Fast computation of an alternating sum

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1 Introduction

In this note we present an algorithm to determine the sum

$$S_\alpha(n) = \sum_{j=1}^{n} (-1)^{\lfloor j\alpha \rfloor}, \qquad \alpha \text{ irrational},$$

where, as usual, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Our algorithm is simple and fast; it consists of two simple operations, and the number of operations needed to evaluate $S_\alpha(n)$ is of order $\log(n)$.

The sum $S_\alpha(n)$ has been studied before in [1], where it was shown that $S_\alpha(n)$ is unbounded for irrational α, and that on the other hand the equality $S_\alpha(n) = 0$ holds for infinitely many n. So this sum has some of the characteristics of a random walk.

However, this random-like sum incorporates some remarkable symmetry properties. For instance, if you calculate $S_{\sqrt{2}}(n)$ for increasing n, and keep track of those n for which $S_{\sqrt{2}}(n)$ attains a value for the first time, then a recurrence relation between the n is displayed. More specifically, $S_{\sqrt{2}}(0) = 0$ is the first new value. The next new value occurs at $n = 1$, for which $S_{\sqrt{2}}(1) = -1$, and then $S_{\sqrt{2}}(3) = 1$, $S_{\sqrt{2}}(8) = -2$, etc. The first few extremes occur at 0,1,3,8,20,49,119,288,... In [4] it was conjectured that these numbers satisfy the recurrence relation $n_{i+1} = 2n_i + n_{i-1} + 1$. At the end of this paper we will see that this is the case indeed.

In order to compute $S_\alpha(n)$ efficiently, we looked for patterns in the plus and minus signs of the terms $(-1)^{\lfloor j\alpha \rfloor}$. We observed two kinds of patterns: 'repetitions' and 'reflections'. Both patterns induce an operation in the algorithm.

- A *repetition* occurs for a number n if $(-1)^{\lfloor j\alpha \rfloor}$ is equal to $(-1)^{\lfloor (n+j)\alpha \rfloor}$ for all $1 \leq j \leq n$. This implies that $S_\alpha(n+k) = S_\alpha(n) + S_\alpha(k)$ for $1 \leq k \leq n$, which is one of the operations in the algorithm.

- A *reflection* occurs for a number n if $(-1)^{\lfloor j\alpha \rfloor}$ and $(-1)^{\lfloor (n-j)\alpha \rfloor}$ have opposite signs for all $1 \leq j < n/2$. This implies that $S_\alpha(n-1) = S_\alpha(n-k) - S_\alpha(k-1)$ for $1 \leq k \leq n/2$, which is the other operation.

For which n does a repetition or a reflection take place? Assume that, for some n, $n\alpha$ is very close to an even integer $2m$. Then $(n+j)\alpha \approx 2m + j\alpha$ and $(n-j)\alpha \approx 2m - j\alpha$, which makes it plausible that

$$\begin{aligned}(-1)^{\lfloor (n+j)\alpha \rfloor} &= (-1)^{\lfloor 2m+j\alpha \rfloor} = (-1)^{2m+\lfloor j\alpha \rfloor} = (-1)^{\lfloor j\alpha \rfloor},\\ (-1)^{\lfloor (n-j)\alpha \rfloor} &= (-1)^{\lfloor 2m-j\alpha \rfloor} = (-1)^{2m-\lfloor j\alpha \rfloor -1} = -(-1)^{\lfloor j\alpha \rfloor}.\end{aligned}$$

Apparently, repetitions and reflections are likely to occur if $n\alpha \approx 2m$, or, in other words, if a rational m/n is a very good approximation of $\alpha/2$.

The best rational approximations of $\alpha/2$ are the so-called convergents of the continued fraction of $\alpha/2$. The next section contains a brief review of continued fractions. Since $S_{\alpha+2}(n)$ is equal to $S_\alpha(n)$, we may restrict ourselves to $-1 < \alpha < 1$. Furthermore, $S_{-\alpha}(n) = -S_\alpha(n)$ if α is irrational, so we may even assume that $0 < \alpha < 1$. Hence, we only consider continued fractions of irrationals between 0 and 1/2.

