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Estimates for the uniform norm of complex polynomials in the unit disk

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Dedicated to the memory of Erhard Schmidt

Let $\|\cdot\|$ denote the uniform norm in the unit disk of the complex plane $\mathbb{C}$. The main result in this note is as follows: *For any complex polynomial P of degree at most n and any $\alpha \in \mathbb{C}$ the inequality*

$$\|P\| \leq (n+1)(\|z\,P(z)+\alpha\| - |\alpha|)$$

holds. For any $\alpha \neq 0$ the factor $n+1$ is best possible, and we determine the cases of equality.

1 Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. $\mathcal{P}_n$ denotes the set of complex polynomials of degree at most n and $\|\cdot\|$ stands for the uniform norm in $\mathbb{D}$: $\|P\|$ is the maximum modulus of the polynomial P over the boundary of the unit disk. The main result in this note is the following theorem.

Theorem 1.1 *For $P \in \mathcal{P}_n$ and $\alpha \in \mathbb{C}$ we have*

$$\|P\| \leq (n+1)\left(\|z\,P(z)+\alpha\| - |\alpha|\right). \tag{1.1}$$

For no n and no α the constant $n+1$ can be replaced by anything smaller without violating the conclusion. On the other hand, the only polynomial for which we have equality in (1.1) *is $P \equiv 0$.*

This result of course is equivalent with

$$\|Q - Q(0)\| < n(\|Q\| - |Q(0)|)$$

for every non-constant polynomial $Q \in \mathcal{P}_n$, $n \geq 2$. As we will see, Theorem 1.1 is a simple consequence of Corollary 1.3 to the following result.

Theorem 1.2 *For $n \geq 2$ and $x \in \mathbb{C}$ let*

$$H_n(x) := \begin{pmatrix} 1 & x & x & \dots & x \\ \overline{x} & 1 & x & \cdots & x \\ \overline{x} & \overline{x} & 1 & \cdots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{x} & \overline{x} & \overline{x} & \cdots & 1 \end{pmatrix}$$

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be a Hermitian $n \times n$ matrix. Let $\Delta_n \subset \mathbb{C}$ be the set of those x for which $H_n(x)$ is positive semi-definite. Then Δ_n is the closed convex hull of the Jordan curve

$$\left\{ -e^{i\varphi} \frac{\sin \frac{\varphi}{n}}{\sin\left(\frac{n-1}{n}\varphi\right)} \; : \; |\varphi| \le \pi \right\}. \tag{1.2}$$

Note that Theorem 1.2 contains the solution to a problem raised in the American Math. Monthly by one of us [2]. See Fig. 1 for typical cases of the shape of $\partial\Delta_n$.

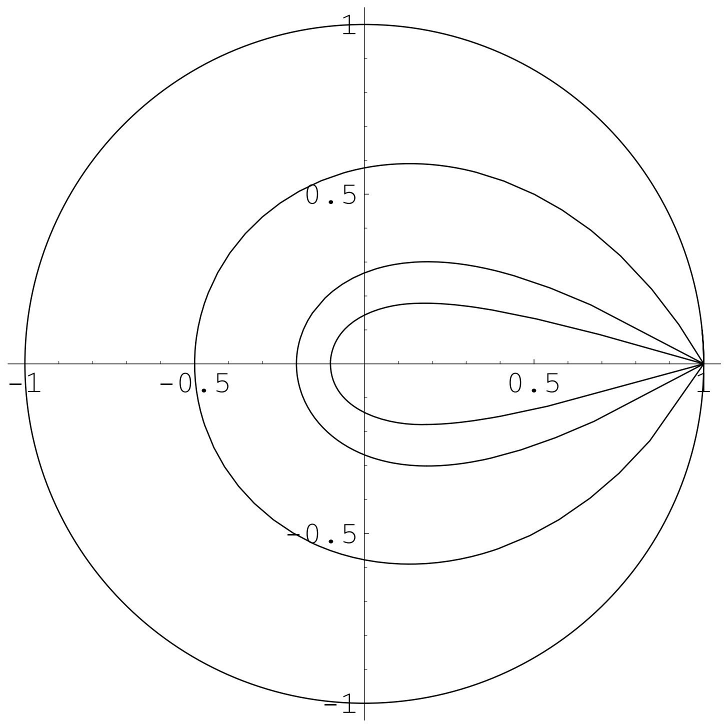

Fig. 1 $\partial\Delta_n$ for $n = 2, 3, 6, 11$

In the sequel we write $D_k(x) := \det H_k(x)$.

