JOURNAL OF APPROXIMATION THEORY **44**, 73–81 (1985)

Inequalities for Polynomials with Two Equal Coefficients

C. FRAPPIER* AND Q. I. RAHMAN

Department of Mathematics, University of Montreal, Montreal, Quebec H3C 3J7, Canada

AND

ST. RUSCHEWEYH

Mathematisches Institut, Universität Würzburg, 8700 Würzburg, West Germany

Communicated by T. J. Rivlin

Received April 2, 1984

1.

1.1. An expression of the form $\sum_{v=-n}^{n} c_v e^{iv\theta}$, where the c_v's are arbitrary complex numbers will be referred to as a trigonometric polynomial of degree n. By a polynomial of degree n we will mean the finite sum $\sum_{v=0}^{n} a_v z^v$, where $a_v \in \mathbb{C}$ $(v = 0, 1, ..., n)$.

According to Bernstein's inequality if t is a trigonometric polynomial of degree n such that

$$|t(\theta)| \leqslant 1 \qquad \text{for} \quad \theta \in \mathbb{R} \tag{1}$$

then (for references see [6])

$$|t'(\theta)| \leqslant n \qquad \text{for} \quad \theta \in \mathbb{R}. \tag{2}$$

In (2), equality holds *if and only if*

$$t(\theta) = c_{-n} e^{-in\theta} + c_n e^{in\theta}, \qquad |c_{-n}| + |c_n| = 1.$$

It was shown by van der Corput and Schaake [1] that in the case when $t(\theta)$ is *real for real values of* θ the much stronger conclusion

$$|t'(\theta) \pm int(\theta)| = \sqrt{\{t'(\theta)\}^2 + n^2\{t(\theta)\}^2} \leqslant n \tag{3}$$

* Research supported by La fondation du prêt d'honneur inc.

0021-9045/85 $3.00

holds for all $\theta \in \mathbb{R}$. Inequality (3) is sharp for each θ; in fact, all real trigonometric polynomials of the form

$$t(\theta) = c_{-n} e^{-in\theta} + c_n e^{in\theta} \qquad (c_{-n} = \bar{c}_n, |c_n| = \tfrac{1}{2})$$

are extremal. The example $t(\theta) = e^{\pm in\theta}$ shows that for an arbitrary trigonometric polynomial of degree n the quantity $|t'(\theta) \pm int(\theta)|$ can be as large as $2n$, which is trivially its upper bound.

If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that

$$|p(z)| \leqslant 1 \qquad \text{for} \quad |z| = 1 \tag{4}$$

then $p(e^{i\theta}) = t(e^{i\theta})$, where t is a trigonometric polynomial of degree n satisfying (1) and so

$$|p'(z)| \leqslant n \qquad \text{for} \quad |z| = 1. \tag{5}$$

Here, equality holds if and only if

$$p(z) = a_n z^n \qquad (|a_n| = 1).$$

If $z^n \overline{p(1/\bar{z})} \equiv p(z)$, i.e., $a_k = \bar{a}_{n-k}$ for $0 \leqslant k \leqslant n$, then (for references see [6]) the right-hand side of (5) may be replaced by $n/2$. The question as to what happens if

$$z^n p(1/z) \equiv p(z) \text{ (i.e., } a_k = a_{n-k}) \qquad \text{for} \quad 0 \leqslant k \leqslant n \tag{6}$$

was taken up by Govil, Jain and Labelle [5] but remains unresolved. In [4] we showed that there exists a polynomial of degree n ($\geqslant 2$), namely

$$p(z) = \{(1 - iz)^2 + z^{n-2}(z - i)^2\}/4, \tag{7}$$

satisfying (6) for which

$$\max_{|z|=1} |p'(z)| \geqslant |p'(-i)| = n - 1 \geqslant (n-1) \max_{|z|=1} |p(z)|. \tag{8}$$

This is surprising since (6) is in some sense quite restrictive. It is clear that for a polynomial p satisfying (4) and (6) the sharp upper bound for $|p'(e^{i\theta})|$ would depend not only on n but also on θ. We shall see that for such polynomials

$$|p'(e^{2k\pi i/n})| \leqslant n - 1, \qquad k = 0, 1, ..., n-1, \tag{9}$$

and so the polynomial in (7) happens to be extremal for $\theta = -i$ if $n = 4, 8, 12, ...$. This remains true even if (6) is replaced by the much weaker assumption $a_0 = a_n$. In fact, we prove

