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Certain subclasses of uniformly harmonic β-starlike functions of complex order

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Abstract. In this paper, we introduce a family of harmonic parabolic starlike functions of complex order in the unit disc and coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination are determined for functions in this family. Further results on integral transforms are discussed. Consequently, many of our results are either extensions or new approaches to those corresponding previously known results.

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1. Introduction and definitions

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

Let $\mathcal{S_H}$ denote the family of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{S_H}$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

The harmonic function $f = h + \bar{g}$ for $g \equiv 0$ reduces to an analytic function $f = h$. Also let $\mathcal{S}_{\overline{\mathcal{H}}}$ be the subclass of $\mathcal{S_H}$ consisting of functions of the form $f = h + \overline{g}$, where the analytic functions h and g as

$$h(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k|\, z^k. \tag{1.2}$$

In 1984 Clunie and Sheil-Small [3] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several papers related on $\mathcal{S}_{\mathcal{H}}$ and its subclasses. Jahangiri [7], Silverman[9], Silverman and Silvia [10] studied the harmonic starlike functions. Frasin [5], Frasin and Murugusundaramoorthy [6] and Dixit et al. [4] extended the results by defining the subclasses by means convolution (or Hadamard product) also, see [2, 13].

Recently, Yalçin and Öztürk [12] defined the class $S_{\overline{\mathcal{H}}}(b)$ consisting of functions $f = h + \bar{g} \in S_{\overline{\mathcal{H}}}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right\} > 0,\ b \in \mathbb{C}\setminus\{0\},\ z \in \mathcal{U}. \tag{1.3}$$

Also, they gave necessary and sufficient conditions for the functions to be in $S_{\overline{\mathcal{H}}}(b)$.

Furthermore, Stephen et al. [11] studied a harmonic parabolic starlike functions of complex order denoted by $S_{\overline{\mathcal{H}}}(b, \alpha)$ consisting of functions $f = h + \bar{g} \in S_{\overline{\mathcal{H}}}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left((1 + e^{i\gamma})\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - (1 + e^{i\gamma}) - 1\right)\right\} > \alpha,\ z \in \mathcal{U}, \tag{1.4}$$

where $0 \le \alpha < 1$, $\gamma \in \mathbb{R}$ and $b \in \mathbb{C}\setminus\{0\}$ with $|b| \le 1$. Also, they gave necessary and sufficient conditions for the functions to be in $S_{\overline{\mathcal{H}}}(b, \alpha)$.

Motivated by Frasin and Murugusundaramoorthy [6], Yalçin and Öztürk [12] and Stephen et al. [11], we define a new class of uniformly harmonic $\beta-$ starlike functions of complex order $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$, the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left((1 + \beta e^{i\gamma})\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) + \overline{g_t(z)}} - \beta e^{i\gamma} - 1\right)\right\} > \alpha,\ z \in \mathcal{U}, \tag{1.5}$$

where $b \in \mathbb{C}\setminus\{0\}$, $\alpha (0 \le \alpha < 1)$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \le t \le 1$, $\Phi(z) = z + \sum\limits_{k=2}^{\infty} \lambda_k z^k$ and $\Psi(z) = z + \sum\limits_{k=2}^{\infty} \mu_k z^k$ are analytic in $\mathcal{U}$ with the conditions $\lambda_k \ge 0$, $\mu_k \ge 0, \beta \ge 0$ and $\gamma \in \mathbb{R}$. The operator " $*$ " stands for the convolution of two power series. We further let $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ denote the subclass of $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_{\overline{\mathcal{H}}}$.

