarchy are the corresponding flows on the space of smooth function $\mathbf{u}(z) : \mathbb{R} \rightarrow \mathfrak{h}$. Note that the function $\mathbf{u}(z)$ does not really have to be smooth everywhere; it may have singularities at certain points.

In principle, we have now answered the question posed in the Introduction as to where commuting derivations acting on rings of differential polynomials come from. But we would like to have explicit formulas for these derivations. From the construction itself we know that the first of them, ∂_1 , is just ∂_z . But we don't know the rest. On the other hand, the mKdV hierarchies have been previously defined by other methods, and we want to make sure that our definition coincides with the earlier definitions.

These issues will be the subject of the next two lectures. We will first write the derivations ∂_n in the so-called zero-curvature form and then compare our definition of the mKdV hierarchy with the other definition to see that they are equivalent.

LECTURE 2

2.1. Zero curvature representation. Let $\omega_i^\vee \in \mathfrak{h}$ be the fundamental coweights that satisfy: $(\alpha_i, \omega_j^\vee) = \delta_{i,j}$. Define \mathbf{u} to be the element

$$\sum_{i=1}^{\ell} \omega_i^\vee \otimes u_i \in \mathfrak{h} \otimes \mathbb{C}[\mathbf{U}].$$

Denote by $\widehat{\mathfrak{g}}$ the completion of \mathfrak{g} , which is the inverse limit of $\mathfrak{g}/(\mathfrak{g}^0 \otimes t^m \mathbb{C}[t] \cap \mathfrak{g})$. Thus, if $\mathfrak{g} = \overline{\mathfrak{g}}[t, t^{-1}]$, then $\widehat{\mathfrak{g}} = \overline{\mathfrak{g}}((t))$, and in general $\widehat{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{g}^0((t))$.

For each $K \in N_+/A_+$, $Kp_{-n}K^{-1}$ is a well-defined element of $\widehat{\mathfrak{g}}$. Thus we obtain an algebraic map $N_+/A_+ \rightarrow \widehat{\mathfrak{g}}$, or, equivalently, an element of $\widehat{\mathfrak{g}} \otimes \mathbb{C}[N_+/A_+]$.¹ By abuse of notation, we denote this element by $Kp_{-n}K^{-1}$, and its projection on $\mathfrak{b}_- \otimes \mathbb{C}[N_+/A_+]$ by $(Kp_{-n}K^{-1})_-$. Since $\mathbb{C}[N_+/A_+] \simeq \mathbb{C}[u_i^{(n)}]$, by Theorem 1.1, we can apply to it $1 \otimes \partial_m$, which we denote simply by ∂_m .

More explicitly, recall that the Lie algebra $\widehat{\mathfrak{g}}$ can be realized as a Lie subalgebra of the Lie algebra $\mathfrak{g}^0((t))$, where \mathfrak{g}^0 is a finite-dimensional simple Lie algebra; e.g., in the non-twisted case, $\mathfrak{g}^0 = \overline{\mathfrak{g}}$. If we choose a basis in \mathfrak{g}^0 , we can consider an element of $\widehat{\mathfrak{g}}$ as a matrix, whose entries are Laurent power series. The entries of the matrix $Kp_{-n}K^{-1}$ are Laurent series in t whose coefficients are regular functions on N_+/A_+ . Hence, under the isomorphism $\mathbb{C}[N_+/A_+] \simeq \mathbb{C}[u_i^{(n)}]$, each coefficient corresponds to a differential polynomial in u_i 's. Applying ∂_m to $Kp_{-n}K^{-1}$ means applying ∂_n to each of these coefficients.

We are going to prove the following result.

¹More precisely, it lies in the completed tensor product, e.g., if $\widehat{\mathfrak{g}} = \overline{\mathfrak{g}}((t))$, then by $\widehat{\mathfrak{g}} \otimes \mathbb{C}[N_+/A_+]$ we mean $(\overline{\mathfrak{g}} \otimes \mathbb{C}[N_+/A_+])((t))$.

Theorem 2.1 ([8], Theorem 2).

$$(2.1) \quad [\partial_z + p_{-1} + \mathbf{u}, \partial_n + (Kp_{-n}K^{-1})_-] = 0.$$

The equation (2.1) can be rewritten as

$$(2.2) \quad \partial_n \mathbf{u} = \partial_z (Kp_{-n}K^{-1})_- + [p_{-1} + \mathbf{u}, (Kp_{-n}K^{-1})_-].$$

As explained above, this equation expresses $\partial_n u_i$ in terms of differential polynomials in u_i 's. Since, by construction, ∂_n commutes with $\partial_1 \equiv \partial_z$, formula (2.2) uniquely determines ∂_n as an evolutionary derivation of $\mathbb{C}[u_i^{(n)}]$. Thus, we obtain an explicit formula for ∂_n .

Note that formula (2.1) looks like the zero curvature condition on a connection defined on a two-dimensional space – hence the name “zero curvature representation”.

Remark 2.1. One should be careful in distinguishing the two variables: t and z , and the two factors, $\widehat{\mathfrak{g}}$ and $\mathbb{C}[N_+/A_+]$, in the formulas below. The variable z is a “dynamical variable” connected with $\mathbb{C}[N_+/A_+]$ due to its isomorphism with $\mathbb{C}[u_i^{(n)}]$; in particular, ∂_z is the derivation of p_{-1}^R of $\mathbb{C}[N_+/A_+]$. On the other hand, t is just the formal variable entering the definition of the affine algebra $\widehat{\mathfrak{g}}$. We remark that in earlier works on soliton theory, t was denoted by λ^{-1} , and λ was called the *spectral parameter*. \square

2.2. Proof of Theorem 2.1. We will actually prove a stronger result:

Proposition 2.1. For $K \in N_+/A_+$,

$$(2.3) \quad [\partial_m + (Kp_{-m}K^{-1})_-, \partial_n + (Kp_{-n}K^{-1})_-] = 0, \quad \forall m, n \in I.$$

Let us first explain how to derive Theorem 2.1 from Proposition 2.1. We need to specialize (2.3) to $m = 1$ and to determine $(Kp_{-1}K^{-1})_-$ explicitly.

Lemma 2.1.

$$(2.4) \quad (Kp_{-1}K^{-1})_- = p_{-1} + \mathbf{u}.$$

Proof. It is clear that $(Kp_{-1}K^{-1})_- = p_{-1} + x$, where $x \in \mathfrak{h}$. Hence we need to show that $x = \mathbf{u}$, or, equivalently, that $(\alpha_i, x) = u_i, i = 1, \dots, \ell$. We can rewrite the latter formula as $u_i = (\alpha_i, (Kp_{-1}K^{-1})_-)$, and hence as $u_i = (\alpha_i, Kp_{-1}K^{-1})$. But this is exactly the definition of the function \tilde{u}_i on N_+/A_+ , which is the image of $u_i \in \mathbb{C}[\mathbf{U}]$ under the isomorphism $\mathbb{C}[\mathbf{U}] \rightarrow \mathbb{C}[N_+/A_+]$. \square

Now specializing $m = 1$ in formula (2.3) and using Lemma 2.1 we obtain formula (2.1).

In order to prove formula (2.3), we need to find an explicit formula for the action of ∂_n on $Kp_{-m}K^{-1}$.

It follows from formula (1.3) that

$$(2.5) \quad a^R \cdot xvx^{-1} = [(xax^{-1})_+, xvx^{-1}], \quad a, v \in \mathfrak{g}.$$

If a and v are both elements of \mathfrak{a} , then formula (2.5) does not change if we multiply x from the right by an element of A_+ . Denote by K the coset of x in N_+/A_+ . Then we can write:

$$(2.6) \quad \partial_n \cdot KvK^{-1} = [(Kp_{-n}K^{-1})_+, KvK^{-1}], \quad v \in \mathfrak{a}.$$

Proof of Proposition 2.1. Substituting $v = p_{-m}$ into formula (2.6), we obtain:

$$\partial_n \cdot Kp_{-m}K^{-1} = [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}].$$

Hence

$$\partial_n \cdot (Kp_{-m}K^{-1})_- = [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]_- = [(Kp_{-n}K^{-1})_+, (Kp_{-m}K^{-1})_-]_-.$$

Therefore we obtain:

$$\begin{aligned} & [\partial_m + (Kp_{-m}K^{-1})_-, \partial_n + (Kp_{-n}K^{-1})_-] \\ &= \partial_m \cdot (Kp_{-n}K^{-1})_- - \partial_n \cdot (Kp_{-n}K^{-1})_- + [(Kp_{-m}K^{-1})_-, (Kp_{-n}K^{-1})_-] \\ &= [(Kp_{-m}K^{-1})_+, (Kp_{-n}K^{-1})_-]_- - [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]_- \\ &+ [(Kp_{-m}K^{-1})_-, (Kp_{-n}K^{-1})_-]_-. \end{aligned}$$

Adding up the first and the last terms, we obtain

$$\begin{aligned} & [Kp_{-m}K^{-1}, (Kp_{-n}K^{-1})_-]_- - [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]_- = \\ &= [Kp_{-m}K^{-1}, Kp_{-n}K^{-1}]_- = 0, \end{aligned}$$

and Proposition 2.1 is proved. \square

2.3. Recurrence relation. Now we have at our disposal explicit formulas for equations of the mKdV hierarchy. But from the practical point of view, it is still difficult to compute explicitly terms like $(Kp_{-n}K^{-1})_-$, since the coset K has been defined in Lecture 1 in a rather abstract way. In this section we will exhibit a simple property of K , which will enable us to compute everything via a straightforward recursive algorithm.

Given a vector space W , we will write $W[\mathbf{U}]$ for $W \otimes \mathbb{C}[\mathbf{U}]$.

Recall that for any $v \in \mathfrak{a}$, KvK^{-1} denotes the element of $\widehat{\mathfrak{g}}[\mathbf{U}] = \widehat{\mathfrak{g}} \otimes \mathbb{C}[\mathbf{U}]$, determined by the embedding $N_+/A_+ \rightarrow \widehat{\mathfrak{g}}$, which maps K to KvK^{-1} .

Lemma 2.2.

$$(2.7) \quad [\partial_z + p_{-1} + \mathbf{u}, KvK^{-1}] = 0, \quad \forall v \in \mathfrak{a}.$$

Proof. Using formula (2.6) and Lemma 2.1 we obtain:

$$\begin{aligned} \partial_z(KvK^{-1}) &= [(Kp_{-1}K^{-1})_+, KvK^{-1}] \\ &= -[(Kp_{-1}K^{-1})_-, KvK^{-1}] \\ &= -[p_{-1} + \mathbf{u}, KvK^{-1}]. \end{aligned}$$

\square

Now suppose that $\mathcal{V} \in \widehat{\mathfrak{g}}[\mathbf{U}]$ satisfies

$$(2.8) \quad [\partial_z + p_{-1} + \mathbf{u}(z), \mathcal{V}] = 0.$$

Then we can decompose \mathcal{V} with respect to the principal gradation on $\widehat{\mathfrak{g}}$ (the principal gradation on $\widehat{\mathfrak{g}}$ should not be confused with the gradation on $\mathbb{C}[\mathbf{U}] = \mathbb{C}[N_+/A_+]$):

$$\mathcal{V} = \sum_{m \geq -n} \mathcal{V}_m, \quad \mathcal{V}_{-n} \neq 0,$$

and write equation (2.8) in components. Recall that $\deg p_{-1} = -1$, $\deg \mathbf{u} = 0$, and $\deg \partial_z = 0$.

The first equation that we obtain, in degree $-n - 1$, reads:

$$[p_{-1}, \mathcal{V}_{-n}] = 0.$$

By Proposition 1.2, it implies that $\mathcal{V}_{-n} \in \mathfrak{a}$. Hence $\mathcal{V}_{-n} = p_{-n}$ times an element of $\mathbb{C}[\mathbf{U}]$, where $n \in I \cup -I$.

Other equations have the form

$$(2.9) \quad \partial_z \mathcal{V}_m + [p_{-1}, \mathcal{V}_{m-1}] + [\mathbf{u}, \mathcal{V}_m] = 0, \quad m \geq -n.$$

Recall from Proposition 1.2 that $\mathfrak{g} = \mathfrak{a} \oplus \text{Im}(\text{ad} p_{-1})$. So we can split each $x \in \widehat{\mathfrak{g}}$ into a sum $x = x^0 + x^\perp$, where $x^0 \in \mathfrak{a}$ and $x^\perp \in \text{Im}(\text{ad} p_{-1})$. Furthermore, we can split our equations (2.9) into two parts, lying in \mathfrak{a} and $\text{Im}(\text{ad} p_{-1})$. Then we obtain two equations:

$$\begin{aligned} \partial_z \mathcal{V}_m^0 + [p_{-1}, \mathcal{V}_{m+1}^0] + [\mathbf{u}, \mathcal{V}_m^0] &= 0, \\ \partial_z \mathcal{V}_m^\perp + [p_{-1}, \mathcal{V}_{m+1}^\perp] + [\mathbf{u}, \mathcal{V}_m^\perp] &= 0. \end{aligned}$$

But clearly, $[p_{-1}, \mathcal{V}_{m+1}^0] = 0$, $[\mathbf{u}, \mathcal{V}_m^0] = 0$, and $[p_{-1}, \mathcal{V}_{m+1}^\perp] = 0$. Also, $[p_{-1}, \mathcal{V}_{m+1}^\perp]^\perp = [p_{-1}, \mathcal{V}_{m+1}^\perp]$. Hence we obtain:

$$(2.10) \quad \partial_z \mathcal{V}_m^0 + [\mathbf{u}, \mathcal{V}_m^\perp]^0 = 0,$$

$$(2.11) \quad \partial_z \mathcal{V}_m^\perp + [p_{-1}, \mathcal{V}_{m+1}^\perp] + [\mathbf{u}, \mathcal{V}_m^\perp]^\perp = 0.$$

We already know that $\mathcal{V}_{-n} \in \mathfrak{a}[\mathbf{U}]$, and so $\mathcal{V}_{-n}^\perp = 0$. Now we can find \mathcal{V}_{-n}^0 and \mathcal{V}_{-n+1}^\perp . First we obtain from equation (2.10):

$$\partial_z \mathcal{V}_{-n}^0 = -[\mathbf{u}, \mathcal{V}_{-n}^\perp] = 0,$$

which implies that $\mathcal{V}_{-n} = p_{-n}$ times a *constant* factor. Without loss of generality we assume that $\mathcal{V}_{-n} = p_{-n}$. Next, we obtain from (2.11),

$$[p_{-1}, \mathcal{V}_{-n+1}^\perp] = -[\mathbf{u}, \mathcal{V}_{-n}^\perp]^\perp,$$

which uniquely determines \mathcal{V}_{-n+1}^\perp , since $\text{ad} p_{-1}$ is invertible on $\text{Im}(\text{ad} p_{-1})$ (see Proposition 1.2).

Now assume by induction that we know \mathcal{V}_k^0 , $-n \leq k \leq m - 1$ and \mathcal{V}_k^\perp , $-n \leq k \leq m$. Then we obtain from formulas (2.10) and (2.11):

$$(2.12) \quad \partial_z \mathcal{V}_m^0 = -[\mathbf{u}, \mathcal{V}_m^\perp]^0,$$

$$(2.13) \quad \mathcal{V}_{m+1}^\perp = (\text{ad} p_{-1})^{-1} \left(-\partial_z \mathcal{V}_m^\perp - [\mathbf{u}, \mathcal{V}_m^\perp]^\perp \right).$$

Consider $[\mathbf{u}, \mathcal{V}_m^\perp]^0 \in \mathfrak{a}$. It can only be non-zero if $m \in \pm I$. In that case it is equal to $p_m \otimes P_m$, $P_m \in \mathbb{C}[\mathbf{U}]$, or $p_m^1 \otimes P_m^1 + p_m^2 \otimes P_m^2$, if the multiplicity of the exponent is 2. If each P_m^i is a total derivative, $P_m^i = \partial_z Q_m^i$, then we find from equation (2.12):

$$\mathcal{V}_m^0 = \sum_i p_m^i \otimes Q_m^i.$$

After that we can solve equation (2.13) for \mathcal{V}_{m+1}^\perp . But we know that $Kp_{-n}K^{-1} = p_{-n} + \dots$ does satisfy equation (2.9). Hence we know for sure that each P_m^i is a total derivative, and equation (2.12) can be resolved (i.e., the antiderivative can be taken whenever we need it). Therefore we can find all \mathcal{V}_m 's following this algorithm.