THEOREM 1. *Let* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n $(\geqslant 2)$ *satisfying* (4). *If* $a_0=a_n$, *then*

$$|p'(e^{i\theta})| \leqslant (n-1)+|a_0|\,|e^{in\theta}-1| \qquad \text{for} \quad \theta \in \mathbb{R}, \tag{10}$$

and so in particular

$$|p'(e^{i\theta})| \leqslant n-1 \qquad \text{if} \quad e^{in\theta}=1. \tag{10'}$$

Remark. The example

$$p_\omega(z)=\{\omega^{n-2}(\omega-z)^2+z^{n-2}(\omega+z)^2\}/4, \qquad \omega^n=1,$$

shows that in (10′) equality can hold at any of the n-th roots of unity for *all* $n\geqslant 2$.

As a global upper bound for $|p'(e^{i\theta})|$, inequality (10) gives us only the trivial value n. But we will show how it can be used to obtain:

THEOREM 2. *Under the conditions of Theorem* 1 *we have*

$$|p'(z)| \leqslant n-\frac{1}{2}+\frac{1}{2(n+1)} \qquad \text{for} \quad |z|=1. \tag{11}$$

If t is a trigonometric polynomial of degree n then

$$e^{in\theta}t(\theta)=p_1(e^{i\theta}), \qquad e^{-in\theta}t(\theta)=p_2(e^{-i\theta})$$

where p_1 and p_2 are polynomials of degree $2n$. Thus Theorems 1 and 2 readily imply:

COROLLARY 1. *Let* $t(\theta)=\sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$ *be a trigonometric polynomial of degree* n *satisfying* (1). *If* $c_{-n}=c_n$ (*which is the case if for example* t *is a cosine polynomial*), *then*

$$|t'(\theta)\pm int(\theta)| \leqslant 2n-1+2\,|c_n|\,|\sin n\theta| \qquad \text{for} \quad \theta\in\mathbb{R}, \tag{12}$$

and so in particular

$$|t'(k\pi/n)\pm int(k\pi/n)| \leqslant 2n-1, \qquad k=0, 1,..., 2n-1. \tag{13}$$

Further

$$|t'(\theta)\pm int(\theta)| \leqslant 2n-\frac{1}{2}+\frac{1}{2(2n+1)} \qquad \text{for} \quad \theta\in\mathbb{R}. \tag{14}$$

It is easily seen that

$$|t'(k\pi/n)+int(k\pi/n)|=2n-1$$

for the trigonometric polynomial

$$t(\theta)=t_{n,k}(\theta)=e^{-in\theta}\{(1-e^{i(\theta-(k\pi/n))})^2 + e^{2i(k\pi/n)}\,e^{2i(n-1)\theta}(1+e^{i(\theta-(k\pi/n))})^2\}/4$$

which satisfies (1) and for which $c_{-n}=c_n=\frac{1}{4}$. We have

$$|t'(k\pi/n)-int(k\pi/n)|=2n-1$$

for

$$t\colon \theta\mapsto \overline{t_{n,k}(\theta)}.$$

1.2. It was proved by Duffin and Schaeffer [3] that if f is an entire function of exponential type τ satisfying

$$|f(x)|\leqslant 1 \qquad \text{for} \quad x\in\mathbb{R} \tag{15}$$

and is real on the real axis, then

$$|f'(x)\pm i\tau f(x)|\leqslant\tau \qquad \text{for} \quad x\in\mathbb{R}. \tag{16}$$

