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# On a discrete norm for polynomials

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ABSTRACT

Given $n+1$ angles $0 \leq \theta_0 < \theta_1 \cdots < \theta_n \leq \pi$, we discuss various extremal problems over the class of polynomials $\mathcal{P}_n$ endowed with the norm

$$|p|_n = \max_{0\leq j\leq n} \left| \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2} \right|.$$



## 1. Introduction

Let $\mathcal{P}_n$ denote the class of polynomials of degree at most $n$ with complex coefficients. To $p(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n$, we associate the polynomial $P(z) = \sum_{k=0}^{n} a_k T_k(z)$ where $T_k$ is the $k$th Chebyshev polynomial [1]. Given $n+1$ angles $0 \leq \theta_0 < \theta_1 < \theta_2 \cdots < \theta_n \leq \pi$, an application of the Lagrange interpolation formula at the $n+1$ nodes

$$-1 \leq \cos(\theta_n) < \cos(\theta_{n-1}) \cdots < \cos(\theta_1) < \cos(\theta_0) \leq 1$$

yields

$$P(\cos\theta) = \sum_{j=0}^{n} L_j(\cos\theta) P(\cos\theta_j), \quad 0 \leq \theta \leq 2\pi, \tag{1.1}$$

where

$$L_j(z) := \frac{W(z)}{(z - \cos\theta_j) W'(\cos\theta_j)} \in \mathcal{P}_n, \quad 0 \leq j \leq n,$$

with

$$W(z) := \prod_{j=0}^{n} \left(z - \cos\theta_j\right) \tag{1.2}$$

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are the fundamental polynomials of this interpolation process. Setting

$$L_j(z) = \sum_{k=0}^{n} a_{k,j} T_k(z) \quad \text{and} \quad \ell_j(z) = \sum_{k=0}^{n} a_{k,j} z^k$$

we readily obtain from (1.1) for all $0 \le \theta \le \pi$

$$p(e^{i\theta}) + p(e^{-i\theta}) = \sum_{j=0}^{n} \left(\ell_j(e^{i\theta}) + \ell_j(e^{-i\theta})\right) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}$$

and, in particular, for any $p \in \mathcal{P}_n$,

$$p(e^{i\theta}) = \sum_{j=0}^{n} \ell_j(e^{i\theta}) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}. \tag{1.3}$$

Therefore, for any linear functional $\mathcal{L}$ over $\mathcal{P}_n$, we have

$$\mathcal{L}(p) = \sum_{j=0}^{n} \mathcal{L}(\ell_j) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}, \quad p \in \mathcal{P}_n. \tag{1.4}$$

Equipping $\mathcal{P}_n$ with the norm

$$|p|_n := \max_{0 \le j \le n} \left| \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2} \right|,$$

we obtain from (1.4) that

$$\max_{p \in \mathcal{P}_n, |p|_n = 1} |\mathcal{L}(p)| \le \sum_{j=0}^{n} |\mathcal{L}(\ell_j)|. \tag{1.5}$$

Indeed, equality holds in (1.5): there exists a polynomial

$$P(z) := \sum_{k=0}^{n} a_k T_k(z) \in \mathcal{P}_n$$

such that

$$P(\cos\theta_j) = \begin{cases} \overline{\mathcal{L}(\ell_j)}/|\mathcal{L}(\ell_j)| & \text{if } \mathcal{L}(\ell_j) \neq 0, \\ \text{any complex number with modulus } \le 1 & \text{if } \mathcal{L}(\ell_j) = 0. \end{cases} \tag{1.6}$$

Then the associated polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ clearly satisfies $|p|_n = 1$ and $\mathcal{L}(p) = \sum_{j=0}^{n} |\mathcal{L}(\ell_j)|$. It is also clear that the extremal polynomials $\widetilde{p} \in \mathcal{P}_n$, i.e., those for which $|\widetilde{p}|_n = 1$ and

$$|\mathcal{L}(\widetilde{p})| = \sum_{j=0}^{n} |\mathcal{L}(\ell_j)|,$$

are fully determined by the interpolation condition (1.6). In particular, if $\mathcal{L}(\ell_j) \neq 0$ for all $0 \le j \le n$, there exists a unique (up to a multiplicative constant of modulus 1) extremal polynomial. Any linear problem over the space $\mathcal{P}_n$ with the discrete norm $|\ |_n$ for an arbitrary system $\{\theta_j : j = 0, \ldots, n\}$ given as above is explicitly solvable.