Note that the only ambiguity is introduced when we take the antiderivative, since we then have the freedom of adding $c_m \cdot p_m$ to \mathcal{V}_m^0 , where $c_m \in \mathbb{C}$. This simply reflects the fact that adding $Kp_mK^{-1} = p_{-m} + \dots$ with $m > -n$ to $Kp_{-n}K^{-1}$, we do not violate (2.9). Thus, we have proved the following

Proposition 2.2. *Let \mathcal{V} be an element of $\widehat{\mathfrak{g}}[\mathbf{U}]$ satisfying*

$$(2.14) \quad [\partial_z + p_{-1} + \mathbf{u}, \mathcal{V}] = 0.$$

Then $\mathcal{V} = KvK^{-1}$, where $v \in \mathfrak{a}$. Moreover, if $\mathcal{V} = p_{-n} +$ terms of degree higher than $-n$ with respect to the principal gradation on \mathfrak{g} then $v = p_{-n} +$ terms of degree higher than $-n$.

Now recall that each \mathcal{V}_n is an element of $\widehat{\mathfrak{g}}[\mathbf{U}]$, i.e., its matrix entries (in a particular representation of $\widehat{\mathfrak{g}}$) are Laurent power series in t , with coefficients from the ring $\mathbb{C}[\mathbf{U}]$. That ring has a \mathbb{Z} -gradation (which should not be confused with the principal gradation on \mathfrak{g} itself!). The corresponding gradation on $\mathbb{C}[N_+/A_+]$ is defined by the vector field $-\rho^\vee$. Using this gradation, for each n we can distinguish a canonical solution \mathcal{V}_n of the recurrence relations (2.9) that equals precisely $Kp_{-n}K^{-1}$.

Lemma 2.3. *Let $x \in \mathfrak{g}$ be homogeneous of degree k . Then the function $f \in \mathbb{C}[N_+/A_+]$ given by the formula $f(K) = (x, Kp_{-n}K^{-1})$ is a homogeneous element of $\mathbb{C}[N_+/A_+]$ of degree $n - m$.*

Proof. We have to show that $\rho^\vee \cdot f = (m - n)f$. But

$$\begin{aligned} (\rho^\vee \cdot f)(K) &= (x, [(K\rho^\vee K^{-1})_+, Kp_{-n}K^{-1}]) \\ &= (x, [K\rho^\vee K^{-1}, Kp_{-n}K^{-1}]) - (x, [(K\rho^\vee K^{-1})_-, Kp_{-n}K^{-1}]) \\ &= (x, K[\rho^\vee, p_{-n}]K^{-1}) + ([\rho^\vee, x], Kp_{-n}K^{-1}) \\ &= (m - n)(x, Kp_{-n}K^{-1}) = (m - n)f(K), \end{aligned}$$

and the lemma is proved. \square

Our recurrence relations (2.10) and (2.11) imply that if x has degree m , then (x, \mathcal{V}_n) has a term of degree $n - m$ in $\mathbb{C}[\mathbf{U}] \simeq \mathbb{C}[N_+/A_+]$ and possibly other terms of smaller degrees which result from addition of constants when taking anti-derivatives at the previous steps of the recursive procedure. It is clear that there is a unique solution \mathcal{V}_n of equation (2.8), such that (x, \mathcal{V}_n) is homogeneous of degree $n - m$, and this solution is $Kp_{-n}K^{-1}$.

Now we have an algorithm for finding $Kp_{-n}K^{-1}$. Hence we can construct explicitly the n th equations of the mKdV hierarchy by inserting $(Kp_{-n}K^{-1})_-$ into equation (2.1). In the next section we will do that in the case of $L\mathfrak{sl}_2$.

2.4. **Example of $L\mathfrak{sl}_2$.** We have

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2}u & 0 \\ 0 & -\frac{1}{2}u \end{pmatrix}.$$

Hence

$$(2.15) \quad \partial_z + p_{-1} + \mathbf{u} = \partial_z + \begin{pmatrix} \frac{1}{2}u & t^{-1} \\ 1 & -\frac{1}{2}u \end{pmatrix}.$$

The first equation of the hierarchy is

$$[\partial_z + p_{-1} + \mathbf{u}, \partial_n + (Kp_{-n}K^{-1})_-] = 0.$$

By formula (2.4), $(Kp_{-1}K^{-1})_- = p_{-1} + \mathbf{u}$. Hence the first equation reads

$$[\partial_z + p_{-1} + \mathbf{u}, \partial_1 + p_{-1} + \mathbf{u}] = 0,$$

which is equivalent to

$$\partial_1 \mathbf{u} = \partial_z \mathbf{u},$$

as we already know (note that this is true for an arbitrary \mathfrak{g}).

Now we consider the next equation in the case of $L\mathfrak{sl}_2$, corresponding to $n = 3$. To write it down, we have to compute $(Kp_{-3}K^{-1})_-$. In order to do that we compute the first few terms of $Kp_{-3}K^{-1}$ following the algorithm of the previous section.

We introduce a basis of $L\mathfrak{sl}_2$ homogeneous with respect to the principal gradation:

$$p_{2j+1} = \begin{pmatrix} 0 & t^j \\ t^{j+1} & 0 \end{pmatrix}, \quad q_{2j+1} = \begin{pmatrix} 0 & t^j \\ -t^{j+1} & 0 \end{pmatrix},$$

$$r_{2j} = \begin{pmatrix} t^j & 0 \\ 0 & -t^j \end{pmatrix}.$$

Then we write

$$Kp_{-3}K^{-1} = p_{-3} + R_{-2}r_{-2} + P_{-1}p_{-1} + Q_{-1}q_{-1} + R_0r_0 + \dots$$

The other terms are irrelevant for us now as we are only interested in $(Kp_{-3}K^{-1})_-$.

Equation (2.8) reads

$$[\partial_z + p_{-1} + \frac{1}{2}ur_0, \partial_3 + p_{-3} + R_{-2}r_{-2} + P_{-1}p_{-1} + Q_{-1}q_{-1} + R_0r_0 + \dots] = 0.$$

Rewriting it in components, we obtain the following equations:

$$2R_{-2} - u = 0, \quad \partial_z R_{-2} - 2Q_{-1} = 0,$$

$$\partial_z P_{-1} + uQ_{-1} = 0, \quad \partial_z Q_{-1} + uP_{-1} - 2R_0.$$

We find from these equations:

$$R_{-2} = \frac{1}{2}u, \quad Q_{-1} = \frac{1}{4}\partial_z u, \quad P_{-1} = -\frac{1}{8}u^2, \quad R_0 = -\frac{1}{16}u^3 + \frac{1}{8}\partial_z^2 u.$$

The equation of the mKdV hierarchy corresponding to $n = 3$ now reads:

$$\partial_3 \begin{pmatrix} \frac{1}{2}u & t^{-1} \\ 1 & -\frac{1}{2}u \end{pmatrix} = \left[\begin{pmatrix} \frac{1}{2}ut^{-2} - \left(\frac{1}{16}u^3 - \frac{1}{8}\partial_z^2 u\right) & t^{-2} + \left(-\frac{1}{8}u^2 + \frac{1}{4}\partial_z u\right)t^{-1} \\ t^{-1} - \left(\frac{1}{8}u^2 + \frac{1}{4}\partial_z u\right) & -\frac{1}{2}ut^{-2} + \left(\frac{1}{16}u^3 - \frac{1}{8}\partial_z^2 u\right) \end{pmatrix}, \partial_z + \begin{pmatrix} \frac{1}{2}u & t^{-1} \\ 1 & -\frac{1}{2}u \end{pmatrix} \right].$$

This is equivalent to the equation

$$(2.16) \quad \partial_3 u = \frac{3}{8}u^2 \partial_z u - \frac{1}{4}\partial_z^3 u,$$

which is the *mKdV equation*. It is related to the KdV equation (0.1) by a change of variables (see Lecture 4).

LECTURE 3

In the previous lecture we have written down explicitly the equations of the mKdV hierarchy in the zero curvature form. In this lecture we will discuss another approach to these equations, which is due to Drinfeld and Sokolov and goes back to Zakharov and Shabat. We will then establish the equivalence between the two approaches.

3.1. Generalities on zero curvature equations. Let us look at the general zero curvature equation

$$(3.1) \quad [\partial_z + p_{-1} + \mathbf{u}, \partial_\tau + V] = 0.$$

The element p_{-1} has degree -1 with respect to the principal gradation of \mathfrak{g} , while \mathbf{u} has degree 0 (with respect to the gradation on \mathfrak{g}). This makes finding an element V that satisfies (3.1) a non-trivial problem. Indeed, equation (3.1) can be written as

$$(3.2) \quad \partial_\tau \mathbf{u} = [\partial_z + p_{-1} + \mathbf{u}, V].$$

The left hand side of (3.2) has degree 0 . Therefore V should be such that the expression in the right hand side of (3.2) is concentrated in degree 0 .

Such elements can be constructed by the following trick (see [31, 7, 29]). Suppose we found some $\mathcal{V} \in \widehat{\mathfrak{g}}[\mathbf{U}]$ which satisfies

$$(3.3) \quad [\partial_z + p_{-1} + \mathbf{u}, \mathcal{V}] = 0.$$

We can split \mathcal{V} into the sum $\mathcal{V}_+ + \mathcal{V}_-$ of its components of positive and non-positive degrees with respect to the principal gradation. Then $V = \mathcal{V}_-$ has the property that the right hand side of (3.2) has degree 0 . Indeed, from (3.3) we find

$$[\partial_z + p_{-1} + \mathbf{u}(z), \mathcal{V}_-] = -[\partial_z + p_{-1} + \mathbf{u}(z), \mathcal{V}_+],$$

which means that both commutators have neither positive nor negative homogeneous components. Therefore equation (3.2) makes sense. Thus, the problem of finding elements V such that (3.2) makes sense, and hence constructing the mKdV hierarchy, has been reduced to the problem of finding solutions of equation (3.3).

In Proposition 2.2 we described the solutions of equation (3.3) in terms of the isomorphism $\mathbf{U} \simeq N_+/A_+$, which attaches $K \in N_+/A_+$ to \mathbf{u} .

There are other ways of solving this problem. In the next section we will discuss the approach of Drinfeld and Sokolov [7], closely related to the dressing method of Zakharov and Shabat [31]. Another approach, proposed by Wilson [29], will be mentioned briefly later.

3.2. Drinfeld-Sokolov approach. Consider the group $N_+[\mathbf{U}]$ of \mathbf{U} -points of N_+ . Thus, in a representation of N_+ elements of $N_+[\mathbf{U}]$ are matrices, whose entries are differential polynomials in u_i 's.

Proposition 3.1 ([7], Prop. 6.2). *There exists an element $M \in N_+[\mathbf{U}]$, such that*

$$(3.4) \quad M^{-1}(\partial_z + p_{-1} + \mathbf{u}(z))M = \partial_z + p_{-1} + \sum_{i \in I} h_i p_i,$$

where $h_i \in \mathbb{C}[\mathbf{U}]$, $\forall i \in I$. M is defined uniquely up to right multiplication by an element of $A_+[\mathbf{U}]$.

Proof. Since the exponential map $\hat{\mathfrak{n}}_+ \rightarrow N_+$ is an isomorphism, we can write $M = \exp \mathbf{m}$, where $\mathbf{m} \in \hat{\mathfrak{n}}_+[\mathbf{U}]$. For any $\alpha \in \mathfrak{g}$, we have:

$$M^{-1}\alpha M = \sum_{n \geq 0} \frac{(-1)^n}{n!} (\text{ad } \mathbf{m})^n \cdot \alpha.$$

Decompose \mathbf{m} with respect to the principal gradation:

$$\mathbf{m} = \sum_{j > 0} \mathbf{m}_j.$$

Then for the n th component of equation (3.4) reads

$$(3.5) \quad [p_{-1}, \mathbf{m}_1] + \mathbf{u} = 0,$$

for $n = 0$, and

$$(3.6) \quad [p_{-1}, \mathbf{m}_{n+1}] + \text{terms involving } \mathbf{m}_i, i \leq n, \text{ and } \mathbf{u} = h_n p_n$$

for $n > 0$ (here we set $h_n = 0$, if $n \notin I$). Since $\mathbf{u} \in \text{Im}(\text{ad } p_{-1})$, we can solve equation (3.5) and start the inductive process. At the n th step of induction we would have found $\mathbf{m}_i, i \leq n$ and $h_i, i < n$. We can now split the n th equation (3.6) into two parts, lying in \mathfrak{a} and $\text{Im}(\text{ad } p_{-1})$. We can then set $\mathbf{m}_{n+1} = (\text{ad } p_{-1})^{-1}$ of the $\text{Im}(\text{ad } p_{-1})$ part, and $h_n p_n =$ the \mathfrak{a} part of the equation (everything evidently works even if the multiplicity of n is 2). Clearly, both \mathbf{m}_n and h_n is local. This inductive procedure gives us a unique solution M of equation (3.4), such that each \mathbf{m}_{n+1} lies in $\text{Im}(\text{ad } p_{-1})$. \square

Now observe that formula (3.4) implies:

$$(3.7) \quad [\partial_z + p_{-1} + \mathbf{u}, MvM^{-1}] = 0, \quad \forall v \in \mathfrak{a}.$$

Hence the zero curvature equations

$$(3.8) \quad [\partial_z + p_{-1} + \mathbf{u}, \partial_{\tau_n} + (Mp_{-n}M^{-1})_-] = 0$$

make sense. Drinfeld and Sokolov called these equations the mKdV hierarchy associated to \mathfrak{g} . We will now show that these equations coincide with our equations (2.1).

3.3. Equivalence of two constructions. In both [7] and [12], one assigns to a jet $(u_i^{(n)}) \in \text{Spec } \mathbb{C}[\mathbf{U}]$, a coset in N_+/A_+ . We now show that they coincide.

Theorem 3.1 ([8], Theorem 3). *The cosets M and K in N_+/A_+ assigned in [7] and [12], respectively, to a jet $(u_i^{(n)})$, coincide.*

Proof. According to formula (3.7),

$$[\partial_z + p_{-1} + \mathbf{u}, Mp_{-1}M^{-1}] = 0.$$

Since $Mp_{-1}M^{-1} = p_{-1} +$ terms of degree higher than -1 with respect to the principal gradation on \mathfrak{g} , we obtain from Lemma 2.2 that $Mp_{-1}M^{-1} = KvK^{-1}$, where $v \in \mathfrak{a}$. Let us show that this implies that $v = p_{-1}$ and that $K = M$ in N_+/A_+ .

Indeed, from the equality

$$Mp_{-1}M^{-1} = KvK^{-1}$$

we obtain that $(K^{-1}M)p_{-1}(K^{-1}M)^{-1}$ lies in \mathfrak{a} . We can represent $K^{-1}M$ as $\exp y$ for some $y \in \widehat{\mathfrak{n}}_+$. Then $(K^{-1}M)p_{-1}(K^{-1}M)^{-1} = v$ can be expressed as a linear combination of multiple commutators of y and p_{-1} :

$$e^y p_{-1} (e^y)^{-1} = \sum_{n \geq 0} \frac{1}{n!} (\text{ad } y)^n \cdot p_{-1}.$$

We can write $y = \sum_{j > 0} y_j$, where y_j is the homogeneous component of y of principal degree j . It follows from Proposition 1.2 that $\widehat{\mathfrak{n}}_+ = \widehat{\mathfrak{a}}_+ \oplus \text{Im}(\text{ad } p_{-1})$. Therefore each y_j can be further split into a sum of $y_j^0 \in \widehat{\mathfrak{a}}_+$ and $y_j^1 \in \text{Im}(\text{ad } p_{-1})$.