2 Continued fractions

Every irrational β, with $0 < \beta < 1/2$, can be represented as an infinite continued fraction

$$\beta = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}}} \qquad n_1 \in \mathbb{Z}_{>1}, \qquad n_i \in \mathbb{Z}_{>0} \quad \text{for } i \geq 2,$$

which is denoted by $[0; n_1, n_2, n_3, ...]$. The truncation $r_i = [0; n_1, n_2, ..., n_i]$ is called the ith convergent of β. The r_i are rational numbers, and their numerators and denominators can be constructed from simple recurrence relations. If we define

$$\begin{array}{llll} p_{-1} = 1 & p_0 = 0 & & p_{i+2} = n_{i+2}p_{i+1} + p_i \\ q_{-1} = 0 & q_0 = 1 & & q_{i+2} = n_{i+2}q_{i+1} + q_i \end{array}$$

then $r_i = p_i/q_i$. By induction one can prove the equality

$$p_i q_{i+1} - p_{i+1} q_i = (-1)^{i-1},$$

which implies that p_i and q_i are relatively prime. Moreover, the recurrence relation for the denominators q_i implies that $q_i < q_{i+1}$ for $i \geq 0$.

The following classical result indicates that convergents of β provide good rational approximations of β with relatively small denominators. For a proof see [3, page 58].

Proposition 1 If a rational m/n lies between β and one of its convergents $r_i = p_i/q_i$, then $n \geq q_{i+1} + q_i$.

Proposition 1 will be an important ingredient of the proofs for the equations that constitute the algorithm.

3 The algorithm

Consider an irrational α, with $0 < \alpha < 1$. Let $q_0, q_1, q_2, ...$ be the successive denominators of the convergents of $\alpha/2$. The following equation is based on repetitions.

Equation 1 $\quad S_\alpha(n) = S_\alpha(n - q_i) + S_\alpha(q_i) \qquad q_i \leq n < (q_{i+1} + q_i)/2.$

Proof. It is sufficient to prove that

$$(-1)^{\lfloor (kq_i+j)\alpha \rfloor} = (-1)^{\lfloor j\alpha \rfloor}, \qquad q_i < kq_i + j < (q_{i+1} + q_i)/2, \qquad 0 \leq j < q_i.$$

Suppose that this equation does not hold for some k and j. Then we have to prove that $kq_i + j$ is greater than or equal to $(q_{i+1} + q_i)/2$. Let $kq_i + j$ be the smallest number for which the repetitive pattern breaks down. In this case the equation still holds for $(k-1)q_i + j$, and we conclude that

$$(-1)^{\lfloor ((k-1)q_i+j)\alpha \rfloor} \neq (-1)^{\lfloor (kq_i+j)\alpha \rfloor}.$$

We rewrite this inequality. Since p_i/q_i is a convergent of $\alpha/2$, the difference between $\alpha/2$ and p_i/q_i is small. Putting $\epsilon = \alpha/2 - p_i/q_i$, we get $((k-1)q_i + j)\alpha = (kq_i + j)\alpha - 2q_i\epsilon - 2p_i$. Since $2p_i$ is even, it follows that

$$(-1)^{\lfloor (kq_i+j)\alpha - 2q_i\epsilon \rfloor} \neq (-1)^{\lfloor (kq_i+j)\alpha \rfloor}.$$

Hence, there must be an integer m between $(kq_i + j)\alpha - 2q_i\epsilon$ and $(kq_i + j)\alpha$. This implies that $m/2(kq_i + j)$ lies between $\alpha/2 - \epsilon = p_i/q_i$ and $\alpha/2$. According to Proposition 1 we then have $2(kq_i + j) \geq q_{i+1} + q_i$, which is what we wanted to prove. □

Equation 1 reduces the effort to compute $S_\alpha(n)$ considerably. If one knows $S_\alpha(n)$ for $n \leq q_{i-1}$, then by Equation 1 $S_\alpha(n)$ is known for $n < (q_i + q_{i-1})/2$. However, Equation 1 by itself does not yet constitute a fast algorithm for calculating the $S_\alpha(n)$. For this purpose it should produce, from the values $S_\alpha(n)$ for $n \leq q_{i-1}$, the values for $n \leq q_i$. The following equation, which is based on reflections, closes the gap between $(q_i + q_{i-1})/2$ and q_i.

Equation 2 $\quad S_\alpha(n) = S_\alpha(q_i - n - 1) + S_\alpha(q_i - 1) \qquad q_i/2 \leq n < q_i.$

Proof. First we show that

$$(-1)^{\lfloor j\alpha \rfloor} = (-1)^{\lfloor j2p_i/q_i \rfloor} \qquad \text{for } 1 \leq j < q_i \text{ and } j \neq q_i/2.$$

Here, the argument is similar to that for Equation 1. Suppose that the equation is not true for some particular j:

$$(-1)^{\lfloor j\alpha \rfloor} \neq (-1)^{\lfloor j2p_i/q_i \rfloor}$$

Since both $j\alpha$ and $j2p_i/q_i$ are non-integral (because $j \neq q_i/2$ and $j \neq q_i$), there must be an integer m in between. In other words, $m/2j$ lies between $\alpha/2$ and p_i/q_i. Proposition 1 then tells us that $2j \geq q_{i+1} + q_i > 2q_i$, and we have a contradiction.