We note that by specializing the functions Φ, Ψ and parameters β, γ and t we obtain the well-known harmonic univalent functions as well as many new ones. For example,

1. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, 1, 1; 1) = G_{\overline{\mathcal{H}}}(\alpha)$ (Rosy et al. [8])
2. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, \beta, 1; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\alpha, \beta; t)$ (Ahuja et al. [1])
3. $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, 1; 1) = \overline{\mathcal{HS}}(\Phi, \Psi; \alpha, \beta)$ (Frasin and Murugusundaramoorthy [6])

4. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, 1, b; 1) = \mathcal{S}_{\overline{\mathcal{H}}}(b, \alpha)$ (Stephen et al [11])
5. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, 0, b; 1) = \mathcal{S}_{\overline{\mathcal{H}}}(b)$ (Yalçin and Öztürk [12])
6. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, 0, 1; 1) = S^*_{\overline{\mathcal{H}}}(\alpha)$ (Jahangiri [7].)
   For $\alpha = 0$ the class $S^*_{\overline{\mathcal{H}}}(\alpha)$ was studied by Silverman and Silvia [10], for $\alpha = 0$ and $b_1 = 0$ see [9, 10].

In this paper, we obtain coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for functions in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. Further results on integral transforms are also discussed.

2. Coefficient inequalities

Our first theorem gives a sufficient condition for functions in $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]}{(1-\alpha)|b|} |a_k| + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]}{(1-\alpha)|b|} |b_k| \le 1, \tag{2.1}$$

where $\beta \ge 0$, $0 \le \alpha < 1$, $0 \le t \le 1$, $k(1-\alpha)|b| \le [(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]$ and $k(1-\alpha)|b| \le [(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]$ for $k \ge 2$ then $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$.

Proof. To prove that $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$, we only need to show that if (2.1) holds, then the required condition (1.5) is satisfied. For (1.5), we can write

$$Re\left\{1 + \frac{1}{b}\left((1+\beta e^{i\gamma})\left(\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) + \overline{g_t(z)}}\right) - \beta e^{i\gamma} - 1\right)\right\} \ge \alpha.$$

Using the fact that $Re\omega \ge \alpha$ if and only if $|1-\alpha+\omega| \ge |1+\alpha-\omega|$, it suffices to show that

$$\begin{aligned}
&\Big|\left[(2-\alpha)b - (\beta e^{i\gamma}+1)\right]\left(h_t(z) + \overline{g_t(z)}\right) \\
&\qquad + \left(\beta e^{i\gamma}+1\right)\left(h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}\right)\Big| \\
&\qquad - \Big|\left[\alpha b + (\beta e^{i\gamma}+1)\right]\left(h_t(z) + \overline{g_t(z)}\right) \\
&\qquad - \left(\beta e^{i\gamma}+1\right)\left(h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}\right)\Big|
\end{aligned}$$

$$\geq 2(1-\alpha)|b||z| - \sum_{k=2}^{\infty} 2\left[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|\right]|a_k||z|^k$$
$$- \sum_{k=1}^{\infty} 2\left[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|\right]|b_k||z|^k$$
$$\geq 2(1-\alpha)|b||z| \left\{1 - \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k||z|^{k-1} \right.$$
$$\left. - \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k||z|^{k-1} \right\}$$
$$\geq 2(1-\alpha)|b| \left\{1 - \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k| \right.$$
$$\left. - \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k| \right\} \geq 0,$$

which implies that $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$. The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{(1-\alpha)|b|}{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|\ ]} x_k z^k$$
$$+ \sum_{k=1}^{\infty} \frac{(1-\alpha)|b|}{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|\ ]} \overline{y_k z^k},$$

where $\sum\limits_{k=2}^{\infty} |x_k| + \sum\limits_{k=1}^{\infty} |y_k| = 1$,shows that the coefficient bound given by (2.1) is sharp. □

In the following theorem, it is shown that the bound (2.1) is also necessary for functions $f = h + \bar{g}$, where h and g are of the form (1.2).

Theorem 2.2. *Let $f = h + \bar{g}$ be so that h and g are given by (1.2). Then $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|}|a_k|$$
$$+ \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|}|b_k| \leq 1, \tag{2.2}$$

where $\beta \geq 0$, $0 \leq \alpha < 1$, $0 \leq t \leq 1$, $k(1-\alpha)|b| \leq [(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]$ and $k(1-\alpha)|b| \leq [(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]$ for $k \geq 2$.