Suppose that y does not lie in $\widehat{\mathfrak{a}}_+$. Let j_0 be the smallest number such that $y_{j_0}^1 \neq 0$. Then the term of smallest degree in $e^y p_{-1} (e^y)^{-1}$ is $[y_{j_0}^1, p_{-1}]$ which lies in $\text{Im}(\text{ad } p_{-1})$ and is non-zero, because $\text{Ker}(\text{ad } p_{-1}) = \widehat{\mathfrak{a}}_+$. Hence $e^y p_{-1} (e^y)^{-1}$ can not be an element of $\widehat{\mathfrak{a}}_+$.

Therefore $y \in \widehat{\mathfrak{a}}_+$ and so $K^{-1}M \in A_+$, which means that K and M represent the same coset in N_+/A_+ , and that $v = p_{-1}$. \square

Remark 3.1. In the proof of Theorem 3 of [8] one has to replace p_{-n} by p_{-1} . \square

From the analytic point of view, Proposition 3.1 produces the same thing as Theorem 1.1: it associates to a smooth function $\mathbf{u}(z) : \mathbb{R} \rightarrow \mathfrak{h}$, a smooth function $M(z) : \mathbb{R} \rightarrow N_+/A_+$ (cf. Sect. 1.8). What we have just shown is that the two constructions are equivalent, and $K(z) = M(z)$.

Having established that $K = M$, we see that the two definitions of mKdV hierarchy: (2.1) and (3.8), coincide. Thus, now we have two equivalent ways of constructing these equations. One of them is based on Theorem 1.1 identifying $\text{Spec } \mathbb{C}[u_i^{(n)}]$ with N_+/A_+ , and the other uses the ‘‘dressing operator’’ M defined by formula (3.4). One can then use an element of N_+/A_+ (K or M) to solve equation (3.3), and produce the mKdV equation (3.2).

Wilson [29] proposed another approach, in which one solves (3.3) directly without constructing first the dressing operator. Recall from Sect. 2.3 that when one tries to solve equation (3.3), one has to be able to take the antiderivative at certain steps. If

we know in advance the existence of solution (as we do in the two other approaches), then this is insured automatically. Wilson [29] gave an argument, which demonstrates directly that the antiderivatives can be taken at each step. Thus, he proved the existence of solution \mathcal{V} of (3.3) of the form $\mathcal{V} = p_{-n} +$ terms of higher degree. He then constructed the equations of the mKdV hierarchy using these \mathcal{V} . We now see that all three approaches are equivalent.

In the next two sections we will give another, more illuminating, explanation of the fact that $K = M$.

Remark 3.2. There exist generalizations of the mKdV hierarchies which are associated to abelian subalgebras of \mathfrak{g} other than \mathfrak{a} (see Remark 1.3). It is known that the Drinfeld-Sokolov approach can be applied to these generalized hierarchies [5, 19]. On the other hand, our approach can also be applied; in the case of the non-linear Schrödinger hierarchy, which corresponds to the homogeneous abelian subalgebra of \mathfrak{g} , this has been done by Feigin and the author [13]. The results of this lecture can be extended to establish the equivalence between the two approaches in this general context. \square

3.4. Realization of $\mathbb{C}[N_+]$ as a polynomial ring. The approach to the mKdV and affine Toda equations used in [12] and here is based on Theorem 1.1 which identifies $\mathbb{C}[N_+/A_+]$ with the ring of differential polynomials $\mathbb{C}[u_i^{(n)}]_{i=1,\dots,\ell;n \geq 0}$. In this section we add to the latter ring new variables corresponding to A_+ and show that the larger ring thus obtained is isomorphic to $\mathbb{C}[N_+]$. In this and the following two sections we follow closely [8].

Consider $u_i^{(n)}, i = 1, \dots, \ell; n \geq 0$, as A_+ -invariant regular functions on N_+ . Recall that

$$u_i(x) = (\alpha_i, xp_{-1}x^{-1}), \quad x \in N_+,$$

Now choose an element χ of $\tilde{\mathfrak{h}}$, such that $(\chi, C) \neq 0$. Introduce the regular functions $\chi_n, n \in I$, on N_+ by the formula:

$$(3.9) \quad \chi_n(x) = (\chi, xp_{-n}x^{-1}), \quad x \in N_+.$$

Theorem 3.2 ([8], Theorem 5). $\mathbb{C}[N_+] \simeq \mathbb{C}[u_i^{(n)}]_{i=1,\dots,\ell;n \geq 0} \otimes \mathbb{C}[\chi_n]_{n \in I}$.

Proof. Let us show that the functions $u_i^{(n)}$'s and χ_n 's are algebraically independent. In order to do that, let us compute the values of the differentials of these functions at the origin. Those are elements of the cotangent space to the origin, which is isomorphic to the dual space $\widehat{\mathfrak{n}}_+^*$ of $\widehat{\mathfrak{n}}_+$.

Using the invariant inner product on $\tilde{\mathfrak{g}}$ we identify $\widehat{\mathfrak{n}}_+^*$ with \mathfrak{n}_- . By Proposition 1.2 $\mathfrak{n}_- = \mathfrak{a}_- \oplus (\mathfrak{a}_+)^{\perp}$. Recall that with respect to the principal gradation, $(\mathfrak{a}_+)^{\perp} = \bigoplus_{j>0} (\mathfrak{a}_+)^{\perp}_{-j}$, where $\dim(\mathfrak{a}_+)^{\perp}_{-j} = \ell$, and $\text{ad } p_{-1} : (\mathfrak{a}_+)^{\perp}_{-j} \rightarrow (\mathfrak{a}_+)^{\perp}_{-j-1}$ is an isomorphism for all $j > 0$.

By construction of u_i 's given in the proof of Theorem 1.1, $du_i|_1, i = 1, \dots, \ell$, form a basis of $(\mathfrak{a}_+)^{\perp}_{-1}$, and hence $du_i^{(n)}|_1, i = 1, \dots, \ell$, form a basis of $(\mathfrak{a}_+)^{\perp}_{-n-1}$. Thus, the covectors $du_i^{(n)}|_1, i = 1, \dots, \ell; n \geq 0$, are linearly independent. Let us show now that the covectors $d\chi_n|_1$ are linearly independent from them and among themselves. For

that it is sufficient to show that the pairing between $dF_m|_1$ and p_n is non-zero if and only if $n = m$. But we have:

$$(3.10) \quad p_n^R \cdot (\chi, xp_{-m}x^{-1}) = (\chi, x[p_n, p_{-m}]x^{-1}).$$

Since $p_n, n \in \pm I$, generate a Heisenberg subalgebra in $\tilde{\mathfrak{g}}$ (see [22], Lemma 14.4),

$$(3.11) \quad [p_n, p_{-m}] = \alpha_n C \delta_{n,m},$$

where $\alpha_n \neq 0, \forall n \in I$. Therefore the pairing between $dF_m|_1$ and p_n is equal to $\alpha_n(\chi, C)\delta_{n,-m}$, and it is non-zero if and only if $n = m$.

Thus, the functions $u_i^{(n)}$'s and χ_n 's are algebraically independent. Hence we have an embedding $\mathbb{C}[u_i^{(n)}]_{i=1, \dots, \ell; n \geq 0} \otimes \mathbb{C}[\chi_n]_{n \in I} \rightarrow \mathbb{C}[N_+]$. But the characters of the two spaces with respect to the principal gradation are both equal to

$$\prod_{n \geq 0} (1 - q^n)^{-\ell} \prod_{i \in I} (1 - q^i)^{-1}.$$

Hence this embedding is an isomorphism. \square

3.5. Another proof of Theorem 3.1. Let $c_n = (\alpha_n(\chi, C))^{-1}$, where $\alpha_n, n \in I$, denote the non-zero numbers determined by formula (3.11).

Proposition 3.2. *Let K be an element of N_+ . We associate to it another element of N_+ ,*

$$\bar{K} = K \exp \left(- \sum_{n \in I} c_n p_n \chi_n(K) \right)$$

In any finite-dimensional representation of N_+ , \bar{K} is represented by a matrix whose entries are Taylor series with coefficients in the ring of differential polynomials in $u_i, i = 1, \dots, \ell$

The map $N_+ \rightarrow N_+$ which sends K to \bar{K} is constant on the right A_+ -cosets, and hence defines a section $N_+/A_+ \rightarrow N_+$.

Proof. Each entry of

$$\bar{K} = K \exp \left(- \sum_{n \in I} c_n p_n \chi_n(K) \right)$$

is a function on N_+ . According to Theorem 1.1 and Theorem 3.2, to prove the proposition it is sufficient to show that each entry of \bar{K} is invariant under the right action of \mathfrak{a}_+ . By formula (3.10) we obtain for each $m \in I$:

$$p_m^R \cdot \chi_n = c_n^{-1} \delta_{n,-m},$$

and hence

$$\begin{aligned} p_m^R \cdot \left(K \exp \left(- \sum_{n \in I} c_n p_n \chi_n(K) \right) \right) = \\ K p_m \exp \left(- \sum_{n \in I} c_n p_n \chi_n \right) + K \exp \left(- \sum_{n \in I} c_n p_n \chi_n \right) (-p_m) = 0. \end{aligned}$$

Therefore \bar{K} is right \mathfrak{a}_+ -invariant.

To prove the second statement, let a be an element of A_+ and let us show that $\overline{Ka} = \overline{K}$. We can write: $a = \exp\left(\sum_{i \in I} \alpha_i p_i\right)$. Then according to formulas (3.9) and (3.10), $\chi_n(Ka) = (\chi, Kap_{-n}a^{-1}K^{-1}) = (\chi, Kp_{-n}K^{-1}) + c_n^{-1}\alpha_n = \chi_n(K) + c_n^{-1}\alpha_n$. Therefore

$$\overline{Ka} = Ka \exp\left(-\sum_{n \in I} \alpha_n p_n - \sum_{n \in I} c_n p_n \chi_n(K)\right) = \overline{K}.$$

□

Consider now the matrix \overline{K} . According to Proposition 3.2, the entries of \overline{K} are Taylor series with coefficients in differential polynomials in u_i 's. In other words, $\overline{K} \in N_+[U]$. Hence we can apply to \overline{K} any derivation of $\mathbb{C}[u_i^{(n)}]$, in particular, $\partial_n = p_{-n}^R$.

Lemma 3.1. *In any finite-dimensional representation of N_+ , the matrix of \overline{K} satisfies:*

$$(3.12) \quad \overline{K}^{-1}(\partial_n + (\overline{K}p_{-n}\overline{K}^{-1})_-)\overline{K} = \partial_n + p_{-n} - \sum_{i \in I} c_i(p_{-n}^R \cdot \chi_i)p_i.$$

Proof. Using formula (1.3), we obtain:

$$\begin{aligned} \overline{K}^{-1}(\partial_n + (\overline{K}p_{-n}\overline{K}^{-1})_-)\overline{K} &= \partial_n + \overline{K}^{-1}(p_{-n}^R\overline{K}) + \overline{K}^{-1}(\overline{K}p_{-n}\overline{K}^{-1})_-\overline{K} \\ &= \partial_n + \overline{K}^{-1}(\overline{K}p_{-n}\overline{K}^{-1})_+\overline{K} - \sum_{i \in I} c_i(p_{-n}^R \cdot \chi_i)p_i + \overline{K}^{-1}(\overline{K}p_{-n}\overline{K}^{-1})_-\overline{K} \\ &= \partial_n + p_{-n} - \sum_{i \in I} c_i(p_{-n}^R \cdot \chi_i)p_i, \end{aligned}$$

which coincides with (3.12). □

Let now K be the point of N_+/A_+ assigned to the jet $(u_i^{(n)})$ by Theorem 1.1. Then \overline{K} is a well-defined element of N_+ corresponding to K under the map $N_+/A_+ \rightarrow N_+$ defined in Proposition 3.2. By construction, \overline{K} lies in the A_+ -coset K .

According to Lemma 2.1, $(\overline{K}p_{-1}\overline{K}^{-1})_- = p_{-1} + \mathbf{u}$. Letting $n = 1$ in formula (3.12), we obtain:

$$\overline{K}^{-1}(\partial_z + p_{-1} + \mathbf{u}(z))\overline{K} = \partial_z + p_{-1} - \sum_{i \in I} c_i(p_{-1}^R \cdot \chi_i)p_i.$$

This shows that \overline{K} gives a solution to equation (3.4), and hence lies in the A_+ -coset of the Drinfeld-Sokolov dressing operator M . Therefore the cosets of K and M coincide, and this completes our second proof of Theorem 3.2.

3.6. Baker-Akhiezer function. In this section we adopt the analytic point of view. Recall that in Lecture 1 we constructed a map, which assigns to every smooth function $\mathbf{u}(z) : \mathbb{R} \rightarrow \mathfrak{h}$, a smooth function $K(z) : \mathbb{R} \rightarrow N_+/A_+$, with the property that the action of n th mKdV flow on $\mathbf{u}(z)$ translates into the action of p_{-n}^R on N_+/A_+ . One can ask whether it is possible to lift this map to the one that assigns to \mathbf{u} a function $\mathbb{R} \rightarrow N_+$ with the same property. Our results from the previous section allow us to lift the function $K(z)$ to the function $\overline{K}(z) : \mathbb{R} \rightarrow N_+$, using the section $N_+/A_+ \rightarrow N_+$.

But one can check easily that the action of the mKdV flows does not correspond to the action of \mathfrak{a}_-^R on $\overline{K}(z)$. However, its modification

$$\tilde{K}(z) = \overline{K}(z) \exp \left(\sum_{n \in I} c_n p_n \chi_n(\overline{K}) \right)$$

does satisfy the desired property.

Unfortunately, $\chi_n \notin \mathbb{C}[u_i^{(n)}] \subset \mathbb{C}[N_+]$, so $\tilde{K}(z)$ can not be written in terms of differential polynomials in u_i (i.e, the jets of $\mathbf{u}(z)$), so $\tilde{K}(z)$ is “non-local”. Nevertheless, one can show that $H_n = p_{-1}^R \cdot \chi_n$ does belong to $\mathbb{C}[u_i^{(n)}]$ (in fact, this H_n can be taken as the density of the hamiltonian of the n th mKdV equation, see Lecture 5). Therefore we can write:

$$\tilde{K}(z) = \overline{K}(z) \exp \left(\sum_{n \in I} c_n p_n \int_{-\infty}^z H_n dz \right).$$

Now let $\mathbf{u}(\mathbf{t})$, where $\mathbf{t} = \{t_i\}_{i \in I}$ and t_i 's are the times of the mKdV hierarchy, be a solution of the mKdV hierarchy. The Baker-Akhiezer function $\Psi(\mathbf{t})$ associated to $\mathbf{u}(\mathbf{t})$, is by definition a solution of the system of equations

$$(3.13) \quad (\partial_n + (K p_{-n} K^{-1})_-) \Psi = 0, \quad \forall n \in I.$$

In particular, for $n = 1$ we have:

$$(3.14) \quad (\partial_z + p_{-1} + \mathbf{u}(z)) \Psi = 0.$$

From formula (3.12) we obtain the following explicit formula for the solution of the system (3.13) with the initial condition is $\Psi(\mathbf{0}) = \tilde{K}(\mathbf{0})$:

$$\Psi(\mathbf{t}) = \tilde{K}(\mathbf{t}) \exp \left(- \sum_{i \in I} p_{-i} t_i \right).$$

But by construction,

$$\tilde{K}(\mathbf{t}) = \left(\tilde{K}(\mathbf{0}) \Gamma(\mathbf{t}) \right)_+,$$

where

$$\Gamma(\mathbf{t}) = \exp \left(\sum_{i \in I} p_{-i} t_i \right)$$

and g_+ denotes the projection of $g \in B_- \cdot N_+ \subset G$ on N_+ (it is well-defined for almost all t_i 's). Hence we obtain:

$$\Psi(\mathbf{t}) = \left(\tilde{K}(\mathbf{0}) \Gamma(\mathbf{t}) \right)_+ \Gamma(\mathbf{t})^{-1}.$$

Similar formula for the Baker-Akhiezer function has been obtained by G. Segal and G. Wilson [28, 30]. Following the works of the Kyoto School [4], they showed that Baker-Akhiezer functions associated to solutions of the KdV equations naturally “live” in the Sato Grassmannian, and that the flows of the hierarchy become linear in these terms (see also [3]).