This result generalizes inequality (3) of van der Corput and Schaake since a trigonometric polynomial $t(\theta)=\sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$ is an entire function of exponential type n of the complex variable θ. A cosine polynomial being an *even* entire function of exponential type one might wonder if Corollary 1 admits an extension to such functions. It turns out that the best possible upper bound is the trivial bound 2τ. To see this let ε be an arbitrary positive number less than τ (there is nothing to prove in the case $\tau=0$) and consider the even entire function

$$f_{\tau,\varepsilon}(z)=e^{-i\tau z}\{(1-ie^{i\varepsilon z})^2+e^{2i(\tau-\varepsilon)z}(e^{i\varepsilon z}-i)^2\}/4$$

which is of exponential type τ and for $x\in\mathbb{R}$

$$\begin{aligned}|f_{\tau,\varepsilon}(x)| &\leqslant \tfrac{1}{4}(|1-ie^{i\varepsilon x}|^2+|e^{i\varepsilon x}-i|^2)\\ &=\tfrac{1}{4}(|e^{i\varepsilon x}+i|^2+|e^{i\varepsilon x}-i|^2)\\ &\leqslant 1.\end{aligned}$$

Further, it is easily checked that

$$\left|f'_{\tau,\varepsilon}\left(\frac{(4k-1)\pi}{2\varepsilon}\right)+i\tau f_{\tau,\varepsilon}\left(\frac{(4k-1)\pi}{2\varepsilon}\right)\right|>2\tau-\varepsilon, \qquad k=0,\,\pm 1,\,\pm 2,\ldots.$$

We have

$$\left|f'\left(\frac{(4k-1)\pi}{2\varepsilon}\right)-i\tau f\left(\frac{(4k-1)\pi}{2\varepsilon}\right)\right|>2\tau-\varepsilon, \qquad k=0,\ \pm 1,\ \pm 2,...$$

for

$$f: z\mapsto \overline{f_{\tau,\varepsilon}(\bar{z})}.$$

1.3. If p is a polynomial of degree n such that

$$|p(x)|\leqslant 1 \qquad \text{for} \quad -1\leqslant x\leqslant 1 \tag{17}$$

then $p(\cos\theta)$ is a cosine polynomial t of degree n satisfying (1) and so as a special case of Corollary 1 we obtain

COROLLARY 2. *Let* $T_n(x)=\cos n \arccos x$ *be the* nth *Chebyshev polynomial of the first kind. If* $p(x)=\sum_{\nu=0}^{n} a_\nu x^\nu$ *is a polynomial of degree* n *satisfying* (17), *then*

$$|np(x)\pm i\sqrt{1-x^2}\,p'(x)| \leqslant 2n-1+\frac{1}{2^{n-1}}|a_n|\sqrt{1-(T_n(x))^2}, \qquad -1\leqslant x\leqslant 1, \tag{18}$$

and so in particular

$$\left|np\left(\cos\frac{k\pi}{n}\right)\pm i\sin\frac{k\pi}{n}\,p'\left(\cos\frac{k\pi}{n}\right)\right| \leqslant 2n-1, \qquad k=0,1,...,n-1. \tag{19}$$

Further

$$|np(x)\pm i\sqrt{1-x^2}\,p'(x)|\leqslant 2n-\frac{1}{2}+\frac{1}{4(n+1)}, \qquad -1\leqslant x\leqslant 1. \tag{20}$$

It is clear from the context that inequality (19) is sharp.

1.4. *A lower bound for* $\max_{|z|=1}|p'(z)|$.

Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n ($\geqslant 2$) such that $a_0=a_n$ and $\max_{|z|=1}|p(z)|=1$. The example $p(z)=z$ shows that for such a polynomial $\max_{|z|=1}|p'(z)|$ *may be as small as* 1. On the other hand, we have

THEOREM 3. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n such that* $|a_0|=|a_n|$ *and* $\max_{|z|=1}|p(z)|=1$, *then*

$$\max_{|z|=1}|p'(z)| \begin{cases} \geqslant 1+\dfrac{n-3}{n+1}|a_0| & \text{if } n\geqslant 3 \\ \geqslant 1 & \text{if } n=2. \end{cases}$$

2. An Interpolation Formula

For the proof of Theorem 1 we need the following

LEMMA 1. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n $(\geqslant 3)$ *then for all real* γ *we have*

$$\begin{aligned} &a_0+((n-1)\,p(z)-zp'(z)+a_n z^n-2a_0)\,e^{i\gamma} \\ &\quad+(zp'(z)-p(z)-2a_n z^n+a_0)\,e^{2i\gamma}+a_n z^n\,e^{3i\gamma} \\ &=\frac{1}{n-2}\,e^{i\gamma}\sin^2(\gamma/2) \\ &\quad\times\sum_{k=1}^{n-2}\frac{e^{-(2k\pi+\gamma)i/(n-2)}}{\sin^2((2k\pi+\gamma)/2(n-2))}\,p(ze^{(2k\pi+\gamma)i/(n-2)}), \end{aligned} \tag{21}$$

with

$$\frac{1}{n-2}\sin^2(\gamma/2)\sum_{k=1}^{n-2}\frac{1}{\sin^2((2k\pi+\gamma)/2(n-2))}=n-2. \tag{22}$$