Proof. The 'if part' follows from Theorem 2.1 upon noting that the functions h and g in $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$ are of the form (1.2), then $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. For the 'only if' part, we wish to show that $f \notin \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ if the condition (2.2) does not hold. Note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (1.2) be in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is that

$$
\begin{aligned}
& Re\left\{\frac{[b-(1+\beta e^{i\gamma})][h_t(z)+\overline{g_t(z)}] +(1+\beta e^{i\gamma})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]}{b[h_t(z)+\overline{g_t(z)}]}-\alpha\right\} \\
= \; & Re\left\{\frac{(1-\alpha)bz-\sum\limits_{k=2}^{\infty}[(1-\alpha)bt+(1+\beta e^{i\gamma})(\lambda_k-t)]|a_k|z^k -\sum\limits_{k=1}^{\infty}[(1+\beta e^{i\gamma})(\mu_k+t)-(1-\alpha)bt]|b_k|\bar{z}^k}{b(z-\sum\limits_{k=2}^{\infty}t|a_k|z^k+\sum\limits_{k=1}^{\infty}t|b_k|\bar{z}^k)}\right\} \\
= \; & Re\left\{\frac{(1-\alpha)|b|^2-\sum\limits_{k=2}^{\infty}[(1-\alpha)bt+(1+\beta e^{i\gamma})(\lambda_k-t)]\bar{b}|a_k|z^{k-1} -\frac{\bar{z}}{z}\sum\limits_{k=1}^{\infty}[(1+\beta e^{i\gamma})(\mu_k+t)-(1-\alpha)bt]\bar{b}|b_k|\bar{z}^{k-1}}{|b|^2(1-\sum\limits_{k=2}^{\infty}t|a_k|z^{k-1}+\sum\limits_{k=1}^{\infty}t|b_k|\bar{z}^{k-1})}\right\} \\
\geq \; & 0.
\end{aligned}
$$

If we choose z to be real $z \to 1^-$ and since $Re(-e^{i\gamma}) \geq -|e^{i\gamma}| = -1$, the above inequality reduces to

$$
\frac{(1-\alpha)|b|^2-\sum\limits_{k=2}^{\infty}[(1-\alpha)bt+(1+\beta)(\lambda_k-t)]\bar{b}|a_k|r^{k-1} -\sum\limits_{k=1}^{\infty}[(1+\beta)(\mu_k+t)-(1-\alpha)bt]\bar{b}|b_k|r^{k-1}}{|b|^2(1-\sum\limits_{k=2}^{\infty}t|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}t|b_kr^{k-1})} \geq 0. \qquad (2.3)
$$

If the condition (2.2) does not hold then the numerator in (2.3) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.3) is negative. This contradicts the condition for $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. Hence the proof is complete. $\square$

3. Extreme points and distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$.

Theorem 3.1. *Let f be given by (1.2). Then $f\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ if and only if*

$$f(z)=\sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_k(z)), \tag{3.1}$$

where $h_1(z)=z$, $h_k(z)=z-\frac{(1-\alpha)|b|}{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}z^k$ $(k=2,3,...)$, $g_k(z)=z+\frac{(1-\alpha)|b|}{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}\overline{z}^k$ $(k=1,2,3,...)$, $\sum\limits_{k=1}^{\infty}(X_k+Y_k)=1$, $X_k\geq 0$, $Y_k\geq 0$. In particular, the extreme points of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (3.1), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_k(z)) \\ &= \sum_{k=1}^{\infty}(X_k+Y_k)\,z-\sum_{k=2}^{\infty}\frac{(1-\alpha)|b|}{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}X_kz^k \\ &\qquad +\sum_{k=1}^{\infty}\frac{(1-\alpha)|b|}{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}Y_k\overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}{(1-\alpha)|b|}\left(\frac{(1-\alpha)|b|}{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}\right)X_k \\ &\quad+\sum_{k=1}^{\infty}\frac{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}{(1-\alpha)|b|}\left(\frac{(1-\alpha)|b|}{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}\right)Y_k \\ &=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1 \end{aligned}$$