We have come to the same conclusion in a different way. We have identified the mKdV variables directly with coordinates on the big cell of $B_- \setminus G/A_+$, and constructed

a map $\mathbf{u}(z) \rightarrow K(z)$. It is then straightforward to check, as we did above, that the Baker-Akhiezer function is simply a lift of this map to $B_- \setminus G$. In the case of KdV hierarchy, we obtain a map to $\overline{G}[t^{-1}] \setminus G$ (see the next lecture), which for $\overline{G} = SL_n$ is the formal version of the Grassmannian that Segal-Wilson had considered.

LECTURE 4

In this lecture we define the generalized KdV hierarchy associated to an affine algebra \mathfrak{g} . We then show the equivalence between our definition and that of Drinfeld-Sokolov.

Throughout this lecture we restrict ourselves to non-twisted affine algebras.

4.1. The left action of \overline{N}_+ . Let $\overline{\mathfrak{n}}_+$ be the Lie subalgebra of \mathfrak{n}_+ generated by $e_i, i = 1, \dots, \ell$. Thus, $\overline{\mathfrak{n}}_+$ is the upper nilpotent subalgebra of the simple Lie algebra $\overline{\mathfrak{g}}$ – the “constant subalgebra” of $\overline{\mathfrak{g}}$, whose Dynkin diagram is obtained by removing the 0th nod of the Dynkin digram of $\overline{\mathfrak{g}}$. Let \overline{N}_+ be the corresponding subgroup of N_+ . The group \overline{N}_+ acts on the main homogeneous space N_+/A_+ from the left.

Lemma 4.1. *The action of \overline{N}_+ on N_+/A_+ is free.*

Proof. We need to show that for each $K \in N_+/A_+$, the map τ from $\overline{\mathfrak{n}}_+$ to the tangent space of N_+/A_+ at each point K is injective. But the tangent space at K is naturally isomorphic to $\widehat{\mathfrak{n}}_+/(K\mathfrak{a}K^{-1})$, and τ is the composition of the embedding $\overline{\mathfrak{n}}_+ \rightarrow \widehat{\mathfrak{n}}_+$ and the projection $\widehat{\mathfrak{n}}_+ \rightarrow \widehat{\mathfrak{n}}_+/(K\mathfrak{a}K^{-1})$. Thus, we need to show that $\overline{\mathfrak{n}}_+ \cap K\mathfrak{a}K^{-1} = 0$ for any $K \in N_+/A_+$. This is obvious because each element of $\overline{\mathfrak{n}}_+$ is a constant element of \mathfrak{g} , i.e. does not depend on t , while each element of $K\mathfrak{a}K^{-1}$ does have a t -dependence. \square

4.2. The KdV jet space. Now we show that $\mathbb{C}[\overline{N}_+ \setminus N_+/A_+]$ can be identified with a ring of differential polynomials.

We define the functions $v_i : N_+ \rightarrow \mathbb{C}, i = 1, \dots, \ell$, by the formula:

$$(4.1) \quad v_i(K) = (f_0, Kp_{-d_i}K^{-1}), \quad K \in N_+.$$

It is clear that these function are right A_+ -invariant. We also find:

$$(e_k^L \cdot v_i)(K) = (f_0, [e_k, Kp_{-d_i}K^{-1}]) = ([f_0, e_k], Kp_{-d_i}K^{-1}) = 0$$

for all $k = 1, \dots, \ell$. Therefore v_i is left $\overline{\mathfrak{n}}_+$ -invariant, and hence left \overline{N}_+ -invariant. Thus, each v_i gives rise to a regular function on $\overline{N}_+ \setminus N_+/A_+$. Let us compute the degree of v_i . We obtain:

$$\begin{aligned} (-\rho^\vee \cdot v_i)(K) &= -(f_0, [(K\rho^\vee K^{-1})_+, Kp_{-d_i}K^{-1}]) \\ &= -(f_0, [K\rho^\vee K^{-1}, Kp_{-d_i}K^{-1}]) + (f_0, [(K\rho^\vee K^{-1})_+, Kp_{-d_i}K^{-1}]) \\ &= -(f_0, K[\rho^\vee, p_{-1}]K^{-1}) + ([f_0, \rho^\vee], Kp_{-d_i}K^{-1}) \\ &= (d_i + 1)(f_0, Kp_{-d_i}K^{-1}) = (d_i + 1)v_i(K). \end{aligned}$$

Hence v_i is homogeneous of degree $d_i + 1$.

Now denote $v_i^{(n)} = (p_{-1}^R)^n \cdot v_i$.

Theorem 4.1.

$$\mathbb{C}[\overline{N}_+ \setminus N_+/A_+] \simeq \mathbb{C}[v_i^{(n)}]_{i=1, \dots, \ell; n \geq 0}.$$

Proof. We follow the proof of Theorem 1.1. First we prove the algebraic independence of the functions $v_i^{(n)}$. To do that, we need to establish the linear independence of the values of their differentials $dv_i^{(n)}|_{\bar{1}}$ at the double coset $\bar{1}$ of the identity element. Note that $dv_i^{(n)}|_{\bar{1}} = (\text{ad } p_{-1})^n \cdot dv_i|_{\bar{1}}$.

Using the invariant inner product on $\tilde{\mathfrak{g}}$, we identify the cotangent space to $\bar{1}$ with $\mathcal{N} = (\mathfrak{a}_+ \oplus \bar{\mathfrak{n}}_+)^{\perp} \cap \mathfrak{n}_-$. It is a graded subspace of \mathfrak{n}_- with respect to the principal gradation. Moreover, the action of $\text{ad } p_{-1}$ on \mathcal{N} is free. One can choose homogeneous elements $\beta_1, \dots, \beta_{\ell} \in \mathcal{N}$, such that $\{(\text{ad } p_{-1})^n \beta_i\}_{i=1, \dots, \ell; n \geq 0}$ is a basis in $(\mathfrak{a}_+ \oplus \bar{\mathfrak{n}}_+)^{\perp} \cap \mathfrak{n}_+^*$. It is also easy to see that $\deg \beta_i = -(d_i + 1)$. This follows from Proposition 1.2 and the fact [23] that the set of degrees of a elements of a homogeneous basis in $\bar{\mathfrak{n}}_-$ is $\cup_{i=1}^{\ell} \{1, \dots, d_i\}$.

We claim that $dv_i|_{\bar{1}}$ can be taken as the elements β_i . Let us compute $dv_i|_{\bar{1}}$. From formula (4.1) we find, in the same way as in the proof of Theorem 1.1:

$$(x, dv_i|_{\bar{1}}) = -(f_0, [x, p_{-d_i}]) = ([f_0, p_{-d_i}], x) \quad \forall x \in \mathfrak{n}_+.$$

Hence

$$(4.2) \quad dv_i|_{\bar{1}} = [f_0, p_{-d_i}]$$

as an element of $\mathcal{N} \subset \mathfrak{n}_-$. This formula shows that the degree of $dv_i|_{\bar{1}}$ equals $d_i + 1$ as we already know.

According to Proposition 1.2, $(\mathfrak{a}_+)^{\perp} \subset \mathfrak{n}_-$ has an ℓ -dimensional component $(\mathfrak{a}_+)^{\perp}_{-j}$ of each negative degree $-j$ (with respect to the principal gradation). Furthermore, since $\bar{\mathfrak{n}}_-$ has no homogeneous components of degrees less than or equal to $-h$, minus the Coxeter number, we obtain that $(\mathfrak{a}_+)^{\perp}_{-j} = \mathcal{N}_{-j}$ for all $j \geq h$. Now let

$$(4.3) \quad \gamma_i = (\text{ad } p_{-1})^{h-d_i-1} \cdot dv_i|_{\bar{1}} = (\text{ad } p_{-1})^{h-d_i-1} \cdot [f_0, p_{-d_i}] \in (\mathfrak{a}_+)^{\perp}_{-h} = \mathcal{N}_{-h}$$

for all $i = 1 \dots, \ell$. Recall that the operator $\text{ad } p_{-1} : (\mathfrak{a}_+)^{\perp}_{-j} \rightarrow (\mathfrak{a}_+)^{\perp}_{-j-1}$ is an isomorphism. If the vectors γ_i are linearly independent, then the covectors

$$dv_i^{(j-d_i-1)}|_{\bar{1}} = (\text{ad } p_{-1})^{j-d_i-1} \cdot [f_0, p_{-d_i}] \in \mathcal{N}_{-j},$$

are linearly independent for each $j > 1$ (here we ignore $dv_i^{(n)}|_{\bar{1}}$, if $n < 0$). We see that the linear independence of the covectors $\{dv_i^{(n)}|_{\bar{1}}\}_{i=1, \dots, \ell; n \geq 0}$ is equivalent to the following

Proposition 4.1. *The vectors $\{\gamma_i\}_{i=1, \dots, \ell}$ are linearly independent.*

The proof is given in the Appendix. This completes the proof of the algebraic independence of the functions $v_i^{(n)}$. Hence we obtain an injective homomorphism $\mathbb{C}[v_i^{(n)}] \rightarrow \mathbb{C}[\bar{\mathcal{N}}_+ \setminus \mathcal{N}_+ / A_+]$. The fact that it is an isomorphism follows in the same way as in the proof of Theorem 1.1, from the computation of characters. Clearly,

$$(4.4) \quad \text{ch } \mathbb{C}[v_i^{(n)}] = \prod_{i=1}^{\ell} \prod_{n_i > d_i} (1 - q^{n_i})^{-1}.$$

On the other hand, since \overline{N}_+ acts freely on N_+/A_+ ,

$$\text{ch } \mathbb{C}[\overline{N}_+ \setminus N_+/A_+] = \text{ch } \mathbb{C}[N_+/A_+] (\text{ch } \mathbb{C}[\overline{N}_+])^{-1},$$

and

$$\text{ch } \mathbb{C}[\overline{N}_+] = \prod_{i=1}^{\ell} \prod_{n_i=1}^{d_i} (1 - q^{n_i})^{-1},$$

by [23] (here we use $-\rho^\vee$ as the gradation operator). Hence $\text{ch } \mathbb{C}[\overline{N}_+ \setminus N_+/A_+]$ is also given by formula (4.4), and the theorem is proved. \square

Remark 4.1. One can prove that $\mathbb{C}[\overline{N}_+ \setminus N_+/A_+]$ is isomorphic to a ring of differential polynomials without using formula (4.1) for the functions v_i . However, the proof given above is shorter and more explicit. \square

4.3. KdV hierarchy. Theorem 4.1 means that just like N_+/A_+ , the double quotient $\overline{N}_+ \setminus N_+/A_+$ can be identified with the space of ∞ -jets of an ℓ -tuple of functions. We call this space the KdV jet space and denote it by \mathbf{V} . The vector fields p_{-n}^R still act on $\overline{N}_+ \setminus N_+/A_+$ and hence give us an infinite set of commuting evolutionary derivations on $\mathbb{C}[\mathbf{V}]$. This is, by definition, the *KdV hierarchy* associated to $\overline{\mathfrak{g}}$.

The natural projection $N_+/A_+ \rightarrow \overline{N}_+ \setminus N_+/A_+$ gives us a map $\mathbf{U} \rightarrow \mathbf{V}$, which amounts to expressing each v_i as a differential polynomial in u_j 's. This map is called the *generalized Miura transformation*. It can be thought of as a change of variables transforming the mKdV hierarchy into the KdV hierarchy.

4.4. Drinfeld-Sokolov reduction. In this section we give another definition of the generalized KdV hierarchy following Drinfeld and Sokolov. We will show in the next section that the two definitions are equivalent.

Denote by \mathbf{Q} the space of ∞ -jets of functions $\mathbf{q} : \mathcal{D} \rightarrow \overline{\mathfrak{b}}_+$, where $\overline{\mathfrak{b}}_+$ is the finite-dimensional Borel subalgebra $\mathfrak{h} \oplus \overline{\mathfrak{n}}_+$ (recall that $\overline{\mathfrak{n}}_+$ is generated by $e_i, i = 1, \dots, \ell$). If we choose a basis $\{\omega_i^\vee, i = 1, \dots, \ell\} \cup \{e_\alpha, \alpha \in \Delta_+\}$ of $\overline{\mathfrak{b}}_+$, then we obtain the corresponding coordinates $q_\alpha^{(n)}, q_i^{(n)}, n \geq 0$ on \mathbf{Q} . Thus, $\mathbb{C}[\mathbf{Q}]$ is a ring of differential polynomials equipped with an action of ∂_z .

Let $C(\overline{\mathfrak{g}})$ be vector the space of operators of the form

$$(4.5) \quad \partial_z + p_{-1} + \mathbf{q},$$

where

$$\mathbf{q} = \sum_{i=1}^{\ell} \omega_i^\vee \otimes q_i + \sum_{\alpha \in \Delta_+} e_\alpha \otimes q_\alpha \in \overline{\mathfrak{b}}_+ \otimes \mathbb{C}[\mathbf{Q}].$$

The group $\overline{N}_+[\mathbf{Q}]$ acts naturally on $C(\overline{\mathfrak{g}})$:

$$(4.6) \quad x \cdot (\partial_z + p_{-1} + \mathbf{q}) = x(\partial_z + p_{-1} + \mathbf{q})x^{-1} = \partial_z + [x, p_{-1} + \mathbf{q}] - \partial_z x.$$

Note that

$$p_{-1} = \overline{p}_{-1} + e_{\max} \otimes t^{-1},$$

where

$$(4.7) \quad \bar{p}_{-1} = \sum_{j=1}^{\ell} \frac{(\alpha_j, \alpha_j)}{2} f_j \in \bar{\mathfrak{n}}_-$$

and e_{\max} is a generator of the one-dimensional subspace of $\bar{\mathfrak{g}}$ corresponding to the maximal root.

We have a direct sum decomposition $\bar{\mathfrak{b}}_+ = \bigoplus_{i \geq 0} \bar{\mathfrak{b}}_{+,i}$ with respect to the principal gradation. The operator $\text{ad } \bar{p}_{-1}$ acts from $\bar{\mathfrak{b}}_{+,i+1}$ to $\bar{\mathfrak{b}}_{+,i}$ injectively for all $i > 1$. Hence we can choose for each $j > 0$ a vector subspace $S_j \subset \bar{\mathfrak{b}}_{+,j}$, such that $\bar{\mathfrak{b}}_{+,j} = [\bar{p}_{-1}, \bar{\mathfrak{b}}_{+,j+1}] \oplus S_j$. Note that $S_j \neq 0$ if and only if j is an exponent of $\bar{\mathfrak{g}}$, and in that case $\dim S_j$ is the multiplicity of the exponent j . In particular, $S_0 = 0$. Let $S = \bigoplus_{j \in E} S_j \subset \bar{\mathfrak{n}}_+$, where E is the set of exponents of $\bar{\mathfrak{g}}$. Then, by construction, S is transversal to the image of the operator $\text{ad } \bar{p}_{-1}$ in $\bar{\mathfrak{n}}_+$.