The discrete norm $|p|_n$ for the choice $\{\theta_j\} = \{j\pi/n\}_{j=0}^{n}$, in combination with the linear functional $\mathcal{L}_\theta$, for $\theta \in [0, \pi]$ fixed and,

$$\mathcal{L}_\theta(p) := \frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}, \quad p \in \mathcal{P}_n,$$

seems to be of particular interest. In this case formula (1.4) becomes

$$\frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{j=0}^{n} c_n(j, \theta) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2} \tag{1.7}$$

with

$$c_n(j, \theta) = \begin{cases} \dfrac{1}{2n} \dfrac{1 - \cos(n\theta)}{1 - \cos(\theta)}, & j = 0, \\[2ex] \dfrac{(-1)^j}{n} \dfrac{\cos(j\pi) - \cos(n\theta)}{\cos(j\pi/n) - \cos(\theta)}, & 1 \le j \le n-1, \\[2ex] \dfrac{1}{2n} \dfrac{-1 + (-1)^n \cos(n\theta)}{1 + \cos(\theta)}, & j = n. \end{cases}$$

This formula has been derived in previous work [2–4] and it has also been shown that

$$\sum_{j=0}^{n} |c_n(j,\theta)| \le n, \quad 0 \le \theta \le \pi. \tag{1.8}$$

The limiting case $\theta \to 0$, when applied to $p(e^{i\varphi}z)$, leads to

$$e^{i\varphi}p'(e^{i\varphi}) = \sum_{j=0}^{n} c_n(j,0)\frac{p(e^{i(\varphi+j\pi/n)}) + p(e^{i(\varphi-j\pi/n)})}{2}, \tag{1.9}$$

and to

$$e^{i\varphi}p'(e^{i\varphi}) - \frac{n}{2}p(e^{i\varphi}) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} c_n(j,0)\frac{p(e^{i(\varphi+j\pi/n)}) + p(e^{i(\varphi-j\pi/n)})}{2}. \tag{1.10}$$

Formula (1.10) has various consequences, some of which are (even improvements of) classical results: for example if $p \in \mathcal{P}_n$ and $p_\varphi(z) = p(e^{i\varphi}z)$, we obtain

$$\left| e^{i\varphi}p'(e^{i\varphi}) - \frac{n}{2}p(e^{i\varphi}) \right| \le \frac{n}{2}|p_\varphi|_n, \quad 0 \le \theta \le 2\pi,$$

which is a refinement of the classical Bernstein inequality for polynomials in the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$.

It can also be shown that (1.10) contains the Marcel Riesz interpolation formula for trigonometric polynomials: if $t(\varphi)$ is a trigonometric polynomial of degree $n$ then $p(e^{i\varphi}) := e^{in\varphi}t(\varphi)$ is in $\mathcal{P}_{2n}$. Formula (1.10), with $n$ replaced by $2n$, and applied to this polynomial $p$ turns out to be nothing but the Marcel Riesz formula.

The famous inequality of Duffin and Schaeffer for the first derivative of polynomials in the interval $[-1, 1]$ also follows from (1.9); see [4]. For matters concerning polynomial inequalities and interpolation formulae, we refer the reader to the book of Rahman and Schmeisser [5].

The norm $|p|_n$ (with $\{\theta_j\} = \{j\pi/n\}_{j=0}^{n}$) naturally should be compared to other norms on $\mathcal{P}_n$, for example to $|p|_{\mathbb{D}} := \sup_{z\in\mathbb{D}} |p(z)|$ or else $\|p\|_n := \max_{0\le j\le 2n-1} |p(e^{ij\pi/n})|$. It is a consequence of a very beautiful theorem of Rakhmanov and Shekhtman [6] (see also [7] for related results) that for some absolute constant $K$ (not depending on $n$)

$$\|p\|_n \le |p|_{\mathbb{D}} \le K\|p\|_n, \quad p \in \mathcal{P}_n.$$

This has recently been extended by Sheil-Small [8] and Dubinin [9]. It is also known [10] that

$$|p'|_{\mathbb{D}} \le n\|p\|_n, \quad p \in \mathcal{P}_n.$$

We shall prove the following three results.