Corollary 1.3 *For all $n \ge 2$ we have $1 \in \partial\Delta_n$. Furthermore, for all $x \in \partial\Delta_n \setminus \{1\}$ we have $D_n(x) = 0$ and $D_k(x) > 0,\ k = 2, \ldots, n-1$.*

Corollary 1.4 *Let $n \ge 2$. The set Δ_n contains the disk $\left\{x \, : \, |x| \le \frac{1}{n-1}\right\}$, but no larger one centered at the origin.*

The proof of Theorem 1.2 will be given in Section 2, followed by the proof of Theorem 1.1 in Section 3.

2 Proof of Theorem 1.2

Let $\mathcal{R}_n$ be the set of all x such that $H_n(x)$ is positive definite. Then $\mathcal{R}_n$ is an open convex set in $\mathbb{C}$, and Δ_n is the closure of $\mathcal{R}_n$. Note that, in particular, $x = 1$ belongs to $\partial\Delta_n$ for all $n \ge 2$.

Lemma 2.1 *We have $\mathcal{R}_2 = \mathbb{D}$ and*

$$\mathcal{R}_n = \left\{ \frac{y}{1-\overline{y}} \; : \; \mathrm{Re}\,(y) < \frac{1}{2} \text{ and } y \in \mathcal{R}_{n-1} \right\}, \quad n \ge 3.$$

P r o o f. $\mathcal{R}_2 = \mathbb{D}$ is obvious. Set

$$T(x) := \frac{x - |x|^2}{1 - |x|^2}, \quad x \in \mathbb{D},$$

$T_0(x) := x$ and $T_k(x) := T(T_{k-1})(x),\ x \in \mathbb{D},\ k \in \mathbb{N}$. We have $D_2(x) = 1 - |x|^2$. Observe also that

$$D_n(x) = \left(1 - |x|^2\right)^{n-1} D_{n-1}(T(x)).$$

To see this, just multiply the last row of the matrix $H_n(x)$ by x, substract it from the $n-1$ other rows, and then factor out $1 - |x|^2$ in each of these $n-1$ rows. From this recurrence relation we arrive at the representation

$$D_n(x) = \prod_{k=0}^{n-2} \left(1 - |T_k(x)|^2\right)^{n-k-1}, \quad n \geq 2.$$

This implies that for $n \geq 2$ we have $x \in \mathcal{R}_n$ if and only if $|T_k(x)| < 1,\ 0 \leq k \leq n-2$.

Note that $T(\mathbb{D}) = \left\{y : \operatorname{Re} y < \frac{1}{2}\right\}$ and that T is injective in $\mathbb{D}$ with $T^{(-1)}(y) = y/(1-\overline{y})$. From this it is immediate that $x \in \mathcal{R}_n$ if and only if $y/(1-\overline{y}) := T^{(-1)}(x) \in \mathcal{R}_{n-1}$, and the assertion of Lemma 2.1 follows. □

We now prove the theorem. Formula (1.2) for $\partial\Delta_n$ is obvious for $n = 2$. Assume it to be true for some $n \geq 2$. Then, for $|\varphi| \leq \pi$, we have

$$y = -e^{i\varphi} \frac{\sin \frac{\varphi}{n}}{\sin\left(\frac{n-1}{n}\varphi\right)} \in \partial\mathcal{R}_n,$$

and $\operatorname{Re} y < \frac{1}{2}$ if and only if $|\varphi| \leq \frac{n\pi}{n+1}$. Furthermore a little calculation yields

$$\frac{y}{1-\overline{y}} = \frac{\dfrac{-e^{i\varphi}\sin\left(\frac{\varphi}{n}\right)}{\sin\left(\frac{(n-1)\varphi}{n}\right)}}{1 + \dfrac{e^{-i\varphi}\sin\left(\frac{\varphi}{n}\right)}{\sin\left(\frac{(n-1)\varphi}{n}\right)}} = \frac{-e^{i(n+1)\varphi/n}\sin\left(\frac{\varphi}{n}\right)}{\sin\left(\frac{(n+1)\varphi}{n}\right)}, \quad |\varphi| \leq \frac{n\pi}{n+1}.$$