Proposition 4.2 ([7]). *The action of $\bar{N}_+[\mathbf{Q}]$ on $C(\bar{\mathfrak{g}})$ is free. Furthermore, each $\bar{N}_+[\mathbf{Q}]$ -orbit contains a unique operator (4.5) satisfying the condition that $\mathbf{q} \in S[\mathbf{Q}]$.*

Proof. Denote by $C'(\bar{\mathfrak{g}})$ the space of operators of the form

$$\partial_z + \bar{p}_{-1} + \mathbf{q}, \quad \mathbf{q} \in \bar{\mathfrak{b}}_+ \otimes \mathbb{C}[\mathbf{Q}].$$

The group $\bar{N}_+[\mathbf{Q}]$ acts on $C'(\bar{\mathfrak{g}})$ by the formula analogous to (4.6). We have an isomorphism $C(\bar{\mathfrak{g}}) \rightarrow C'(\bar{\mathfrak{g}})$, which sends $\partial_z + p_{-1} + \mathbf{q}$ to $\partial_z + \bar{p}_{-1} + \mathbf{q}$. Since $[x, e_{\max}] = 0, \forall x \in \bar{\mathfrak{n}}_+$, this isomorphism commutes with the action of $\bar{N}_+[\mathbf{Q}]$. Thus, we can study the action of $\bar{N}_+[\mathbf{Q}]$ on $C'(\bar{\mathfrak{g}})$ instead of $C(\bar{\mathfrak{g}})$.

We claim that each element of $C(\bar{\mathfrak{g}})$ can be uniquely represented in the form

$$(4.8) \quad \partial_z + \bar{p}_{-1} + \mathbf{q} = \exp(\text{ad } U) \cdot (\partial_z + \bar{p}_{-1} + \mathbf{q}^0),$$

where $U \in \bar{\mathfrak{n}}_+[\mathbf{Q}]$ and $\mathbf{q}^0 \in S[\mathbf{Q}]$. Decompose with respect to the principal gradation: $U = \sum_{j \geq 0} U_j$, $\mathbf{q} = \sum_{j \geq 0} \mathbf{q}_j$, $\mathbf{q}^0 = \sum_{j \geq 0} \mathbf{q}_j^0$. Equating the homogeneous components of degree j in both sides of (4.8), we obtain that $\mathbf{q}_i^0 + [U_{i+1}, \bar{p}_{-1}]$ is expressed in terms of $\mathbf{q}_i, \mathbf{q}_1^0, \dots, \mathbf{q}_{i-1}^0, U_1, \dots, U_i$. The direct sum decomposition $\bar{\mathfrak{b}}_{+,i} = [\bar{p}_{-1}, \bar{\mathfrak{b}}_{+,i+1}] \oplus S_i$ then allows us to determine uniquely \mathbf{q}_i^0 and U_{i+1} . Hence U and \mathbf{q}^0 satisfying equation (4.8) can be found uniquely by induction, and lemma follows. \square

From the analytic point of view, Proposition 4.2 means that every first order differential operator (4.5), where $\mathbf{q} : \mathbb{R} \rightarrow \bar{\mathfrak{b}}_+$ is a smooth function, can be brought to the form (4.5), where \mathbf{q} takes values in $S \subset \bar{\mathfrak{n}}_+$, by gauge transformation with a function $x(z) : \mathbb{R} \rightarrow \bar{N}_+$. Moreover, $x(z)$ depends on the entries of \mathbf{q} only through their derivatives at z (i.e., it is local).

Remark 4.2. The statement of the lemma remains true if we replace the ring of differential polynomials $\mathbb{C}[\mathbf{Q}]$ by any differential ring R . For example, we can take $R = \mathbb{C}[[z]]$. Then the quotient of $C(\bar{\mathfrak{g}})$ by the action of $\bar{N}_+((z))$ is what Beilinson and Drinfeld call the space of *opers* on the formal disc. This space can be defined intrinsically without choosing a particular uniformizing parameter z (using the notion of connection on the formal disc). In this form it has been generalized by Beilinson and Drinfeld to the

situation where the formal disc is replaced by any algebraic curve. For instance, in the case of $\bar{\mathfrak{g}} = \mathfrak{sl}_2$, the notion of oper coincides with that of projective connection. \square

It is easy to see from the proof of Proposition 3.1 that its statement remains true if we replace the operator $\partial_z + p_{-1} + \mathbf{u}(z)$ by operator (4.5). Using this fact, Drinfeld and Sokolov construct in [7] the zero-curvature equations for the operator (4.5) in the same way as for the mKdV hierarchy using formulas (3.4). Drinfeld and Sokolov show that these equations preserve the corresponding $\overline{N}_+[\mathbf{Q}]$ -orbits (see [7], Sect. 6.2). Thus, they obtain a system of compatible evolutionary equations on $\overline{N}_+[\mathbf{Q}]$ -orbits in $C(\bar{\mathfrak{g}})$, which they call the generalized KdV hierarchy corresponding to $\bar{\mathfrak{g}}$.

These equations give rise to evolutionary derivations acting on the ring of differential polynomials $\mathbb{C}[s_i^{(n)}]_{i=1,\dots,\ell;n \geq 0}$. Indeed, the space S is ℓ -dimensional. Let us choose homogeneous coordinates s_1, \dots, s_ℓ of S . Then according to Proposition 4.2, the KdV equations can be written as partial differential equations on s_i 's. It is easy to see that the first of them is just ∂_z itself. Hence others give rise to evolutionary derivations of $\mathbb{C}[s_i^{(n)}]_{i=1,\dots,\ell;n \geq 0}$.

4.5. Equivalence of two definitions. Recall that \mathbf{Q} is the space of ∞ -jets of \mathbf{q} with coordinates $q_\alpha^{(n)}, q_i^{(n)}, n \geq 0$. Denote by \mathbf{S} the space of ∞ -jets of \mathbf{q}^0 with coordinates $s_i^{(n)}, i = 1, \dots, \ell, n \geq 0$.

Note that we have a natural embedding $\iota : \mathbf{U} \rightarrow \mathbf{Q}$, which sends $u_i^{(n)}$ to $q_i^{(n)}$. Let $\mu : \mathbf{U} \rightarrow \mathbf{S}$ be the composition of the embedding ι and the projection $\mathbf{Q} \rightarrow \mathbf{S}$.

Proposition 3.1, suitably modified for operators of the form (4.5), gives us a map $\nu : \mathbf{Q} \rightarrow N_+/A_+$ and hence a map $\tilde{\nu} : \mathbf{S} \rightarrow \overline{N}_+ \setminus N_+/A_+$. According to Theorem 3.1, the composition of the embedding ι with the map ν coincides with the isomorphism $\mathbf{U} \simeq N_+/A_+$ constructed in Theorem 1.1. Hence the maps ν and $\tilde{\nu}$ are surjective.

Proposition 4.3. *The map $\tilde{\nu} : \mathbf{S} \rightarrow \overline{N}_+ \setminus N_+/A_+$ is an isomorphism.*

Proof. By construction, the homomorphism $\tilde{\nu}$ is surjective and homogeneous with respect to the natural \mathbb{Z} -gradations. Hence, it suffices to show that the characters of the spaces $\mathbb{C}[\mathbf{S}]$ and $\mathbb{C}[\overline{N}_+ \setminus N_+/A_+]$ coincide. Since, by construction, $\deg s_i = d_i + 1$,

$$\text{ch } \mathbb{C}[\mathbf{S}] = \prod_{i=1}^{\ell} \prod_{n_i \geq d_i+1} (1 - q^{n_i})^{-1}.$$

But in the proof of Theorem 4.1 we showed that $\text{ch } \mathbb{C}[\overline{N}_+ \setminus N_+/A_+]$ is given by the same formula. \square

Thus, we have shown that there is an isomorphism of rings

$$\mathbb{C}[v_i^{(n)}]_{i=1,\dots,\ell;n \geq 0} \rightarrow \mathbb{C}[s_i^{(n)}]_{i=1,\dots,\ell;n \geq 0},$$

which preserves the \mathbb{Z} -gradation and the action of ∂_z , and such that it sends the derivations of generalized KdV hierarchy defined in Sect. 4.3 to the derivations defined in Sect. 4.4 following Drinfeld-Sokolov [7]. It also follows that the map μ defined above as the composition of the embedding $\mathbf{U} \rightarrow \mathbf{Q}$ and the projection $\mathbf{Q} \rightarrow \overline{N}_+ \setminus \mathbf{Q} \simeq \mathbf{S}$

coincides with the projection $\mathbf{U} \simeq N_+/A_+ \rightarrow \overline{N}_+ \backslash N_+/A_+ \simeq \mathbf{S}$. This is the generalized Miura transformation.

4.6. **Example of \mathfrak{sl}_2 .** The space $C'(\mathfrak{sl}_2)$ consists of operators of the form

$$\partial_z + \begin{pmatrix} a & b \\ 1 & -a \end{pmatrix}.$$

The group

$$\overline{N}_+ = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

acts on $C'(\mathfrak{sl}_2)$ by gauge transformations (4.6). We write canonical representatives of \overline{N}_+ -orbits in the form

$$\begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix}.$$

We have:

$$\begin{pmatrix} 1 & -\frac{u}{2} \\ 0 & 1 \end{pmatrix} \left(\partial_z + \begin{pmatrix} \frac{u}{2} & 0 \\ 1 & -\frac{u}{2} \end{pmatrix} \right) \begin{pmatrix} 1 & -\frac{u}{2} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix},$$

where

$$(4.9) \quad s = \frac{1}{4}u^2 + \frac{1}{2}\partial_z u.$$

This formula defines the Miura transformation $\mathbb{C}[s^{(n)}] \rightarrow \mathbb{C}[u_i^{(n)}]$. If we apply this change of variables (4.9) to the mKdV equation (2.16) we obtain the equation

$$(4.10) \quad \partial_3 s = \frac{3}{2}s\partial_z s - \frac{1}{4}\partial_z^3 s,$$

which, as expected, closes on s and its derivatives. This equation becomes the KdV equation (0.1) after a slight redefinition of variables: $s \rightarrow -v$, $\tau_3 \rightarrow -4\tau$. Thus, the results of this lecture prove the existence of infinitely many higher KdV flows, i.e., evolutionary derivations on $\mathbb{C}[v^{(n)}]$, which commute with the KdV derivation defined by equation (0.1).

The fact that the KdV and mKdV equations are connected by a change of variables (4.10) was discovered by R. Miura, who also realized that it can be rewritten as

$$\partial_z^2 - s = \left(\partial_z - \frac{u}{2} \right) \left(\partial_z + \frac{u}{2} \right).$$

It appears that it was this observation that triggered the fascinating idea that the KdV equation should be considered as a flow on the space of Sturm-Liouville operators $\partial_z^2 - s(z)$ [15], which led to the concept of Inverse Scattering Method and the modern view of the theory of solitons (see [26]).

4.7. Explicit formulas for the action of \mathfrak{n}_+ . In this section we obtain explicit formulas for the action of the generators $e_i, i = 0, \dots, \ell$, of \mathfrak{n}_+ on $\mathbb{C}[\mathbf{U}]$. This formulas can be used to find the KdV variables $v_i \in \mathbb{C}[\mathbf{U}]$, and we will also need them in the next lecture when we study Toda field theories.

In order to find these formulas, we need a geometric construction of modules contragradient to the Verma modules over \mathfrak{g} and homomorphisms between them. This construction is an affine analogue of Kostant's construction [24] in the case of simple Lie algebras.

For $\lambda \in \mathfrak{h}^*$, denote by \mathbb{C}_λ the one-dimensional representation of \mathfrak{b}_+ , on which $\mathfrak{h} \subset \mathfrak{b}_+$ acts according to its character λ , and $\mathfrak{n}_+ \subset \mathfrak{b}_+$ acts trivially. Let M_λ be the Verma module over \mathfrak{g} of highest weight λ :

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda.$$

we denote by v_λ the highest weight vector of $M_\lambda^*, 1 \otimes 1$.

Denote by $\langle \cdot, \cdot \rangle$ the pairing $M_\lambda^* \times M_\lambda \rightarrow \mathbb{C}$. Let ω be the Cartan anti-involution on \mathfrak{g} , which maps generators e_0, \dots, e_ℓ to f_0, \dots, f_ℓ and vice versa and preserves \mathfrak{h} [22]. It extends to an anti-involution of $U(\mathfrak{g})$. Let M_λ^* be the module contragradient to M_λ . As a linear space, M_λ^* is the restricted dual of M_λ . The action of $x \in \mathfrak{g}$ on $y \in M_\lambda^*$ is defined as follows:

$$\langle x \cdot y, z \rangle = \langle y, \omega(x) \cdot z \rangle, \quad z \in M_\lambda.$$

The module M_λ^* can be realized in the space of regular functions on N_+ (see [12], Sect. 4). Indeed, the \mathfrak{n}_+ -module $\mathbb{C}[N_+]$ (with respect to the right action) is dual to a free module with one generator, and so is each M_λ^* . Hence we can identify M_λ^* and $\mathbb{C}[N_+]$ as \mathfrak{n}_+ -modules for any λ . It is easy to see that for $\lambda = 0$, M_0^* is isomorphic to $\mathbb{C}[N_+]$ on which the action of \mathfrak{g} is defined via the left infinitesimal action of \mathfrak{g} on N_+ by vector fields, described in Lecture 1. For general λ , the action of \mathfrak{g} is given by first order differential operators: for $a \in \mathfrak{g}$ this differential operator is equal to $a^R + f_\lambda(a)$, where $f_\lambda(a) \in \mathbb{C}[N_+]$. The function f_λ is the image of $a \cdot v_\lambda$ under the isomorphism $M_\lambda^* \simeq \mathbb{C}[N_+]$. Hence if a is homogeneous, then $f_\lambda(a)$ is also homogeneous of the same weight.

Here is another, homological, point of view on $f_\lambda(a)$. As a \mathfrak{g} -module, $\mathbb{C}[N_+] = M_0^*$ is coinduced from the trivial representation of \mathfrak{b}_- . By Shapiro's lemma (cf., e.g., [14], Sect. 1.5.4), $H^1(\mathfrak{g}, \mathbb{C}[N_+]) \simeq H^1(\mathfrak{b}_-, \mathbb{C}) \simeq (\mathfrak{b}_- / [\mathfrak{b}_-, \mathfrak{b}_-])^* = \mathfrak{h}^*$. We see that all elements of $H^1(\mathfrak{g}, \mathbb{C}[N_+])$ have weight 0. On the other hand, functions on N_+ can only have negative or 0 weights and the only functions, which have weight 0 are constants, which are invariant with respect to the action of \mathfrak{g} . Therefore the coboundary of any element of the 0th group of the complex, $\mathbb{C}[N_+]$, has a non-zero weight. Hence any cohomology class from $H^1(\mathfrak{g}, \mathbb{C}[N_+])$ canonically defines a one-cocycle f , i.e., a map $\mathfrak{g} \rightarrow \mathbb{C}[N_+]$. Thus, having identified $H^1(\mathfrak{g}, \mathbb{C}[N_+])$ with \mathfrak{h}^* , we can assign to each $\lambda \in \mathfrak{h}^*$ and each $a \in \mathfrak{g}$, a function on N_+ – this is our $f_\lambda(a)$.

The following two results will enable us to compute the action of e_i^L .