**Theorem 1.1.** *There exists a universal constant $M < \infty$ (independent of $n$) such that for any polynomial $p \in \mathcal{P}_n$ we have*

$$|p|_{\mathbb{D}} \le M \log n \max_{0\le j\le n} \left| \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2} \right|.$$

Again using the case where $\{\theta_j\} = \{j\pi/n\}_{j=0}^{n}$, we obtain inequalities for coefficient functionals.

**Theorem 1.2.** *For any $p(z) := \sum_{k=0}^{n} a_k(p)z^k$, we have*

$$|a_0(p)| \le |p|_n, \qquad |a_n(p)| \le |p|_n$$

*and*

$$|a_1(p)| \le |p|_n \begin{cases} \dfrac{2}{n}\cot\left(\dfrac{\pi}{2n}\right), & n \text{ even} \\ \dfrac{2}{n}\csc\left(\dfrac{\pi}{2n}\right), & n \text{ odd } (\ge 3). \end{cases}$$

*Furthermore, for a fixed integer $k \ge 1$, we have*

$$\lim_{n\to\infty} \max_{\substack{p\in\mathcal{P}_n \\ |p|_n=1}} |a_k(p)| = \frac{4}{\pi}.$$

Our final result is dealing with the coefficients $c_n(j,\theta)$ in the formula (1.7). We have seen that for the system $\{\theta_j\} = \{j\pi/n\}_{j=0}^n$ the relation

$$\sum_{j=0}^{n} |c_n(j,\theta)| \le n, \quad 0 \le \theta \le \pi, \tag{1.11}$$

holds. This is obviously also important for possible variants of the Bernstein inequality. We are interested to which extent (1.11) holds for other node systems $\{\theta_j\}$ as well. We have the following theorem.

**Theorem 1.3.** *Let $n$ be odd and for some set of nodes $\{\theta_j\}_{j=0}^n$ assume*

$$\max\left\{\sum_{j=0}^{n} |c_n(j,0)|, \sum_{j=0}^{n} |c_n(j,\pi)|\right\} \le n. \tag{1.12}$$

*Then $\{\theta_j\}_{j=0}^n = \{j\pi/n\}_{j=0}^n$ and (1.11) holds.*

We have numerical evidence that this result is not true for $n$ even. There is also evidence that one cannot replace the condition (1.12) by the weaker one

$$\sum_{j=0}^{n} |c_n(j,0)| \le n$$

to guarantee the validity of the conclusion of Theorem 1.3.

Theorem 1.3 is reminiscent of a result of Duffin and Schaeffer [11] (see also [5, pp. 574–576] for a detailed proof).

## 2. Proof of Theorem 1.1

We shall use the notation

$$\sum_{j=0}^{n}{}'' \alpha_j = \frac{\alpha_0}{2} + \sum_{j=1}^{n-1} \alpha_j + \frac{\alpha_n}{2}.$$

Then, it has been established in [4] that (compare with (1.3))

$$p(e^{i\theta}) = \sum_{j=0}^{n}{}'' \lambda_j(\theta) \left(\frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}\right)$$

where for $0 \le j \le n$

$$\lambda_j(\theta) = \frac{i(-1)^j \sin(\theta)(-e^{in\theta} + (-1)^j)}{n(\cos(j\pi/n) - \cos(\theta))}$$

and using (1.5), (1.6)

$$\begin{aligned} M(n,\theta) &:= \max_{p\in\mathcal{P}_n, |p|_n=1} |p(e^{i\theta})| = \sum_{j=0}^{n}{}'' |\lambda_j(\theta)| \\ &= \frac{2}{n} \sum_{j=0}^{n}{}'' \frac{|\sin(\theta)|\,|\sin(n\theta/2 + j\pi/2)|}{\left|\cos(\theta) - \cos(j\pi/n)\right|}. \end{aligned}$$