Proof. Let γ ($\not\equiv 0 \pmod{2\pi}$) be an arbitrary real number. Further, let z be any complex number and consider the integral

$$I(\rho)=\int_{|\zeta|=\rho}F(\zeta)\,d\zeta$$

where

$$F(\zeta)=\frac{p(\zeta)}{(\zeta-z)^2\zeta(\zeta^{n-2}-e^{i\gamma}z^{n-2})}.$$

Clearly

$$I(\rho)\to a_n \qquad \text{as} \quad \rho\to\infty, \tag{23}$$

whereas the residues of F at its poles z, 0 and $ze^{i(\gamma+2k\pi)/(n-2)}$, $k=1,\dots,n-2$, are

$$-\frac{1}{4}\frac{1}{z^n}\frac{e^{-i\gamma}}{\sin^2(\gamma/2)}\{(1-e^{i\gamma})\,zp'(z)-(1-e^{i\gamma})\,p(z)-(n-2)\,p(z)\},$$

$$-\frac{1}{z^n}e^{-i\gamma}\,a_0$$

and

$$-\frac{1}{4}\frac{1}{z^n}\frac{1}{n-2}e^{-i\gamma}\frac{e^{-(2k\pi+\gamma)i/(n-1)}}{\sin^2((2k\pi+\gamma)/2(n-2))}\,p(ze^{(2k\pi+\gamma)i/(n-2)}),\ k=1,\dots,n-2,$$

respectively. Hence by the theorem of residues

$$\begin{aligned}&4a_n z^n e^{i\gamma}\sin^2(\gamma/2)\\ &\qquad -\{(n-2)\,p(z)+(1-e^{i\gamma})\,p(z)-(1-e^{i\gamma})\,zp'(z)\}+4a_0\sin^2(\gamma/2)\\ &\quad =-\frac{1}{n-2}\sin^2(\gamma/2)\\ &\qquad\times\sum_{k=1}^{n-2}\frac{e^{-(2k\pi+\gamma)i/(n-2)}}{\sin^2((2k\pi+\gamma)/2(n-2))}\,p(ze^{(2k\pi+\gamma)i/(n-2)}),\end{aligned}$$

i.e.,

$$\begin{aligned}&(e^{3i\gamma}-2e^{2i\gamma}+e^{i\gamma})\,a_n z^n\\ &\qquad+\{(n-1)\,e^{i\gamma}p(z)-e^{2i\gamma}p(z)-(e^{i\gamma}-e^{2i\gamma})\,zp'(z)\}\\ &\qquad+(e^{2i\gamma}-2e^{i\gamma}+1)\,a_0\\ &\quad=\frac{1}{n-2}e^{i\gamma}\sin^2(\gamma/2)\\ &\qquad\times\sum_{k=1}^{n-2}\frac{e^{-(2k\pi+\gamma)i/(n-2)}}{\sin^2((2k\pi+\gamma)/2(n-2))}\,p(ze^{(2k\pi+\gamma)i/(n-2)})\end{aligned}$$

which is the same as (21). The assumption "$\gamma\not\equiv 0 \pmod{2\pi}$" can obviously be dropped. Formula (21) when applied to z^{n-1} (or to z) readily leads us to the identity (22).

3.1. *Proof of Theorem* 1. If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n satisfying (4), then (21) in conjunction with (22) and a result of van der Corput and Visser [2] implies that

$$|a_0|+|zp'(z)-p(z)-2a_n z^n+a_0|\leqslant n-2,\qquad |z|=1,\ n\geqslant 3.$$

In the case when $a_0 = a_n$, this latter inequality can be written as

$$|a_n z^n| + |zp'(z) - p(z) - 2a_n z^n + a_0| \leqslant n-2, \qquad |z| = 1, n \geqslant 3$$

from which we readily obtain (10) for $n \geqslant 3$.