Proposition 4.4 ([24], Theorem 2.2). *Consider $\lambda \in \mathfrak{h}^*$ as an element of \mathfrak{h} , using the invariant inner product. Then*

$$(4.11) \quad f_\lambda(a)(x) = (\lambda, xax^{-1}).$$

Proposition 4.5 ([12], Prop. 3). *For any $a \in \mathfrak{g}$ we have:*

$$(4.12) \quad [e_i^L, a^R] = -f_{\alpha_i}(a)e_i^L, \quad i = 0, \dots, \ell.$$

Substituting $\lambda = \alpha_i$ and $a = p_{-1}$ in formula (4.11), we obtain:

$$f_{\alpha_i}(p_{-1})(x) = (\alpha_i, xp_{-1}x^{-1}) = u_i.$$

Hence formula (4.12) gives:

$$(4.13) \quad [e_i^L, p_{-1}^R] = -u_i e_i^L.$$

Let us write

$$e_i^L = \sum_{1 \leq j \leq \ell, n \geq 0} C_{i,j}^{(m)} \frac{\partial}{\partial u_j^{(m)}},$$

where $C_{i,j}^{(m)} \in \mathbb{C}[u_i^{(n)}]_{1 \leq i \leq \ell, n \geq 0}$. Formula (4.13) now gives us recurrence relations for the coefficients of $\partial/\partial u_j^{(m-1)}$ in the vector field e_i :

$$C_{i,j}^{(m)} = -u_i C_{i,j}^{(m-1)} + \partial_z C_{i,j}^{(m-1)},$$

where we have identified p_{-1}^R with ∂_z and used the formula

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial_z \right] = \frac{\partial}{\partial u_i^{(n-1)}},$$

if $n > 0$. We also have, according to formula (1.7),

$$e_i \cdot u_j = -(\alpha_i, \alpha_j).$$

This gives the initial condition for our recurrence relation. Combining them, we obtain the following formula:

$$(4.14) \quad e_i^L = - \sum_{n \geq 0} B_i^{(n)} \partial_i^{(n)},$$

where

$$(4.15) \quad \partial_i^{(n)} = \sum_{1 \leq j \leq \ell} (\alpha_i, \alpha_j) \frac{\partial}{\partial u_j^{(n)}},$$

and $B_i^{(n)}$'s are polynomials in $u_i^{(m)}$'s, which satisfy the recurrence relation:

$$(4.16) \quad B_i^{(n)} = -u_i B_i^{(n-1)} + \partial_z B_i^{(n-1)},$$

with the initial condition $B_i^{(0)} = 1$.

Formula (4.14) is valid for all $i = 0, \dots, \ell$. Using this formula, we can find explicitly the KdV variables $v_i, i = 1, \dots, \ell$, following the proof of Theorem 4.1. Consider, for example, the case $\bar{\mathfrak{g}} = \mathfrak{sl}_2$. In this case $\mathbb{C}[\mathbf{U}] = \mathbb{C}[u^{(n)}]$, and $\mathbb{C}[\mathbf{V}] = \mathbb{C}[v^{(n)}]$ is the subspace of $\mathbb{C}[\mathbf{U}]$, which consists of differential polynomials annihilated by

$$e^L = - \sum_{n \geq 0} B^{(n)} \frac{\partial}{\partial u^{(n)}}.$$

We find from formula (4.16): $B^{(0)} = 1, B^{(1)} = -u$, etc.

The KdV variable v lies in the degree 2 component of $\mathbb{C}[\mathbf{U}]$, i.e., $\text{span}\{u^2, \partial_z u\}$, and we have: $e^L \cdot u^2 = -2u$, $e^L \cdot \partial_z u = u$. Hence

$$(4.17) \quad v = \frac{1}{4}u^2 + \frac{1}{2}\partial_z u$$

is annihilated by e^L and can be taken as a KdV variable. As expected, this agrees with formula (4.9) obtained by means of the Drinfeld-Sokolov reduction.

LECTURE 5

In this lecture we consider the Toda equations and their local integrals of motion. The Toda equation associated to the simple Lie algebra $\bar{\mathfrak{g}}$ (respectively, affine algebra) reads

$$(5.1) \quad \partial_\tau \partial_z \phi_i(z, t) = \sum_{j \in J} (\alpha_i, \alpha_j) e^{-\phi_j(z, t)}, \quad i = 1, \dots, \ell,$$

where J is the set of vertices of the Dynkin diagram of $\bar{\mathfrak{g}}$ (respectively, \mathfrak{g}). Each $\phi_i(z, t), i = 1, \dots, \ell$, is a family of functions in z , depending on the time variable τ , and

$$\phi_0(z, t) = -\frac{1}{a_0} \sum_{i=1}^{\ell} a_i \phi_i(t, \tau).$$

If we set formally $u_i(t) = \partial_z \phi_i(t), i = 1, \dots, \ell$, then equations (5.1) can be rewritten as an equation on $\mathbf{u}(z)$:

$$(5.2) \quad \partial_\tau u_i = \sum_{j \in J} (\alpha_i, \alpha_j) e^{-\phi_j}, \quad i = 1, \dots, \ell,$$

where ϕ_j is now understood as the anti-derivative of u_j .

Certainly, equation (5.2) is not in the form (0.2), since the terms $e^{-\phi_j}$ on the right hand side are not differential polynomials in u_i 's, so our differential algebra formalism can not be applied. However, we can modify our formalism to suit the Toda equations. Moreover, we will see that these equations are closely connected to the derivations e_i^L studied in the previous lecture. After that, we will introduce the notion of local integral of motion of the Toda equation. We will describe these local integrals of motion, and will show that the corresponding hamiltonian flows are the mKdV flows.

5.1. The differential algebra formalism of Toda equation. Let Λ be the $(\ell + 1)$ dimensional lattice spanned by $\alpha_0, \dots, \alpha_\ell$. For any element $\lambda = \sum_{0 \leq i \leq \ell} \lambda_i \alpha_i$ of Λ , define the linear space $\pi_\lambda = \mathbb{C}[u_i^{(n)}] \otimes e^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{0 \leq i \leq \ell} \lambda_i \phi_i$, equipped with an action of ∂_z by the formula

$$(5.3) \quad \partial_z \cdot (P \otimes e^{\bar{\lambda}}) = (\partial_z P) \otimes e^{\bar{\lambda}} + \left(\sum_{0 \leq i \leq \ell} \lambda_i u_i^{(0)} P \right) \otimes e^{\bar{\lambda}},$$

where we put

$$u_0^{(n)} = -\frac{1}{a_0} \sum_{1 \leq i \leq \ell} a_i u_i^{(n)}.$$

This formula means that we set $\partial_z \phi_i = u_i$. In particular, $\pi_0 = \mathbb{C}[u_i^{(n)}]$.

We introduce a \mathbb{Z} -gradation on π_λ by setting $\deg P \otimes e^\lambda = (\rho^\vee, \lambda) + \deg P$, where the gradation on $\mathbb{C}[u_i^{(n)}]$ is the one previously defined (recall that ρ^\vee satisfies $(\rho^\vee, \alpha_i) = 1, i = 0, \dots, \ell$).

An evolutionary operator $X : \pi_\lambda \rightarrow \pi_\mu$ is, by definition, a linear operator $\pi_\lambda \rightarrow \pi_\mu$, which commutes with the action of ∂_z . Defining an evolutionary operator X acting from π_0 to $\bigoplus_{j \in S} \pi_{\lambda_j}$ is the same as defining $X \cdot u_i$ as an element of $\bigoplus_{j \in S} \pi_{\lambda_j}$ for all $j \in S$. Hence given an equation of the form

$$\partial_\tau u_i = \sum_{j \in S} P_{ij},$$

where $P_j \in \pi_{\lambda_j}$, we obtain an evolutionary operator

$$X\{P_{ij}\} = \sum_{j \in S} X_j,$$

where

$$X_j = \sum_{n \geq 0} \sum_{i=1}^{\ell} (\partial_z^n \cdot P_{ij}) \frac{\partial}{\partial u_i^{(n)}}.$$

Toda equation (5.2) gives rise to the evolutionary operator

$$(5.4) \quad \mathcal{H} : \pi_0 \rightarrow \bigoplus_{j \in J} \pi_{-\alpha_j},$$

which is the sum of the terms

$$(5.5) \quad \tilde{Q}_i = \sum_{n \geq 0} (\partial_z^n e^{-\phi_i}) \partial_i^{(n)},$$

where $\partial_i^{(n)}$ is given by formula (4.15). Here \tilde{Q}_i is an evolutionary operator acting from π_0 to $\pi_{-\alpha_i}$. Clearly,

$$\partial_z^n e^{-\phi_i} = b_i^{(n)} e^{-\phi_i},$$

where $b_i^{(n)}$'s are differential polynomials in $u_j^{(n)}$'s, satisfying the recurrence relation

$$b_i^{(n)} = -u_i b_i^{(n-1)} + \partial_z b_i^{(n-1)}.$$

This recurrence relation coincides with the recurrence relation (4.16) for the differential polynomials $B_i^{(n)}$ appearing in formula (4.14) for e_i^L . Hence we obtain the following result.

Proposition 5.1. *Let T_j be the operator of multiplication by $e^{-\phi_j}$ acting from π_0 to $\pi_{-\alpha_j}$. The evolutionary operator \mathcal{H} associated to the Toda equation equals*

$$\mathcal{H} = - \sum_{j \in J} T_j e_j^L.$$

Thus, while the mKdV flows come from the right action of \mathfrak{a}_+ on N_+/A_+ , the Toda flow comes from the left action of the generators of \mathfrak{n}_+ (or $\bar{\mathfrak{n}}_+$) on N_+/A_+ .

Now we want to introduce the notion of local integral of motion of the Toda equation. In order to do that, we first review the general hamiltonian formalism of soliton equations.

5.2. Hamiltonian formalism. In this section we briefly discuss the hamiltonian structure on the space of local functionals. This is a special case of the formalism of generalized hamiltonian structures and of generalized hamiltonian operators developed by Gelfand, Dickey and Dorfman [16, 17, 18] (see [11], Sect. 2; [12], Sect. 2 for review).

First we introduce the concept of *local functional*. Consider the space $F(\mathfrak{h})$ of functions on the circle with values in the Cartan subalgebra \mathfrak{h} of \mathfrak{g} , $\mathbf{u}(z) = (u_1(z), \dots, u_\ell(z))$. Then each differential polynomial $P \in \mathbb{C}[u_i^{(n)}]$ gives rise to a functional on $F(\mathfrak{h})$ that sends $\mathbf{u}(z)$ to

$$\int P(u_i^{(n)}) dz.$$

Such functionals are called local functionals. It is easy to see that this functional is 0 if and only if P is a sum of a total derivative (i.e., $\partial_z Q$ for some $Q \in \mathbb{C}[u_i^{(n)}]$) and a constant. Hence we formally define the space of local functionals \mathcal{F}_0 as the quotient of $\pi_0 = \mathbb{C}[u_i^{(n)}]$ by the subspace $\text{Im } \partial_z \oplus \mathbb{C}$ spanned by total derivatives and constants. We introduce a \mathbb{Z} -gradation on \mathcal{F}_0 by subtracting 1 from the gradation induced from π_0 , and denote by \int the projection $\pi_0 \rightarrow \mathcal{F}_0$.

From now on, to simplify notation, we shall write ∂ for ∂_z .

Introduce variational derivatives:

$$\delta_i P = \sum_{n \geq 0} \sum_{1 \leq j \leq l} (\alpha_i, \alpha_j) (-\partial)^n \frac{\partial P}{\partial u_j^{(n)}}, \quad i = 1, \dots, \ell.$$

Now for each $P \in \pi_0$, define an evolutionary derivation ξ_P of π_0 by the formula.

$$(5.6) \quad \xi_P = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+1} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n)}}$$

Since $\delta_i P = 0, i = 1, \dots, \ell, \forall P \in \text{Im } \partial_z \oplus \mathbb{C}$, we see that ξ_P depends only on the image of P in $\mathcal{F}_0, \int P dz$. So sometimes we shall write $\xi_{\int P dz}$ instead of ξ_P . We can view $\xi : P \rightarrow \xi_P$ as a linear map $\mathcal{F}_0 \rightarrow \text{Der}^{\partial_z}$, where Der^{∂_z} is the space of evolutionary derivations on π_0 (note that it is a Lie algebra).

For instance, we find:

$$(5.7) \quad \partial = \sum_{1 \leq i \leq l, n \geq 0} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}} = \xi_P, \quad P = \frac{1}{2} \sum_{1 \leq i \leq l} u_i u^i,$$

where $u^i, i = 1 \dots, \ell$, are dual to $u_i, i = 1 \dots, \ell$, with respect to the inner product defined by (\cdot, \cdot) .

Now we define a Poisson bracket on \mathcal{F}_0 by the formula

$$(5.8) \quad \left\{ \int P dz, \int R dz \right\} = \int (\xi_P \cdot R) dz.$$

Proposition 5.2 ([16, 17, 18]). *Formula (5.8) defines a Lie algebra structure on \mathcal{F}_0 . Furthermore, the map $\xi : \mathcal{F}_0 \rightarrow \text{Der}^{\partial}$ is a homomorphism of Lie algebras, so that*

$$(5.9) \quad \xi_{\{P,R\}} = [\xi_P, \xi_R], \quad \forall P, R \in \pi_0.$$

Remark 5.1. Using the isomorphism $\pi_0 \simeq \mathbb{C}[N_+/A_+]$, B. Enriquez and the author have expressed the formula for Poisson bracket (5.8) in terms of the corresponding unipotent cosets [9]. \square

Now we extend the map ξ to incorporate the spaces π_λ , following Kupersmidt and Wilson [25, 29].

Let \mathcal{F}_λ be the quotient of π_λ by the subspace of total derivatives and \int be the projection $\pi_\lambda \rightarrow \mathcal{F}_\lambda, P \rightarrow \int Pdz$. We define a \mathbb{Z} -gradation on \mathcal{F}_λ by subtracting 1 from the gradation induced from π_λ .

For any $P \in \mathcal{F}_0$ the derivation $\xi_P : \pi_0 \rightarrow \pi_0$ can be extended to a linear operator on $\oplus_{\lambda \in \Lambda} \pi_\lambda$ by the formula

$$\xi_P = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+1} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n)}} + \sum_{1 \leq i \leq l} \delta_i P \frac{\partial}{\partial \phi_i},$$

where $\partial/\partial \phi_i \cdot (Se^{\bar{\lambda}}) = \lambda_i Se^{\bar{\lambda}}$. This defines a structure of \mathcal{F}_0 -module on π_λ .

For each $P \in \pi_0$ the operator ξ_P commutes with the action of derivative. Hence we obtain the structure of an \mathcal{F}_0 -module on \mathcal{F}_λ , i.e., a map $\{\cdot, \cdot\} : \mathcal{F}_0 \times \mathcal{F}_\lambda \rightarrow \mathcal{F}_\lambda$:

$$\left\{ \int Pdz, \int Rdz \right\} = \int (\xi_P \cdot R) dz.$$

Similarly, any element $R \in \pi_\lambda$ defines a linear operator ξ_R , acting from π_0 to π_λ and commuting with ∂ :

$$(5.10) \quad \xi_{Se^{\bar{\lambda}}} = \sum_{1 \leq i \leq l, n \geq 0} \partial^n \left(\partial(\delta_i S \cdot e^{\bar{\lambda}}) - S \frac{\partial e^{\bar{\lambda}}}{\partial \phi_i} \right) \frac{\partial}{\partial u_i^{(n)}}.$$

The operator ξ_R depends only on the image of R in \mathcal{F}_λ . Therefore it gives rise to a map $\{\cdot, \cdot\} : \mathcal{F}_\lambda \times \mathcal{F}_0 \rightarrow \mathcal{F}_\lambda$. We have for any $P \in \mathcal{F}_0, R \in \mathcal{F}_\lambda$:

$$\int (\xi_R \cdot P) dz = - \int (\xi_P \cdot R) dz.$$

Therefore our bracket $\{\cdot, \cdot\}$ is antisymmetric.

Note that, by construction, the operators ξ_R are evolutionary. Furthermore, formula (5.9) holds for any $P \in \mathcal{F}_0, R \in \oplus_{\lambda \in \Lambda} \mathcal{F}_\lambda$.