and so f is in closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. Conversely, suppose that f is in closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. Setting

$$X_k=\frac{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}{(1-\alpha)|b|}|a_k|,\ k=2,3,\ldots,$$

and

$$Y_k=\frac{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}{(1-\alpha)|b|}|b_k|,\ k=1,2,\ldots,$$

where $\sum\limits_{k=1}^{\infty}(X_k+Y_k)=1$. Then note that by Theorem 2.2, $0\le X_k\le 1$ $(k=2,3,\dots)$ and $0\le Y_k\le 1$ $(k=1,2,3,\dots)$. We define $X_1=1-\sum\limits_{k=2}^{\infty}X_k-\sum\limits_{k=1}^{\infty}Y_k$ and by Theorem 2.2 , $X_1\ge 0$. Consequently, we obtain $f(z)=\sum\limits_{k=1}^{\infty}\left(X_kh_k(z)+Y_kg_k(z)\right)$.

Using Theorem 2.2, it is easily seen that $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ is convex and closed, so closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ is $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. In other words, the statement of Theorem 3.1 is really for $f\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. □

The following theorem gives the distortion bounds for functions in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ which yields a covering result for this class.

Theorem 3.2. *Letf $\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ and $\lambda_2\le\lambda_k$, $\lambda_2\le\mu_k$ for $k\ge 2$, then*

$$\begin{aligned}|f(z)|\le&(1+|b_1|)r\\&+\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right)r^2\end{aligned}$$

and

$$\begin{aligned}|f(z)|\ge&(1-|b_1|)r\\&-\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right)r^2.\end{aligned}$$

Proof. Let $f\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. Taking the absolute value of f, we obtain

$$\begin{aligned}|f(z)|&\le(1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\&\le(1+|b_1|)r+\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\\&\quad\left(\sum_{k=2}^{\infty}\frac{[(\lambda_k-t)(1+\beta)+(1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k|\right.\\&\qquad\left.+\frac{[(\mu_k+t)(1+\beta)-(1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k|\right)r^2\\&=(1+|b_1|)r\\&\quad+\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right.\\&\qquad\left.-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right)r^2.\end{aligned}$$

Similarly

$$\begin{aligned}|f(z)| \quad \geq \quad & (1-|b_1|)r \\ & -\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right. \\ & \left.\qquad -\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right) r^2.\end{aligned}$$

The upper and lower bounds given in Theorem 3.2 are respectively attained for the following functions.

$$\begin{aligned}f(z) = & z+|b_1|\bar{z} \\ & +\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right. \\ & \left.\qquad -\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right) \bar{z}^2\end{aligned}$$

and

$$\begin{aligned}f(z) = & (1-|b_1|)z \\ & -\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right. \\ & \left.\qquad -\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right) z^2.\end{aligned}$$

□

The following covering result follows from the left hand inequality in Theorem 3.2.

Corollary 3.3. *Let f of the form (1.2) be so that $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ and $\lambda_2 \leq \lambda_k$, $\lambda_2 \leq \mu_k$ for $k \geq 2$. Then*

$$\begin{aligned}\Bigg\{\omega : |\omega| < 1 - & \frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]} \\ & -\left[1-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right]|b_1|\Bigg\}.\end{aligned}$$

4. Convolution and convex combinations

In this section, we show that the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$ and

$F(z) = z - \sum\limits_{k=2}^{\infty} |A_k| z^k + \sum\limits_{k=1}^{\infty} |B_k| \overline{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k + \sum_{k=1}^{\infty} |b_k||B_k| \overline{z}^k. \tag{4.1}$$