Since the functions $\lambda_j$ are $2\pi$-periodic and fulfil

$$|\lambda_j(-\theta)| = |\lambda_j(\pi - \theta)| = |\lambda_{n-j}(\theta)|, \quad j = 0, \ldots, n,$$

we find

$$M(n, \theta + 2\pi) = M(n, -\theta) = M(n, \pi - \theta) = M(n, \theta), \quad \theta \in \mathbb{R},$$

so that we can restrict our attention to the range $0 \le \theta \le \pi/2$. We look for upper bounds for the terms in the last sum. When $j = 0$,

$$\frac{1}{n} \frac{|\sin(\theta)|\,|\sin(n\theta/2)|}{|1 - \cos\theta|} = \frac{1}{n}\left|\frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{\theta}{2}\right)\right| \le \frac{1}{n}\left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right| \le 1.$$

When $j = n$,

$$\frac{1}{n}\frac{|\sin(\theta)|\,|\sin(n\theta/2 + n\pi/2)|}{|1-\cos\theta|} = \frac{1}{2n}\frac{|\sin(\theta)|\,|\sin(n\theta/2+\pi/2)|}{|\cos^2(\theta/2)|} \le \frac{1}{2n\cos^2(\theta/2)} \le \frac{1}{n}$$

and for $1 \le j \le n-1$, the terms are

$$\frac{2}{n}\frac{|\sin(\theta)|\,|\sin(n\theta/2 - j\pi/2)|}{|\cos(\theta)-\cos(j\pi/n)|} = \frac{1}{n}\frac{|\sin(\theta)|}{|\sin(j\pi/(2n)+\theta/2)|}\frac{|\sin(n(\theta-j\pi/n)/2)|}{|\sin((\theta-j\pi/n)/2)|}$$

while

$$\begin{aligned}\left|\frac{\sin(\theta)}{\sin(j\pi/(2n)+\theta/2)}\right| &= \frac{|\sin(\theta/2)|}{|\sin(j\pi/(2n)+\theta/2)|}\frac{|\sin(\theta)|}{|\sin(\theta/2)|}\\ &\le 2\frac{|\sin(\theta/2)|}{|\sin(j\pi/(2n)+\theta/2)|}\\ &= 2\frac{\sin(\theta/2)}{\sin(j\pi/(2n)+\theta/2)} \qquad (2.1)\end{aligned}$$

because $\pi/(2n) < j\pi/(2n) + \theta/2 < \pi$ if $1 \le j \le n-1$.

Remark further that

$$\frac{d}{d\theta}\frac{2\sin(\theta/2)}{\sin(j\pi/(2n)+\theta/2)} = \frac{\sin(j\pi/n)}{\sin^2(j\pi/n+\theta/2)} \ge 0$$

and therefore by (2.1) and $\pi/4 < j\pi/2n + \pi/4 < 3\pi/4$

$$\left|\frac{2\sin(\theta/2)}{\sin(j\pi/(2n)+\theta/2)}\right| \le \frac{2\sin(\pi/4)}{\sin(j\pi/(2n)+\pi/4)} \le 2.$$

This leads to

$$M(n,\theta) \le 1 + \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1}\left|\frac{\sin(n(\theta-j\pi/n)/2)}{\sin((\theta-j\pi/n)/2)}\right|.$$

We now write $\theta = s\pi/n + \varepsilon\pi/n$ where $s$ is an integer, $0 \le s < n-1$ and $-1/2 \le \varepsilon < 1/2$. Then

$$\begin{aligned}M(n,\theta) &\le 1 + \frac{1}{n} + \frac{2}{n}\frac{|\sin(n(\theta-s\pi/n)/2)|}{|\sin((\theta-s\pi/n)/2)|} + \frac{2}{n}\sum_{\substack{j=1\\ j\ne s}}^{n-1}\frac{|\sin(n(\theta-j\pi/n)/2)|}{|\sin((\theta-j\pi/n)/2)|}\\ &\le 3 + \frac{1}{n} + \frac{2}{n}\sum_{\substack{j=1\\ j\ne s}}^{n-1}\frac{|\sin(n(\theta-j\pi/n)/2)|}{|\sin((\theta-j\pi/n)/2)|}\end{aligned}$$

and given our representation we also have for any $1 \le j \le n-1$,

$$\left|\frac{\theta - j\pi/n}{2}\right| \le \frac{\pi}{2} \quad \text{so that } \frac{|\sin(\theta/2 - j\pi/(2n))|}{|\theta/2 - j\pi/(2n)|} \ge \frac{2}{\pi}.$$