Now we can prove Equation 2. Since $(q_i - j)2p_i/q_i = 2p_i - j2p_i/q_i$, we have

$$(-1)^{\lfloor (q_i-j)2p_i/q_i \rfloor} = -(-1)^{\lfloor j2p_i/q_i \rfloor}, \qquad \text{for } 1 \leq j < q_i/2.$$

By the equality that has just been deduced, we may replace $2p_i/q_i$ by α:

$$(-1)^{\lfloor (q_i-j)\alpha \rfloor} = -(-1)^{\lfloor j\alpha \rfloor}, \qquad \text{for } 1 \le j < q_i/2,$$

which immediately implies Equation 2. □

The algorithm is nearly complete. We already know the operations to reduce the n in $S_\alpha(n)$, but in order to compute $S_\alpha(n)$ we still need to know its values at the denominators q_i and $q_i - 1$. The values $S_\alpha(q_i)$ can be obtained efficiently from the reflection principle.

Lemma 2

$$S_\alpha(q_i) = (-1)^i \qquad \text{if } q_i \text{ is odd,}$$
$$S_\alpha(q_i) = 0 \qquad \text{if } q_i \text{ is even.}$$

Proof. In the proof of Equation 2, it was shown that

$$(-1)^{\lfloor (q_i-j)\alpha \rfloor} = -(-1)^{\lfloor j\alpha \rfloor}, \qquad \text{for } 1 \le j < q_i/2.$$

(This equality says that almost all terms of $S_\alpha(q_i)$ cancel.)

First, assume that q_i is odd. Then it follows that $S_\alpha(q_i) = (-1)^{\lfloor q_i\alpha \rfloor}$. If we put $\epsilon = \alpha/2 - p_i/q_i$, then $q_i\alpha = 2p_i + 2q_i\epsilon$. We will show that $2q_i|\epsilon| < 1$.

Suppose that $j|\epsilon| \ge 1$ for some j. Then there lies an integer m between $j\alpha/2$ and $j\alpha/2 - j\epsilon$. So m/j lies between $\alpha/2$ and $\alpha/2 - \epsilon = p_i/q_i$. According to Proposition 1 we then have $j \ge q_{i+1} + q_i > 2q_i$. Thus $2q_i|\epsilon| < 1$.

Hence, $\lfloor q_i\alpha \rfloor = 2p_i$ is even if $\epsilon > 0$, and $\lfloor q_i\alpha \rfloor = 2p_i - 1$ is odd if $\epsilon < 0$. If i is even, then the convergent p_i/q_i approximates $\alpha/2$ from below, so in that case $\epsilon > 0$. If i is odd, then p_i/q_i approximates $\alpha/2$ from above, so that $\epsilon < 0$. This proves that $S_\alpha(q_i) = (-1)^i$.

Next, assume that q_i is even. Then $S_\alpha(q_i) = (-1)^{\lfloor q_i\alpha/2 \rfloor} + (-1)^{\lfloor q_i\alpha \rfloor}$. We claim that these remaining two terms have opposite signs. As above, we have $q_i\alpha/2 = p_i + q_i\epsilon/2$ and $q_i\alpha = 2p_i + q_i\epsilon$. The numerator p_i is odd, because q_i is even. It follows for $\epsilon > 0$ (i.e., for even i) that $\lfloor q_i\alpha/2 \rfloor$ and $\lfloor q_i\alpha \rfloor$ are odd and even respectively. Similarly, if $\epsilon < 0$ (i.e., if i is odd), then they are even and odd respectively. So $S_\alpha(q_i) = 0$. □

So we can formulate Equation 1 as follows.

Equation 1 For $q_i \le n < (q_{i+1} + q_i)/2$ we have

$$S_\alpha(n) = S_\alpha(n - q_i) + (-1)^i \qquad \text{if } q_i \text{ is odd,}$$
$$S_\alpha(n) = S_\alpha(n - q_i) \qquad \text{if } q_i \text{ is even.}$$

To complete the algorithm we calculate $S_\alpha(q_i - 1)$, in order to reduce Equation 2 to a more suitable form.