Using the definition, we show that the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \delta < \alpha < 1$, let $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ and $F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. Then $f * F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$.*

Proof. Let $f(z) = z - \sum\limits_{k=2}^{\infty} |a_k| z^k + \sum\limits_{k=1}^{\infty} |b_k| \overline{z}^k$ and $F(z) = z - \sum\limits_{k=2}^{\infty} |A_k| z^k + \sum\limits_{k=1}^{\infty} |B_k| \overline{z}^k$ be in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. Then the convolution $f * F$ is given by (4.1), from the assertion that $f * F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. We note that $|A_k| \leq 1$ and $|B_k| \leq 1$. In view of Theorem 2.2 and the inequality $0 \leq \delta \leq \alpha < 1$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\delta)|b|t]}{(1-\delta)|b|} |a_k||A_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\delta)|b|t]}{(1-\delta)|b|} |b_k||B_k| \\
&\qquad \leq \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\delta)|b|t]}{(1-\delta)|b|} |a_k| \\
&\qquad\quad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\delta)|b|t]}{(1-\delta)|b|} |b_k| \\
&\qquad \leq \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} |a_k| \\
&\qquad\quad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} |b_k| \\
&\qquad \leq 1,
\end{aligned}$$

by Theorem 2.2, $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. By the same token, we then conclude that $f * F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. □

Next, we show that the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convex combination of its members.

Theorem 4.2. *The class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{ik}| z^k + \sum_{k=1}^{\infty} |b_{ik}| \overline{z}^k.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ik}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ik}| \right) \overline{z}^k.$$

Since,

$$\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} |a_{ik}| \\ + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} |b_{ik}| \leq 1,$$

from the above equation we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} \sum_{i=1}^{\infty} t_i |a_{ik}| \\ &\qquad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} \sum_{i=1}^{\infty} t_i |b_{ik}| \\ &= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} |a_{ik}| \right. \\ &\qquad \left. + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} |b_{ik}| \right\} \\ &\leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

This is the condition required by (2.2) and so $\sum\limits_{i=1}^{\infty} t_i f_i \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. □

5. Class preserving integral operator

Finally, we consider the closure property of the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c[f]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi \ (c > -1).$$

Theorem 5.1. *Let $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. Then $\mathcal{L}_c[f] \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$.*

Proof. From the representation of $\mathcal{L}_c[f]$, it follows that

$$\begin{aligned}
\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c}\int_0^z \xi^{c-1}h(\xi)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}g(\xi)d\xi} \\
&= \frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\xi - \sum_{k=2}^{\infty}|a_k|\xi^k\right)d\xi \\
&\qquad + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\sum_{k=1}^{\infty}|b_k|\xi^k\right)d\xi} \\
&= z - \sum_{k=2}^{\infty}A_k z^k + \sum_{k=1}^{\infty}B_k z^k,
\end{aligned}$$

where $A_k = \frac{c+1}{c+k}\,|a_k|$ and $B_k = \frac{c+1}{c+k}\,|b_k|$. Hence

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[(\lambda_k - t)(1+\beta)+(1-\alpha)t|b|]}{(1-\alpha)|b|}\left(\frac{c+1}{c+k}|a_k|\right) \\
&\qquad + \sum_{k=1}^{\infty}\frac{[(\mu_k + t)(1+\beta)-(1-\alpha)t|b|]}{(1-\alpha)|b|}\left(\frac{c+1}{c+n}|b_k|\right) \\
&\qquad\qquad \le \sum_{k=2}^{\infty}\frac{[(\lambda_k - t)(1+\beta)+(1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k| \\
&\qquad\qquad\qquad + \sum_{k=1}^{\infty}\frac{[(\mu_k + t)(1+\beta)-(1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k| \\
&\qquad\qquad \le 1,
\end{aligned}$$

since $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$, therefore by Theorem 2.2, $\mathcal{L}_c[f] \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. □

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