Therefore

$$\begin{aligned}M(n,\theta) &\le 3 + \frac{1}{n} + \frac{2}{n}\sum_{\substack{j=1\\ j\ne s}}^{n-1}\frac{1}{|\sin((\theta-j\pi/n)/2)|}\\ &\le 3 + \frac{1}{n} + \frac{2\pi}{n}\sum_{\substack{j=1\\ j\ne s}}^{n-1}\frac{1}{|\theta-j\pi/n|}\\ &= 3 + \frac{1}{n} + 2\sum_{\substack{j=1\\ j\ne s}}^{n-1}\frac{1}{|s-j+\varepsilon|}\end{aligned}$$

$$\leq 3+\frac{1}{n}+2\sum_{\substack{j=1\\ j\neq s}}^{n-1}\frac{1}{|s-j|-\varepsilon}$$

$$\leq 3+\frac{1}{n}+8\sum_{\substack{j=1\\ j\neq s}}^{n-1}\frac{1}{2j-1}$$

$$\leq 3+\frac{1}{n}+8\sum_{j=1}^{2n}\frac{1}{j}$$

$$\leq 3+\frac{1}{n}+8\gamma+8\ \log(2n+1),$$

where $\gamma$ is the Euler constant. □

The order of growth $O(\log\ n)$ is sharp since, for odd $n$, the Riemann sums $M(n,\pi/2)$ satisfy

$$M\left(n,\frac{\pi}{2}\right)\geq\frac{2\sqrt{2}}{\pi}\log(n)+O(1).$$

**Remark.** Applying the classical Bernstein inequality to Theorem 1.1 yields

$$|p'|_{\mathbb{D}}\leq Mn\log(n)\,|p|_n,\quad p\in\mathscr{P}_n,$$

with the same constant $M$ as in Theorem 1.1.

## 3. Proof of Theorem 1.2 and related remarks

In this section we always assume $\{\theta_j\}=\{j\pi/n\}_{j=0}^{n}$.

**Proof of Theorem 1.2.** In this case we have (see (1.2))

$$W(z)=-\frac{(1-z^2)T_n'(z)}{n2^{n-1}}$$

and a simple computation together with (1.4) gives

$$a_k=\begin{cases}\dfrac{2}{n}\displaystyle\sum_{j=0}^{n}{}''\cos\left(\frac{kj\pi}{n}\right)\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}, & 1\leq k\leq n-1\\ \dfrac{1}{n}\displaystyle\sum_{j=0}^{n}{}''\cos\left(\frac{kj\pi}{n}\right)\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}, & k=0,n\end{cases}$$

for any polynomial $p(z)=\sum_{k=0}^{n}a_kz^k\in\mathscr{P}_n$. As explained in the introduction we obtain

$$\max_{p\in\mathscr{P}_n,|p|_n=1}|a_0|=\max_{p\in\mathscr{P}_n,|p|_n=1}|a_n|=1$$

with the maximum attained respectively only if $p(z)\equiv a_0$ or $p(z)\equiv a_nz^n$. Similarly, for $n>1$, elementary computations lead to

$$\max_{p\in\mathscr{P}_n,|p|_n=1}|a_1|=\begin{cases}\dfrac{2}{n}\cot\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is even,}\\ \dfrac{2}{n}\csc\left(\dfrac{\pi}{2n}\right) & \text{if } n\geq 3 \text{ is odd}\end{cases}$$

and there will be essentially only one extremal polynomial when $n$ is odd but many more when $n$ is even. For $k\geq 1$ fixed, we have

$$\lim_{n\to\infty}\max_{p\in\mathscr{P}_n,|p|_n=1}|a_k|=2\int_0^1|\cos(k\pi x)|\mathrm{d}x=\frac{4}{\pi}.\quad\square$$