Lemma 3

$$S_\alpha(q_i - 1) = 0 \qquad \text{if } q_i \text{ is odd,}$$
$$S_\alpha(q_i - 1) = (-1)^{i-1} \qquad \text{if } q_i \text{ is even.}$$

Proof. We have $S_\alpha(q_i - 1) = S_\alpha(q_i) - (-1)^{\lfloor q_i\alpha \rfloor}$. Using equalities that have been deduced in the proof of Equation 3, we obtain that $S_\alpha(q_i - 1) = 0$ if q_i is odd and $S_\alpha(q_i - 1) = (-1)^{\lfloor q_i\alpha/2 \rfloor} = (-1)^{i-1}$ if q_i is even. □

Hence, we can formulate Equation 2 as follows.

Equation 2 For $q_i/2 \leq n < q_i$ we have

$$\begin{aligned} S_\alpha(n) &= S_\alpha(q_i - n - 1) && \text{if } q_i \text{ is odd,} \\ S_\alpha(n) &= S_\alpha(q_i - n - 1) + (-1)^{i-1} && \text{if } q_i \text{ is even.} \end{aligned}$$

Combining Equations 1 and 2, we obtain the promised fast algorithm for $S_\alpha(n)$.

4 An example

We demonstrate the use of the algorithm by calculating $S_e(1,000,000)$. Since $0 < e-2 < 1$, we replace e by $e-2$. The continued fraction of $(e-2)/2$ is

$$[0; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, ...],$$

so that the denominators of the first convergents are

$$1,\ 2,\ 3,\ 11,\ 14,\ 25,\ 39,\ 142,\ 465,\ 1537,\ 2002,\ 7543,$$
$$9545,\ 36178,\ 190435,\ 607483,\ 797918,\ 4597073.$$

This is all we need to know in order to apply the algorithm to $S_e(1,000,000)$. From Equation 1 it follows that

$$\begin{array}{lclclcccl} S_e(1,000,000) & = & S_e(797,918) & + & S_e(202,082) & = & & & S_e(202,082) \\ S_e(202,082) & = & S_e(190,435) & + & S_e(11,647) & = & 1 & + & S_e(11,647) \\ S_e(11,647) & = & S_e(9,545) & + & S_e(2,102) & = & 1 & + & S_e(2,102) \\ S_e(2,102) & = & S_e(2,002) & + & S_e(100) & = & & & S_e(100). \end{array}$$

According to Equation 2, reflection with respect to 142 yields

$$S_e(100) = S_e(41) + 1 = S_e(39) + S_e(2) + 1 = 2.$$

Since we picked up four ones on the way, we find $S_e(1,000,000) = 4$.

5 A recurrence relation

Using the results from Section 3, we can prove the conjecture from [4], saying that the numbers n where $S_{\sqrt{2}}(n)$ attains a new value satisfy the recurrence relation $n_{i+1} = 2n_i + n_{i-1} + 1$.

Since $0 < 2 - \sqrt{2} < 1$, we replace $\sqrt{2}$ by $2 - \sqrt{2}$. The continued fraction of $(2-\sqrt{2})/2$ is $[0; 3, 2, 2, 2, ...]$, so that the denominators $q_0, q_1, q_2, ...$ of the convergents are found by the recurrence relation $q_{i+1} = 2q_i + q_{i-1}$ with $q_0 = 1$ and $q_1 = 3$. This implies that all q_i are odd, so according to Equation 3 we have $S_\alpha(q_i) = (-1)^i$. Then Equations 2′ and 1 yield

$$\begin{aligned} S_\alpha(q_{i+1} - k) &= S_\alpha(k-1), && k \leq q_{i+1}/2, \\ S_\alpha(q_i + l) &= (-1)^i + S_\alpha(l), && q_i + l < q_{i+1}/2. \end{aligned}$$

The first equation implies that extremes do not occur between $q_{i+1}/2$ and q_{i+1}; the value of S_α at $q_{i+1} - k$ has already been attained at $k-1$. The second equation implies that, if j_i and k_i denote the numbers where the ith new minimum and maximum of S_α are attained, then we have the recurrence relations

$$\begin{array}{rclcrl} j_i &=& q_{2i-1} &+& j_{i-1}, & \qquad j_0 = 0, \\ k_i &=& q_{2i} &+& k_{i-1}, & \qquad k_0 = 0. \end{array}$$

Hence, $j_i = q_{2i-1} + q_{2i-3} + \cdots + q_1$ and $k_i = q_{2i} + q_{2i-2} + \cdots + q_2$, from which it is clear that each new minimum is followed by a new maximum and vice versa. It is now straightforward to check that the recurrence relation for the n_i (with $n_{2i-1} = j_i$ and $n_{2i} = k_i$) reads $n_{i+1} = 2n_i + n_{i-1} + 1$. □

In fact, along the same lines we can deduce a similar result for all *quadratic* irrationals α, because these are exactly the irrationals that have a periodic continued fraction (see e.g. [2]).

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