5.3. Local integrals of motion. We obtain from the definition of ξ :

$$\xi_{e^{-\phi_j}} = \sum_{n \geq 0} (\partial^n e^{-\phi_j}) \partial_i^{(n)},$$

which coincides with formula (5.5) for \tilde{Q}_j . Hence

$$\tilde{Q}_j = \left\{ \int e^{-\phi_j} dz, \cdot \right\},$$

is a hamiltonian operator, and we obtain:

Proposition 5.3. *The evolutionary operator \mathcal{H} defined by the Toda equation is hamiltonian:*

$$(5.11) \quad \mathcal{H} = \sum_{j \in J} \left\{ \int e^{-\phi_j} dz, \cdot \right\}.$$

Consider the corresponding operator

$$\{H, \cdot\} : \mathcal{F}_0 \rightarrow \bigoplus_{0 \leq i \leq l} \mathcal{F}_{-\alpha_i},$$

where

$$H = \sum_{j \in J} \int e^{-\phi_j} dz$$

can be viewed as the hamiltonian of the Toda equation (5.2).

We define *local integrals of motion* of the (finite or affine) Toda field theory as local functionals, which lie in the kernel of the operator \mathcal{H} . Thus, local integrals of motion are quantities that are conserved with respect to the Toda equation.

Denote the space of all local integrals of motion associated to $\bar{\mathfrak{g}}$ by $I(\bar{\mathfrak{g}})$ and the space of local integrals of motion associated to \mathfrak{g} by $I(\mathfrak{g})$. Denote by \bar{Q}_i the operator $\mathcal{F}_0 \rightarrow \mathcal{F}_{-\alpha_i}$ induced by \tilde{Q}_i . Then,

$$I(\bar{\mathfrak{g}}) = \bigcap_{i=1}^{\ell} \text{Ker}_{\mathcal{F}_0} \bar{Q}_i, \quad I(\mathfrak{g}) = \bigcap_{i=0}^{\ell} \text{Ker}_{\mathcal{F}_0} \bar{Q}_i.$$

Thus, we have a sequence of embeddings: $I(\mathfrak{g}) \subset I(\bar{\mathfrak{g}}) \subset \mathcal{F}_0$. From the fact that \tilde{Q}_i 's are hamiltonian operators, it follows that both $I(\mathfrak{g})$ and $I(\bar{\mathfrak{g}})$ are Poisson subalgebras of \mathcal{F}_0 .

Recall that in the previous lecture we defined an embedding $\mathbb{C}[v_i^{(n)}] \rightarrow \mathbb{C}[u_i^{(n)}]$. Let $\mathcal{W}(\bar{\mathfrak{g}})$ be the quotient of $\mathbb{C}[v_i^{(n)}]$ by the total derivatives and constants. We have an embedding $\mathcal{W}(\bar{\mathfrak{g}}) \rightarrow \mathcal{F}_0$.

The following results describe the spaces $I(\bar{\mathfrak{g}})$ and $I(\mathfrak{g})$.

Theorem 5.1 ([11], (2.4.10)). *The space of local integrals of motion of the Toda equation associated to a simple Lie algebra $\bar{\mathfrak{g}}$ equals $\mathcal{W}(\bar{\mathfrak{g}})$.*

Theorem 5.2 ([12], Theorem 1). *The space of local integrals of motion of the Toda equation associated to an affine Kac-Moody algebra \mathfrak{g} is linearly spanned by elements $I_m \in \mathcal{F}_0, m \in I$, where $\deg I_m = m$.*

The proof of these results is based on the following idea [11]. We need to compute the 0th cohomology of the two-step complex

$$\mathcal{F}_0 \rightarrow \bigoplus_{j \in J} \mathcal{F}_{-\alpha_j},$$

where the differential is given by the sum of $\bar{Q}_i, i \in S$. We will extend this complex further to the right (this certainly does not change its 0th cohomology), and then we will compute the cohomology of the resulting complex by using the Bernstein-Gelfand-Gelfand (BGG) resolution.

5.4. The complexes $F^\bullet(\mathfrak{g})$ and $F^\bullet(\bar{\mathfrak{g}})$. Recall [2, 27] that the dual of the BGG resolution of \mathfrak{g} is a complex $B^\bullet(\mathfrak{g}) = \bigoplus_{j \geq 0} B^j(\mathfrak{g})$, where $B^j(\mathfrak{g}) = \bigoplus_{w \in W, l(w)=j} M_{w(\rho)-\rho}^*$. Here M_λ^* is the module contragradient to the Verma module of highest weight λ , W is the corresponding affine Weyl group, and $l : W \rightarrow \mathbb{Z}_+$ is the length function. The differentials of the complex are described in the Appendix. They commute with the action of \mathfrak{g} . The 0th cohomology of $B^\bullet(\mathfrak{g})$ is one-dimensional and all higher cohomologies of $B^\bullet(\mathfrak{g})$ vanish, so that $B^\bullet(\mathfrak{g})$ is an injective resolution of the trivial representation of \mathfrak{n}_+ .

By construction, the right action of the Lie algebra \mathfrak{g} on this complex commutes with the differentials. Therefore we can take the subcomplex of invariants $F^\bullet(\mathfrak{g}) = B^\bullet(\mathfrak{g})^{\mathfrak{a}_+^R}$ with respect to the action of the Lie algebra \mathfrak{a}_+^R . Since $M_\lambda^* \simeq \mathbb{C}[N_+]$ as an \mathfrak{n}_+ -module (see Sect. 4.7), we can identify $(M_\lambda^*)^{\mathfrak{a}_+^R} \simeq \mathbb{C}[N_+/A_+]$ with $\pi_\lambda = \mathbb{C}[u_i^{(n)}] \otimes e^{\bar{\lambda}}$ using Theorem 1.1. Hence

$$F^0(\mathfrak{g}) = \pi_0, \quad F^1(\mathfrak{g}) = \bigoplus_{i=0}^{\ell} \pi_{-\alpha_i}.$$

Moreover, according to Sect. A.2 of the Appendix, the differential $F^0(\mathfrak{g}) \rightarrow F^1(\mathfrak{g})$ equals \mathcal{H} given by formula (5.11) with $J = \{0, \dots, \ell\}$.

By construction, the differential of the complex $F^\bullet(\mathfrak{g})$ commutes with the action of \mathfrak{a}_-^R , in particular, with the action of $\partial = p_{-1}^R$. It is also clear that the space of constants $\mathbb{C} \subset \pi_0 = F^0(\mathfrak{g})$ is a direct summand in $F^\bullet(\mathfrak{g})$. Hence the quotient $F^\bullet(\mathfrak{g})/(\mathbb{C} \oplus \text{Im } \partial)$ is also a complex. We have, by definition,

$$\mathcal{F}^0(\mathfrak{g}) = \mathcal{F}_0, \quad \mathcal{F}^1(\mathfrak{g}) = \bigoplus_{0 \leq i \leq \ell} \mathcal{F}_{-\alpha_i},$$

and the differential $\bar{\delta}^0 : \mathcal{F}^0(\mathfrak{g}) \rightarrow \mathcal{F}^1(\mathfrak{g})$ is given by

$$\bar{\delta}^0 = \sum_{0 \leq i \leq \ell} \tilde{Q}_i.$$

Therefore the 0th cohomology of the complex $\mathcal{F}^\bullet(\mathfrak{g})$ is isomorphic to the $I(\mathfrak{g})$. This cohomology will be computed in the next section.

5.5. Proof of Theorem 5.2. First we compute the cohomology of the complex $F^\bullet(\mathfrak{g})$.

Lemma 5.1 ([12], Lemma 1). *The cohomology of the complex $F^\bullet(\mathfrak{g})$ is isomorphic to $H^\bullet(\mathfrak{n}_+, \pi_0)$. The action of \mathfrak{a}_-^R on the cohomology of $F^\bullet(\mathfrak{g})$ is trivial.*

Here $H^\bullet(\mathfrak{n}_+, \pi_0)$ denotes the Lie algebra cohomology of \mathfrak{n}_+ with coefficients in $\pi_0 \simeq \mathbb{C}[N_+/A_+]$ (see [14] for the definition). The \mathfrak{n}_+ -module $\mathbb{C}[N_+/A_+]$ is coinduced from the trivial representation of \mathfrak{a}_+ . Hence, by Shapiro's lemma (see [14], Sect. 1.5.4), $H^\bullet(\mathfrak{n}_+, \pi_0) \simeq H^\bullet(\mathfrak{a}_+, \mathbb{C})$. Since \mathfrak{a}_+ is an abelian Lie algebra, $H^\bullet(\mathfrak{a}_+, \mathbb{C})$ is isomorphic to the exterior algebra $\bigwedge^\bullet(\mathfrak{a}_+^*) \simeq \bigwedge^\bullet(\mathfrak{a}_-)$.

Lemma 5.2. *The 0th cohomology of the complex $\mathcal{F}^\bullet(\mathfrak{g})$ is isomorphic to the 1st cohomology of the complex $F^\bullet(\mathfrak{g})$, and hence to \mathfrak{a}_- .*

This readily implies Theorem 5.2. Indeed, with respect to the \mathbb{Z} -gradation on \mathcal{F}_λ , the differentials of the complex are homogeneous of degree 0. Moreover, the corresponding \mathbb{Z} -gradation on cohomology coincides with the one induced by the principal gradation on \mathfrak{a}_+ . Therefore the space $I(\mathfrak{g})$ is linearly spanned by elements $I_m, m \in I$, where $\deg I_m = m$.

The proof of Lemma 5.2 is given in the Appendix. Here we only explain how to construct an integral of motion in the affine Toda field theory starting from a class in the first cohomology of the complex $F^\bullet(\mathfrak{g})$.

Consider such a class $\tilde{H} \in \oplus_{0 \leq i \leq \ell} \pi_{-\alpha_i}$. Since $\partial = p_{-1}^R \in \mathfrak{a}_-^R$ acts trivially on cohomologies of the complex $\mathcal{F}^\bullet(\mathfrak{g})$ (see , $\partial \tilde{H}$ is a coboundary, i.e. there exists $H \in \pi_0$ that $\bar{\delta}^0 \cdot H = \partial \tilde{H}$.

By construction, the element H has the property that $\tilde{Q}_i \cdot H \in \pi_{-\alpha_i}$ is a total derivative for $i = 0, \dots, \ell$. But it itself is not a total derivative, because otherwise \tilde{H} would also be a trivial cocycle. Therefore, $\int H dz \neq 0$. But then $\int H dz$ is an integral of motion of the mKdV hierarchy, because by construction $\bar{\delta}^0 \cdot \int H dz = \int (\bar{\delta}^0 \cdot H) dz = 0$ and hence $\tilde{Q}_i \cdot \int H dz = 0$ for any $i = 0, \dots, \ell$.

We denote by H_m the integral of motion corresponding to $p_{-m} \in \mathfrak{a}_-$ via Lemma 5.2. By Lemma 5.2, $I_m = \int H_m dz, m \in I, \text{span } I(\mathfrak{g})$. Explicit formulas for \tilde{H}_m and H_m can be found in [8], Sect. 5.

5.6. Proof of Theorem 5.1. Let \bar{W} be the Weyl group of $\bar{\mathfrak{g}}$. It is a finite subgroup of the affine Weyl group W of \mathfrak{g} . Consider the subcomplex $F^\bullet(\bar{\mathfrak{g}})$ of $F^\bullet(\mathfrak{g})$, where

$$F^j(\bar{\mathfrak{g}}) = \oplus_{w \in \bar{W}} \pi_{w(\rho) - \rho}.$$

In particular,

$$F^0(\bar{\mathfrak{g}}) = \pi_0, \quad F^1(\bar{\mathfrak{g}}) = \oplus_{i=1}^{\ell} \pi_{-\alpha_i},$$

and the differential $F^0(\bar{\mathfrak{g}}) \rightarrow F^1(\bar{\mathfrak{g}})$ equals \mathcal{H} given by formula (5.11) with $J = \{1, \dots, \ell\}$.

Consider the quotient complex $F^\bullet(\bar{\mathfrak{g}})/(\mathbb{C} \oplus \text{Im } \partial)$. We have, by definition,

$$\mathcal{F}^0(\bar{\mathfrak{g}}) = \mathcal{F}_0, \quad \mathcal{F}^1(\bar{\mathfrak{g}}) = \oplus_{1 \leq i \leq \ell} \mathcal{F}_{-\alpha_i},$$

and the differential $\mathcal{F}^0(\bar{\mathfrak{g}}) \rightarrow \mathcal{F}^1(\bar{\mathfrak{g}})$ is given by $\sum_{0 \leq i \leq \ell} \tilde{Q}_i$. Therefore the 0th cohomology of the complex $\mathcal{F}^\bullet(\bar{\mathfrak{g}})$ is isomorphic to the $I(\bar{\mathfrak{g}})$.

Lemma 5.3 ([11]). *The cohomology of the complex $F^\bullet(\bar{\mathfrak{g}})$ is isomorphic to $H^\bullet(\bar{\mathfrak{n}}_+, \pi_0)$. Hence the 0th cohomology of $F^\bullet(\bar{\mathfrak{g}})$ is isomorphic to $\mathbb{C}[v_i^{(n)}] \subset \mathbb{C}[u_i^{(n)}]$, and the higher cohomologies vanish.*

Theorem 5.1 immediately follows from this lemma. Thus, the space of integrals of motion of the Toda theory associated to $\bar{\mathfrak{g}}$ is the Poisson algebra $\mathcal{W}(\bar{\mathfrak{g}})$, which is a Poisson subalgebra of \mathcal{F}_0 . It is called the *classical \mathcal{W} -algebra* associated to $\bar{\mathfrak{g}}$.

5.7. KdV hierarchy is hamiltonian. Denote by η_m the derivation ξ_{H_m} , where H_m is the density of the m th integral of motion of the affine Toda field theory associated to \mathfrak{g} (it is defined up to adding a total derivative). In particular, we can choose as the one-cocycle \tilde{H}_1 , the vector $\sum_{0 \leq i \leq \ell} e^{-\phi_i}$. Then $\partial \tilde{H}_1 = -\sum_{0 \leq i \leq \ell} u_i e^{-\phi_i}$ and $H_1 = \frac{1}{2} \sum_{1 \leq i \leq \ell} u_i u^i$, where u^i 's are dual to u_i 's with respect to the inner product on \mathfrak{h} . Hence $\eta_1 = \bar{\partial}$, by formula (5.7).

Now we have for each $m \in I$, a hamiltonian vector field η_m on \mathbf{U} . On the other hand, we have a vector field p_{-m}^R coming from the right infinitesimal action of the Lie algebra $\mathfrak{a}_- \subset \mathfrak{g}$ on $N_+/A_+ \simeq \mathbf{U}$; these vector fields define the flows of the mKdV hierarchy.

Theorem 5.3 ([12], Theorem 3). *The vector field η_m coincides with the vector field p_{-m}^R up to a non-zero constant multiple for any $m \in I$.*

This, we can rescale H_m so as to make $\eta_m = p_{-m}^R$. The theorem means that the vector field p_{-m}^R is *hamiltonian*, with the hamiltonian being the m th integral of motion H_m of the affine Toda field theory. Note that we already know it when $m = 1$, since $\eta_1 = \partial = p_{-1}^R$.

This means that the mKdV equations are hamiltonian, that is the m th equation of the mKdV hierarchy can be written as

$$\partial_n u_i(z) = \{u_i(z), \int H_m dz\}$$

(up to constant multiple). In this formula $u_i(z)$ stands for the delta-like functional whose value at $u_i : \mathbb{R} \rightarrow \mathfrak{h}$ is $u_i(z)$ (see [11]). Now recall that $\mathcal{W}(\overline{\mathfrak{g}})$ is a Poisson subalgebra of \mathcal{F}_0 , by Theorem 5.1. It is clear from the definition that H_m can be chosen in such a way that it lies in $\mathbb{C}[v_i^{(n)}] \subset \mathbb{C}[u_i^{(n)}]$, so that $\int H_m dz \in \mathcal{W}(\overline{\mathfrak{g}})$. Then we see that the KdV hierarchy is also hamiltonian, i.e., it can be written as

$$\partial_n v_i(z) = \{v_i(z), \int H_m dz\}.$$

Proposition 5.4. *The integrals of motion of the affine Toda field theory (equivalently, the hamiltonians of the corresponding (m)KdV hierarchy) commute with each other:*

$$\left\{ \int H_n dz, \int H_m dz \right\} = 0$$

in \mathcal{F}_0 for any $n, m \in I$.