The right-hand side of this relation, after a substitution $\varphi \mapsto n\varphi/(n+1)$ coincides with our representation of the boundary points of $\mathcal{R}_{n+1}$.

This proves (1.2). □

P r o o f o f C o r o l l a r y 1.3. The first part of this corollary is obvious from Theorem 1.2. The relation

$$\Delta_n \setminus \{1\} \subset \mathcal{R}_{n-1}, \quad n \geq 2, \tag{2.1}$$

is most easily verified by taking into account the starlikeness of Δ_n with respect to the origin and a comparison of the points of $\partial\mathcal{R}_n$ and $\partial\mathcal{R}_{n-1}$ on any ray emanating from the origin. □

Corollary 1.4 follows immediately from the representation (1.2).

3 Proof of Theorem 1.1

3.1 First part of the proof

By the Carathéodory–Toeplitz theorem it is well-known that to any positive semi-definite Toeplitz $(n+1) \times (n+1)$-matrix with the first row reading $(c_0, c_1, \ldots, c_n)$ with $c_0 = 1$ there exists a function

$$f(z) = \sum_{k=0}^{\infty} c_k\, z^k = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1 - \zeta z},$$

where μ is a Borel probability measure on $\partial\mathbb{D}$. Hence f is analytic in $\mathbb{D}$ and for any $Q(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n$ we find

$$\|(Q * f)\| = \left\| \sum_{k=0}^{n} a_k c_k z^k \right\| = \left\| \int_{\partial\mathbb{D}} Q(\zeta z)\, d\mu(\zeta) \right\| \leq \|Q\|,$$

i.e. the convolution (Hadamard product) of f with any $P \in \mathcal{P}_n$ does not increase the norm. Hence, in view of Theorem1.2, we have for any $x \in \Delta_{n+1}$ and any polynomial $Q \in \mathcal{P}_n$ the inequality

$$|Q(0) + x(Q(z) - Q(0))| \leq \|Q\|, \quad z \in \mathbb{D}. \tag{3.1}$$

Next, since this is true for every x with $|x| = \frac{1}{n}$ and every $z \in \mathbb{D}$, we can conclude that indeed

$$\frac{1}{n}|Q(z) - Q(0)| + |Q(0)| \leq \|Q\|, \quad z \in \mathbb{D},$$

and therefore

$$\|Q - Q(0)\| \leq n(\|Q\| - |Q(0)|). \tag{3.2}$$

In part 2 of this proof we will show that equality in (3.2) can only hold for constant polynomials Q when $n \geq 2$. We keep this in mind to be used after the following reformulation of (3.2). Let $P \in \mathcal{P}_n$ and $\alpha \in \mathbb{C}$ be arbitrary. Set $Q(z) = zP(z) + \alpha$, so that $Q \in \mathcal{P}_{n+1}$ with $Q(0) = \alpha$. Then (3.2) with n replaced by $n+1$ gives

$$\|zP\| \leq (n+1)(\|zP(z) + \alpha\| - |\alpha|),$$

which is (1.1). Clearly, equality can only hold if $Q \equiv$const., which is equivalent to $P \equiv 0$.

3.2 Second part of the proof

Here we prove the claim made in the last paragraph of the previous subsection, actually in a much more general setting.

Theorem 3.1 *Let H be a Toeplitz $(n+1) \times (n+1)$-matrix with $n > 1$, its first row $(c_0, c_1, \ldots, c_n)$, with $c_0 = 1$ and the following properties*

1. $|c_n| < 1$.
2. $\det H = 0$,
3. *All principal minors of H of order $\leq n$ are positive.*

Let $P(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$ be such that

$$\left\| \sum_{k=0}^{n} a_k c_k z^k \right\| = \|P\|.$$

Then $P \equiv$ const.