In the case $n = 2$, $p(z)$ has the form $a_0(z^2+1) + a_1 z$ so that

$$e^{-i\theta} p(e^{i\theta}) = 2a_0 \cos\theta + a_1.$$

Thus

$$|p'(e^{i\theta})| \leqslant |p(e^{i\theta})| + 2\,|a_0 \sin\theta|$$

which gives us the desired estimate.

Proof of Theorem 2. From (10) it readily follows that (11) holds provided $|a_0| \leqslant \frac{1}{4}((n+2/(n+1))$. In case $|a_0| > \frac{1}{4}((n+2)/(n+1))$ we may use the known estimate [7, p. 125]

$$|p'(e^{i\theta})| \leqslant n - \frac{2n}{n+2}\,|a_0|, \qquad \theta \in \mathbb{R},$$

to obtain the desired conclusion.

Proof of Theorem 3. Let

$$P(z) = p(z) - a_0 \qquad \text{and} \qquad Q(z) = z^n\,\overline{P(1/\bar{z})} = \bar{a}_1 z^{n-1} + \bar{a}_2 z^{n-2} + \cdots + \bar{a}_n.$$

From

$$Q(e^{i\theta}) = e^{in\theta}\,\overline{P(e^{i\theta})}, \qquad \theta \in \mathbb{R},$$

it follows that if $|P(e^{i\theta_0})| = \max_{|z|=1} |P(z)| = M$ (say), then

$$|P'(e^{i\theta_0})| \geqslant Mn - |Q'(e^{i\theta_0})|. \tag{24}$$

Further, since Q is a polynomial of degree $n-1$ such that $\max_{|z|=1} |Q(z)| = M$ and $|Q(0)| = |a_n| = |a_0|$, we have

$$|Q'(e^{i\theta})| \leqslant M(n-1) - \frac{2(n-1)}{n+1}\,|a_0|, \qquad \theta \in \mathbb{R}.$$

Thus, we obtain

$$\max_{|z|=1} |P'(z)| \geqslant |P'(e^{i\theta_0})| \geqslant M + \frac{2(n-1)}{n+1}\,|a_0|.$$

This gives us the desired result for $n \geqslant 3$ since $M \geqslant 1 - |a_0|$.

In the case $n=2$ we clearly have

$$\begin{aligned}\max_{|z|=1} |p'(z)| = \max_{|z|=1} |2a_2 z + a_1| &= 2\,|a_2| + |a_1| \\ &= |a_2| + |a_1| + |a_0| \\ &\geqslant \max_{|z|=1} |p(z)| \\ &= 1.\end{aligned}$$

REFERENCES

1. J. G. VAN DER CORPUT AND G. SCHAAKE, Ungleichungen für Polynome und trigonometrische Polynome, *Compositio Math.* **2** (1935), 321–361.
2. J. G. VAN DER CORPUT AND C. VISSER, Inequalities concerning polynomials and trigonometric polynomials, *Nederl. Akad. Wetensch. Proc.* **49**, 383–392 (*Indag. Math.* **8** (1946), 238–247).
3. R. J. DUFFIN AND A. C. SCHAEFFER, Some inequalities concerning functions of exponential type, *Bull. Amer. Math. Soc.* **43** (1937), 554–556.
4. C. FRAPPIER, Q. I. RAHMAN, AND ST. RUSCHEWEYH, New inequalities for polynomials, *Trans. Amer. Math. Soc.*, to appear.
5. N. K. GOVIL, V. K. JAIN, AND G. LABELLE, Inequalities for polynomials satisfying $P(z) \equiv z^n P(1/z)$, *Proc. Amer. Math. Soc.* **57** (1976), 238–242.
6. Q. I. RAHMAN AND G. SCHMEISSER, "Les inégalités de Markoff et de Bernstein," Séminaire de Mathématiques Supérieures, No. 86, Eté, 1981, Les Presses de l'Université de Montréal, Montréal, 1983.
7. ST. RUSCHEWEYH, "Convolutions in Geometric Function Theory," Séminaire de Mathématiques Supérieures, No. 83, Eté, 1981, Les Presses de l'Université de Montréal, Montrèal, 1982.