Proof. Since $p_{-m}, m \in I$, lie in a commutative Lie algebra, they commute with each other; so do the corresponding vector fields. By Theorem 5.3, the same holds for the vector fields $\eta_m, m \in I$: $[\eta_n, \eta_m] = 0$. By formula (5.9), injectivity of the map ξ on \mathcal{F}_0 , and the definition of the vector fields η_m , the corresponding integrals of motion also commute with each other. \square

We conclude that the KdV and mKdV hierarchies are completely integrable hamiltonian systems.

5.8. Quantization. In conclusion of this lecture, I would like to describe the problem of quantization of Toda integrals of motion, which was the original motivation for developing the formalism of these lectures.

The map $\xi : \pi_0 \rightarrow \text{End } \pi_0$ defined in Sect. 5.2 can be quantized in the following sense. Let $\pi_\hbar^\xi = \pi_\lambda[[\hbar]]$. There exists a linear map $\xi^\hbar : \pi_0^\hbar \rightarrow \text{End } \pi_0^\hbar$ such that

- (1) ξ^\hbar factors through $\mathcal{F}_0^\hbar = \mathcal{F}_0[[\hbar]] = \pi_0^\hbar / (\text{Im } \partial \oplus \mathbb{C})[[\hbar]]$;
- (2) the bracket $[\cdot, \cdot]_\hbar : \mathcal{F}_0^\hbar \times \mathcal{F}_0^\hbar \rightarrow \mathcal{F}_0^\hbar$, defined by the formula

$$[\int P dz, \int R dz]_\hbar = \int (\xi_P^\hbar \cdot R) dz,$$

where \int denotes the projection $\pi_0^\hbar \rightarrow \mathcal{F}_0^\hbar$, is a Lie bracket;

- (3) $\xi^\hbar = \hbar \xi^{(1)} + \hbar^2(\dots)$, and the map $\pi_0 \rightarrow \text{End } \pi_0$ induced by $\xi^{(1)}$ coincides with ξ .

Such a map ξ^{\hbar} can be defined using the vertex operator algebra structure on π_0 . This is explained in detail in [11], Sect. 4 (where \hbar is denoted by β^2). The key observation that enables us to construct this map is that π_0 can be viewed as a Fock representation of a Heisenberg algebra with generators $b_i(n), i = 1, \dots, \ell; n \in \mathbb{Z}$, and relations

$$[b_i(n), b_j(m)] = \hbar n(\alpha_i, \alpha_j) \delta_{n, -m}.$$

This module is endowed with a canonical vertex operator algebra structure $Y^{\hbar} : \pi_0^{\hbar} \rightarrow \text{End } \pi_0^{\hbar}[[z, z^{-1}]]$, $P \rightarrow Y^{\hbar}(P, z)$. Given $P \in \pi_0^{\hbar}$, we denote by ξ_P^{\hbar} the linear endomorphism of π_0^{\hbar} given by the residue, i.e. the (-1) st Fourier component, of $Y^{\hbar}(P, z)$. This gives us a map $\xi^{\hbar} : \pi_0^{\hbar} \rightarrow \text{End } \pi_0^{\hbar}$, which satisfies the conditions above.

Thus, the Gelfand-Dickey-Dorfman structure on π_0 can be viewed as a classical limit of the structure of vertex operator algebra on π_0^{\hbar} .

In [11] we also defined quantum deformations of the maps $\xi : \pi_0 \rightarrow \text{End } \pi_{\lambda}$ and $\pi_{\lambda} \rightarrow \text{Hom}(\pi_0, \pi_{\lambda})$. This enabled us to quantize the operators $\tilde{Q}_i : \pi_0 \rightarrow \pi_{-\alpha_i}$ and $\bar{Q}_i : \mathcal{F}_0 \rightarrow \mathcal{F}_{-\alpha_i}$. Hence we can define the space of quantum integrals of motion as the intersection of the kernels of the quantum operators $\bar{Q}_0^{\hbar}, \dots, \bar{Q}_{\ell}^{\hbar}$. This space could *a priori* be “smaller” than the space $I(\mathfrak{g})$ of classical integrals of motion, i.e., it could be that some (or even all) of them do not survive quantization. However, we proved in [11] that all integrals of motion of affine Toda field theory can be quantized.

Our proof was based on the fact that the quantum operators \tilde{Q}_i^{\hbar} in a certain sense generate the quantized universal enveloping algebra $U_q(\mathfrak{n}_+)$, where $q = \exp(\pi i \hbar)$ (recall that the operators \tilde{Q}_i generate $U(\mathfrak{n}_+)$). Using this fact, we were able to deform the whole complex $F^{\bullet}(\mathfrak{g})$ and derive the quantization property from a deformation theory argument (see [11]).

APPENDIX

A.1. Proof of Proposition 4.1. The proposition has been proved by B. Kostant (private communication). The proof given below is different, but it uses the ideas of Kostant’s proof.

Recall that for the non-twisted $\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$,

$$p_{-1} = \bar{p}_{-1} + f_0,$$

where \bar{p}_{-1} is given by formula (4.7) and $f_0 = e_{\max} \otimes t^{-1}$. Here e_{\max} is a generator of the one-dimensional subspace of $\bar{\mathfrak{n}}_+$ corresponding to the maximal root. More generally, for $i = 1, \dots, \ell$, we can also write

$$(5.12) \quad p_{-d_i} = \bar{p}_{-i} \otimes 1 + r_i \otimes t^{-1},$$

where $\bar{p}_{-i} \in \bar{\mathfrak{n}}_-$, and $r_i \in \bar{\mathfrak{n}}_+$. It is clear that $[p_{-d_i}, p_{-d_j}] = 0$ implies

$$(5.13) \quad [\bar{p}_{-i}, \bar{p}_{-j}] = [r_i, r_j] = 0.$$

Since $[x, e_{\max}] = 0$ for all $x \in \bar{\mathfrak{n}}_+$,

$$\begin{aligned} [f_0, p_{-d_i}] &= -[\bar{p}_{-1}, p_{-d_i}] = -[\bar{p}_{-1}, \bar{p}_{-i} + r_i \otimes t^{-1}] \\ &= -[\bar{p}_{-1}, r_i] \otimes t^{-1}. \end{aligned}$$

Furthermore, we find from formula (4.3):

$$\gamma_i = -(\text{ad } \bar{p}_{-1})^{h-d_i} \cdot r_i \otimes t^{-1} \in \mathfrak{h} \otimes t^{-1}.$$

Hence the linear independence of the vectors γ_i is equivalent to the linear independence of the vectors

$$\bar{\gamma}_i = (\text{ad } \bar{p}_{-1})^{h-d_i} \cdot r_i, \quad i = 1, \dots, \ell.$$

Let $\{e, h, f\}$ be the principal \mathfrak{sl}_2 subalgebra of $\bar{\mathfrak{g}}$, such that $f = \bar{p}_{-1}$ and $h = 2\bar{p}^\vee$. Recall from [23] that as a principal \mathfrak{sl}_2 -module, $\bar{\mathfrak{g}}$ splits into the direct sum of irreducible representations R_i of dimension $2d_i + 1$, where $i = 1, \dots, \ell$. The multiplicity of R_i in the decomposition of $\bar{\mathfrak{g}}$ equals the multiplicity of the exponent d_i . Note that different components R_j are mutually orthogonal with respect to the invariant inner product (\cdot, \cdot) on $\bar{\mathfrak{g}}$, and hence every element of $\bar{\mathfrak{g}}$ can be written canonically as a sum of its projections on various R_j 's.

The linear independence of the vectors $\bar{\gamma}_i$ is equivalent to the statement that each r_i has a non-zero projection on R_i (and moreover, if d_i has multiplicity two, then the projections of the corresponding r_i^1, r_i^2 on R_i are linearly independent). Note that $\deg r_i = \ell - d_i$, with respect to the gradation operator \bar{p}^\vee . Thus, it suffices to show that $(r_i, \bar{p}_{-\ell+i}) \neq 0$ (resp., the pairing between $\text{span}(r_i^1, r_i^2)$ and $\text{span}(\bar{p}_{-\ell+i}^1, \bar{p}_{-\ell+i}^2)$ induced by (\cdot, \cdot) is non-degenerate).

Now recall that according to Kac [22], Lemma 14.4, the inverse image of \mathfrak{a} in $\tilde{\mathfrak{g}}$ is a (non-degenerate) Heisenberg Lie subalgebra $\mathfrak{a} \oplus \mathbb{C}C$. Thus, $[p_n, p_{-n}] \neq 0, \forall n \in I$ (and moreover, if n has multiplicity two, the pairing between $\text{span}(p_n^1, p_n^2)$ and $\text{span}(p_{-n}^1, p_{-n}^2)$ induced by the commutator is non-degenerate). But note that

$$(5.14) \quad p_{d_i} = \bar{p}_{-\ell+i} \otimes t + r_{\ell-i} \otimes 1.$$

We find from formulas (5.12) and (5.14) the following commutator in $\tilde{\mathfrak{g}}$:

$$[p_{d_i}^k, p_{-d_i}^l] = (\bar{p}_{-\ell+i}^k, r_i^l)C.$$

The proposition follows.

A.2. The BGG resolution. Vector y from the Verma module M_λ is called *singular vector* of weight $\mu \in \mathfrak{h}^*$, if $\mathfrak{n}_+ \cdot y = 0$ and $x \cdot y = \mu(x)y$ for any $x \in \mathfrak{h}$. We have $M_\lambda \simeq U(\mathfrak{n}_-) \cdot v_\lambda$, where v_λ , which is called the highest weight vector, is a generator of the space \mathbb{C}_λ . This vector is a singular vector of weight λ . Any singular vector of M_λ of weight μ can be uniquely represented as $P \cdot v_\lambda$ for some element $P \in U(\mathfrak{n}_-)$ of weight $\mu - \lambda$. This singular vector canonically defines a homomorphism of \mathfrak{g} -modules $i_P : M_\mu \rightarrow M_\lambda$, which sends $u \cdot v_\mu$ to $(uP) \cdot v_\lambda$ for any $u \in U(\mathfrak{n}_-)$. Denote by i_P^* the dual homomorphism $M_\lambda^* \rightarrow M_\mu^*$.

There is an isomorphism $U(\mathfrak{n}_-) \rightarrow U(\mathfrak{n}_+)$, which maps the generators f_0, \dots, f_ℓ to $-e_0, \dots, -e_\ell$. Denote by \bar{P} the image of $P \in U(\mathfrak{n}_-)$ under this isomorphism.

The homomorphism sending $\alpha \in \mathfrak{n}_+$ to α^L , can be extended in a unique way to a homomorphism from $U(\mathfrak{n}_+)$ to the algebra of differential operators on N_+ . Denote the image of $u \in U(\mathfrak{n}_+)$ under this homomorphism by u^L .

Proposition A.1 ([12], Prop. 2). *If $P \cdot v_\lambda$ is a singular vector in M_λ of weight μ , then the homomorphism $i_P^* : M_\lambda^* \rightarrow M_\mu^*$ is given by the differential operator \bar{P}^L .*

In the case of simple Lie algebras, these homomorphisms were first studied by B. Kostant [24]. Using Proposition A.1, one can explicitly construct the differentials of the dual BGG resolution $B^\bullet(\mathfrak{g})$ of Sect. 5.4.

It is known that for each pair of elements of the Weyl group, such that $w \prec w'$, there is a singular vector $P_{w,w'} \cdot v_{w(\rho)-\rho}$ in $M_{w(\rho)-\rho}$ of weight $w'(\rho) - \rho$. By Proposition A.1, this vector defines the homomorphism $\overline{P}_{w,w'}^L : M_{w(\rho)-\rho}^* \rightarrow M_{w'(\rho)-\rho}^*$.

It is possible to normalize all $P_{w,w'}$'s in such a way that $P_{w'_1,w''} P_{w,w'_1} = P_{w'_2,w''} P_{w,w'_2}$ for any quadruple of elements of the Weyl group, satisfying $w \prec w'_1, w'_2 \prec w''$. Then we obtain: $\overline{P}_{w,w'_1}^L \overline{P}_{w'_1,w''}^L = \overline{P}_{w,w'_2}^L \overline{P}_{w'_2,w''}^L$.

The differential $\delta^j : B^j(\mathfrak{g}) \rightarrow B^{j+1}(\mathfrak{g})$ of the BGG complex can be written as follows:

$$\delta^j = \sum_{l(w)=j, l(w')=j+1, w \prec w'} \epsilon_{w,w'} \overline{P}_{w,w'}^L,$$

where $\epsilon_{w,w'} = \pm 1$ are chosen as in [2, 27].

A.3. Proof of Lemma 5.2. Since $\partial = p_{-1}^R$ commutes with the differentials of the complex $F^\bullet(\mathfrak{g})$, we can consider the double complex

$$(A.3) \quad \mathbb{C} \longrightarrow F^\bullet(\mathfrak{g}) \xrightarrow{\pm p_{-1}} F^\bullet(\mathfrak{g}) \longrightarrow \mathbb{C}.$$

Here $\mathbb{C} \rightarrow \pi_0 \subset F^\bullet(\mathfrak{g})$ and $F^\bullet(\mathfrak{g}) \rightarrow \pi_0 \rightarrow \mathbb{C}$ are the embedding of constants and the projection on constants, respectively. We place \mathbb{C} in dimensions -1 and 2 of our complex, and $F^\bullet(\mathfrak{g})$ in dimensions 0 and 1 .

In the spectral sequence, in which $\pm p_{-1}$ is the 0th differential, the first term is the complex $\mathcal{F}^\bullet(\mathfrak{g})[-1]$, where

$$\mathcal{F}^j(\mathfrak{g}) \simeq \oplus_{l(w)=j} \mathcal{F}_{w(\rho)-\rho}.$$

Indeed, if $\lambda \neq 0$, then in the complex

$$\pi_\lambda \xrightarrow{p_{-1}} \pi_\lambda$$

the 0th cohomology is 0, and the first cohomology is, by definition, the space \mathcal{F}_λ . If $\lambda = 0$, then in the complex

$$\mathbb{C} \longrightarrow \pi_0 \xrightarrow{p_{-1}} \pi_0 \longrightarrow \mathbb{C}$$

the 0th cohomology is 0 and the first cohomology is, by definition, the space \mathcal{F}_0 .

We need to find the 0th cohomology of the complex $\mathcal{F}^\bullet(\mathfrak{g})$, which is the same as the 1st cohomology of the double complex (A.3).

We can compute this cohomology, using the other spectral sequence associated to our double complex. Since $H^\bullet(F^\bullet(\mathfrak{g})) \simeq \wedge^\bullet(\mathfrak{a}_-)$, we obtain in the first term the following complex

$$\mathbb{C} \longrightarrow \wedge^\bullet(\mathfrak{a}_-) \xrightarrow{\pm p_{-1}} \wedge^\bullet(\mathfrak{a}_-) \longrightarrow \mathbb{C}.$$

By Lemma 5.1, the action of p_{-1} on $\wedge^\bullet(\mathfrak{a}_-)$ is trivial and hence the cohomology of the double complex (A.3) is isomorphic to $\wedge^\bullet(\mathfrak{a}_-)/\mathbb{C} \oplus \wedge^\bullet(\mathfrak{a}_-)/\mathbb{C}[-1]$. In particular, we see that the space of Toda integrals is isomorphic to \mathfrak{a}_- .

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