For the proof of this result we invoke two classical results.

Lemma 3.2 (Carathéodory–Fejér–Toeplitz, cf. [1, Sections 9.3 and 9.5]) *Let H be as in Theorem 3.1. Then there exists a unique function $f(z)$ of the form*

$$f(z) = \sum_{k=1}^{n} \frac{\lambda_k}{1 - \zeta_k z},$$

with $\lambda_k > 0,\ \sum_{k=1}^{n} \lambda_k = 1$ and $\zeta_k \in \partial\mathbb{D}$ pairwise distinct, such that

$$f(z) = 1 + \sum_{k=1}^{n} c_k z^k + \mathcal{O}\big(z^{n+1}\big)$$

at the origin.

The following lemma is the so-called Visser inequality [3]. We present a proof which also settles the cases of equality in this inequality.

Lemma 3.3 (Visser's inequality) *Let* $P(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$. *Then* $|a_0| + |a_n| \le \|P\|$, *and equality holds if and only if* $P(z) = a_0 + a_n z^n$.

Proof. We have

$$1 + z^n = \frac{1}{1 - z^n} + \mathcal{O}\big(z^{n+1}\big) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 - \gamma_k z} + \mathcal{O}\big(z^{n+1}\big)$$

with $\gamma_k = e^{2\pi i k/n},\ k = 1, \ldots, n$. Therefore

$$|a_0 + a_n z^n| = \left| P(z) * \frac{1}{1 - z^n} \right| = \left| \frac{1}{n} \sum_{k=1}^{n} P(\gamma_k z) \right| \le \frac{1}{n} \sum_{k=1}^{n} |P(\gamma_k z)| \le \|P\|,$$

which proves the inequality. In the case of equality choose $u \in \partial\mathbb{D}$ such that

$$|a_0 + a_n u^n| = |a_0| + |a_n| = \|P\|.$$

This implies $P(\gamma_k u) = c\|P\|,\ k = 1, \ldots, n$ for some $|c| = 1$. But then, for some constant d,

$$P(z) = c\|P\| + d \prod_{k=1}^{n} (1 - \overline{\gamma_k u} z) = c\|P\| + d(1 - (\overline{u} z)^n) = A + B z^n,$$

which is the assertion. □

Proof of Theorem 3.1. As in the previous proof we have for some suitable $u \in \partial\mathbb{D}$

$$\|P\| = \left| \sum_{k=0}^{n} a_k c_k u^k \right| = |(P * f)(u)| = \left| \sum_{k=1}^{n} \lambda_k P(\zeta_k u) \right| \le \sum_{k=1}^{n} \lambda_k |P(\zeta_k u)| \le \|P\|, \tag{3.3}$$

so that also here $P(\zeta_k u) = c\|P\|$ for some constant c and all k. Hence

$$\begin{aligned} P(z) &= c\|P\| + d \prod_{k=1}^{n} (1 - \overline{\zeta_k u} z) \\ &= (c\|P\| + d) + \cdots + \left(d \prod_{k=1}^{n} (-\overline{\zeta_k u}) \right) z^n. \end{aligned}$$

By Visser's inequality applied to P we now obtain

$$|c\|P\| + d| + \left| d \prod_{k=1}^{n} (-\overline{\zeta_k u}) \right| \le \|P\|,$$

and therefore

$$|c\|P\| + d| \le \|P\| - |d|,$$

which is only possible if $|c\|P\| + d| = \|P\| - |d|$ so that equality holds in our application of Visser's inequality. This shows that $P(z) = a_0 + a_n z^n$, but then, assuming $a_n \ne 0$, we have as in (3.5)

$$\|P\| = |(P * F)(u)| = |a_0 + c_n a_n u^n| < |a_0| + |a_n| = \|P\|,$$

a contradiction. Hence $a_n = 0$ and $P \equiv$ const.