We end this section by looking at a particular extremal problem and set for $n\geq 1$,

$$m_n=\max_{p\in\mathscr{P}_n,|p|_{\mathbb{D}}=1}|a_0+a_1|=\max_{p\in\mathscr{P}_n,\ |p|_{\mathbb{D}}=1}(|a_0|+|a_1|).$$

Not much seems to be known about the sharp size of $m_n$. Rahman [10] has shown that

$$m_n \le \frac{2}{n+1}\cot\left(\frac{\pi}{2(n+1)}\right) \tag{3.1}$$

but the equality cannot hold for infinitely many values of $n$ since by a simple application of the Schwarz lemma we have

$$\sup_f |a_0 + a_1| = \frac{5}{4} < \frac{4}{\pi} = \lim_{n\to\infty} \frac{2}{n+1}\cot\left(\frac{\pi}{2(n+1)}\right)$$

where the sup is taken over all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, holomorphic in $\mathbb{D}$ with $|f|_{\mathbb{D}} = 1$. It is also known [12] that

$$m_n = \inf_Q \sum \left|\text{res}\frac{1}{Q}\right|$$

where the inf is taken over all polynomials $Q(z) = \prod_{j=1}^{n}(1 - e^{i\varphi_j} z)$ with $\varphi_j$ real and $|Q'(0)| = 1$ and the sum is taken over all residues of each such $1/Q$. The choice $Q(z) = (1 - z^{n+1})/(1 - z)$ leads again to (3.1).

The more recent remark [13] is that for any $p(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n$

$$|a_0| + |a_1| \le \max_{|z|=2\cos(\pi/n+2)} |p(z)|.$$

Unfortunately, none of the above results seems to yield an explicit sharp estimate concerning $m_n$ for large values of $n$. The approach in this paper yields an interesting alternative:

$$\mu_n := \max_{p\in\mathcal{P}_n, |p|_n=1} |a_0 + a_1| = \frac{1}{n}\sum_{j=0}^{n}\left|1 + 2\cos\left(\frac{j\pi}{n}\right)\right|$$

with

$$\lim_{n\to\infty} \mu_n = \int_0^1 |1 + 2\cos(\pi x)|\mathrm{d}x + \frac{1}{3} + \frac{2\sqrt{3}}{\pi} = 1.4359\ldots.$$

Remark however that $5/4 < 4/\pi < 1/3 + 2\sqrt{3}/\pi$. It is also not clear whether or not

$$\max_{p\in\mathcal{P}_n, |p|_n=1} (|a_0| + |a_1|) = \max_{p\in\mathcal{P}_n, |p|_n=1} |a_0 + a_1|.$$

## 4. Proof of Theorem 1.3

Suppose we are given a set of angles $\{\theta_j\}$ with $n$ odd and with

$$\max\left\{\sum_{k=0}^{n}|c_n(k,0)|, \sum_{k=0}^{n}|c_n(k,\pi)|\right\} \le n. \tag{4.1}$$

We shall prove that $\{\theta_j\} = \{j\pi/n\}_{j=0}^{n}$. Let first $p(z) \equiv z^n$ in (1.6) with $\theta = 0$. We then have

$$n = \sum_{k=0}^{n} c_n(k,0)\cos(n\theta_k) = (-1)^{n-1} n \sum_{k=0}^{n} c_n(k,\pi)\cos(n\theta_k)$$

and by (4.1) for both, $\theta = 0$ and $\theta = \pi$,

$$\begin{aligned} n &= \left|\sum_{k=0}^{n} c_n(k,\theta)\cos(n\theta_k)\right| \le \sum_{k=0}^{n} |c_n(k,\theta)|\,|\cos(n\theta_k)| \\ &\le \sum_{k=0}^{n} |c_n(k,\theta)| \le n. \end{aligned}$$