We now deal with the case of equality in (3.2) when $n > 1$. Assume that Q is not constant but that equality holds in (3.2). Clearly this cannot happen if $Q(0) = 0$. Choose $\zeta \in \partial\mathbb{D}$ so that $Q(\zeta) - Q(0) = \|Q(z) - Q(0\|$ and let

$$\varphi = \left(\arg \frac{Q(0)}{Q(\zeta) - Q(0)} \mod (2\pi) \right) - \pi.$$

Then

$$x := -e^{i\varphi}\frac{\sin\frac{\varphi}{n+1}}{\sin\left(\frac{n}{n+1}\right)\varphi} \in \partial\Delta_n,$$

by Theorem 1.2. Thus, for our polynomial Q we get

$$|Q(0)| + |x||Q(\zeta) - Q(0)| = |Q(0) + x(Q(\zeta) - Q(0))| \le \|Q\|.$$

Now, if $\varphi \neq 0$ we have $|x| > \frac{1}{n}$, and therefore equality cannot hold in (3.2). This leaves us with the only possibility that $\varphi = 0$ and $x = -\frac{1}{n}$. In this case we indeed have

$$|Q(0) + x(Q(\zeta) - Q(0))| = |Q(0)| + \frac{1}{n}\|Q - Q(0)\| = \|Q\|,$$

and for this x Theorem 3.1 applies: Q must be constant. This completes this part of the proof of Theorem 1.1.

3.3 Third part of the proof

It is a consequence of Corollary 1.4 that the constant $n+1$ in (1.1) is best possible. Here we show with an explicit example that this is indeed the case. Without loss of generality we assume $\alpha > 0$. The Fejér polynomial of degree $n+1$

$$F_{n+1}(z) := \sum_{k=0}^{n+1} \frac{n+2-k}{n+2} z^k \tag{3.4}$$

satisfies

$$\operatorname{Re} F_{n+1}(e^{i\theta}) = \frac{1}{2} + \frac{1}{2(n+1)}\frac{1-\cos(n+2)\theta}{1-\cos(\theta)}$$

and therefore fulfills

$$\min_{z\in\overline{\mathbb{D}}} \operatorname{Re} F_{n+1}(z) = \frac{1}{2}, \tag{3.5}$$

$$\|F_{n+1}\| = F_{n+1}(1) = \frac{n+3}{2}. \tag{3.6}$$

Now, for $\varepsilon > 0$, we set

$$P_\varepsilon(z) := -2\varepsilon\sum_{k=0}^{n}\frac{n+1-k}{n+2}z^k = -2\varepsilon\frac{F_{n+1}(z)-1}{z},$$

so that $\operatorname{Re} zP_\varepsilon(z) \le \varepsilon$ on $\overline{\mathbb{D}}$. Furthermore note that $\|P_\varepsilon\| = -P_\varepsilon(1) = \varepsilon(n+1)$, and choose $\zeta = \zeta(\varepsilon)$ with $|\zeta P_\varepsilon(\zeta) + \alpha| = \|zP_\varepsilon(z) + \alpha\|$. Then $P_\varepsilon \in \mathcal{P}_n$ and we have, for $\varepsilon \to 0$,

$$\begin{aligned}\|zP_\varepsilon(z) + \alpha\| - \alpha &= \sqrt{|\zeta P_\varepsilon(\zeta)|^2 + 2\alpha\operatorname{Re}\zeta P_\varepsilon(\zeta) \ + \alpha^2} - \alpha\\ &\le \sqrt{\varepsilon^2(n+1)^2 + 2\alpha\varepsilon + \alpha^2} - \alpha\\ &= \varepsilon + \mathcal{O}(\varepsilon^2),\end{aligned}$$

so that

$$\lim_{\varepsilon\to 0}\frac{\|P_\varepsilon\|}{\|zP_\varepsilon(z)+\alpha\| - \alpha} \ge \lim_{\varepsilon\to 0}\frac{\varepsilon(n+1)}{\varepsilon + \mathcal{O}(\varepsilon^2)} = n+1,$$

which completes the proof of Theorem 1.1. □

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