It follows that the above equality holds everywhere and in particular for each $0 \le k \le n$

$$|c_n(k,0)| + |c_n(k,\pi)| > 0 \Longrightarrow \cos(n\theta_k) = \pm 1 \Longrightarrow \theta_k = \frac{\ell_k \pi}{n}, \tag{4.2}$$

where $\ell_k$ is an integer, $0 \le \ell_k \le n$. We now write the identity (1.6) as

$$\frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{k\in S} c_n(k,\theta)\frac{p(e^{i\theta_k}) + p(e^{-i\theta_k})}{2} + \sum_{k\in T} c_n(k,\theta)\frac{p(e^{i\theta_k}) + p(e^{-i\theta_k})}{2} \tag{4.3}$$

for all $p \in \mathcal{P}_n$, $0 \le \theta \le \pi$, where

$$T = \left\{ k \mid 0 \le k \le n \text{ and } \theta_k \ne \frac{j\pi}{n} \ \forall j \in \{0, 1, \ldots, n\} \right\}$$

and

$$S = \left\{ k \mid 0 \le k \le n \text{ and } \theta_k = \frac{\ell_k \pi}{n} \text{ for some } \ell_k \in \{0, 1, \ldots, n\} \right\}.$$

By (4.2), $c_n(k, 0) = c_n(k, \pi) = 0$ for each $k \in T$. We shall now assume that $T$ is non-empty, i.e., there exists $\widetilde{k}$ with $0 \le \widetilde{k} \le n$ such that $\theta_{\widetilde{k}}$ is not an integer multiple of $\pi/n$. Then the cardinality of $S$ is at most $n$ and there exists $v$, $0 \le v \le n$ such that $\theta_k \ne v\pi/n$ for all $k \in S$.

Next we define

$$P(z) := \frac{(1 - z^2)T_n'(z)}{z - \cos(v\pi/n)} := \sum_{j=0}^{n} \alpha_{j,v} T_j(z) \in \mathcal{P}_n$$

and the associated $p(z) = \sum_{j=0}^{n} \alpha_{j,v} z^j$. As before we have

$$P(\cos\theta) = \frac{p(e^{i\theta}) + p(e^{-i\theta})}{2}$$

and in particular for all $k \in S$, $p(e^{i\theta_k}) + p(e^{-i\theta_k})/2 = 0$. It follows from (4.3) that

$$\frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{k\in T} c_k(\theta) \frac{p(e^{i\theta_k}) + p(e^{-i\theta_k})}{2}$$

and, taking the limits $\theta \to 0$ and $\theta \to \pi$,

$$p'(1) = \sum_{k\in T} c_n(k, 0) \frac{p(e^{i\theta_k}) + p(e^{-i\theta_k})}{2} = 0,$$

and

$$p'(-1) = \sum_{k\in T} c_n(k, \pi) \frac{p(e^{i\theta_k}) + p(e^{-i\theta_k})}{2} = 0.$$

What remains is to calculate the values $p'(1)$ and $p'(-1)$ explicitly. We have

$$P(\cos\theta) = \frac{\sin^2(\theta) T_n'(\cos\theta)}{\cos(\theta) - \cos(v\pi/n)} = \frac{n\sin(\theta)\sin(n\theta)}{\cos(\theta) - \cos(v\pi/n)}$$

and using $z = e^{i\theta}$, we get

$$\begin{aligned} P(\cos\theta) &= -\frac{n}{2z^n} \frac{(1 - z^2)(1 - z^{2n})}{(1 - ze^{iv\pi/n})(1 - ze^{-iv\pi/n})} \\ &= 2n(-1)^{v+1} \sum_{k=0}^{n}{}'' \cos\left(k\frac{v\pi}{n}\right)\cos(k\theta), \end{aligned} \tag{4.4}$$

hence

$$p(z) = 2n(-1)^{v+1} \sum_{k=0}^{n}{}'' \cos\left(k\frac{v\pi}{n}\right) z^k.$$

This implies

$$p'(1) = \begin{cases} -n^3, & v = 0, \\ \dfrac{n((-1)^{v+1} + 1)}{\cos(\pi v/n) - 1}, & 0 < v \le n, \end{cases}$$

which can be 0 only if $v$ is an even number, and

$$p'(-1) = \begin{cases} \dfrac{n((-1)^n - (-1)^v)}{\cos(\pi v/n) + 1}, & 0 \le v < n, \\ (-1)^n n^3, & v = n, \end{cases}$$

which, under the assumption $v$ even, can be zero only if $n$ is even as well. This contradiction completes the proof. □

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