Finiteness in infinite-valued Łukasiewicz logic

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Abstract. In this paper we deepen Mundici's analysis on reducibility of the decision problem from infinite-valued Łukasiewicz logic $\mathcal{L}_\infty$ to a suitable m-valued Łukasiewicz logic $\mathcal{L}_m$, where m only depends on the length of formulas to be proved. Using geometrical arguments we find a better upper bound for the least integer m such that a formula is valid in $\mathcal{L}_\infty$ if and only if so is in $\mathcal{L}_m$. We also reduce the notion of logical consequence in $\mathcal{L}_\infty$ to the same notion in a suitable finite set of finite-valued Łukasiewicz logics. Finally, we define an analytic and internal sequent calculus for infinite-valued Łukasiewicz logic.

1 Introduction

Many-valued Łukasiewicz logics were introduced in the twenties to extend the scope of classical logic by considering a set of truth-values larger than $\{0, 1\}$.

In particular, these logics suit to model reasoning in the presence of uncertainty and vagueness [13].

Although an infinite number of truth-values can drastically improve expressiveness, many-valued logics are meant to cope with inferences concerned only with a finite numbers of objects. Then naturally the question arises whether it is possible to achieve the expressiveness of infinite-valued Łukasiewicz logic by using a proper finite-valued Łukasiewicz logic.

In [16] it is shown that the decision problem in propositional infinite-valued Łukasiewicz logic can be reduced to the same problem in a suitable set of finite-valued Łukasiewicz logics. More precisely the author proved that a formula φ is valid in $\mathcal{L}_\infty$ if and only if so is in each logic $\mathcal{L}_i$, for $i \in \{2, \ldots, 2^{(2\#(\varphi))^2} + 1\}$, where $\#(\varphi)$ denotes the total number of occurrences of variables in φ.

Strengthening this result, in this paper we show that validity of φ in $\mathcal{L}_\infty$ can be checked in *exactly one* finite-valued logic $\mathcal{L}_m$, for m smaller than $2^{\#(\varphi)} + 1$.

We also consider logical consequence in $\mathcal{L}_\infty$ and prove that for any two finite sets of formulas Γ and Δ of $\mathcal{L}_\infty$ there exists an integer k, only depending on the total number of occurrences of variables in $\Gamma \cup \Delta$, such that $\Gamma \models_{\mathcal{L}_\infty} \Delta$ if and only if $\Gamma \models_{\mathcal{L}_i} \Delta$, for each $i \in \{2, \ldots, k\}$.

From this result we obtain an analytic and internal sequent calculus for infinite-valued Łukasiewicz logic. To the best of our knowledge, this is the first example of a calculus for $\mathcal{L}_\infty$ having these desirable features. Indeed, while the literature contains various calculi for $\mathcal{L}_\infty$, all these calculi fail to be either analytic (see e.g. [18,6]) or *internal* ([12,22,17]). Stated otherwise, all these *non-internal* calculi require manipulations of spurious objects, other than formulas in the logic. In particular, they entail a lot of geometrical or algebraic computations, such as solving integer programs [12], intersecting hyperplanes [22] or determining θ-supports of formulas [17].

The idea behind our calculus is very simple. We add to the axioms and rules of the multi-component sequent calculi for finite-valued Łukasiewicz logics introduced in [3] rules allowing to dynamically increase the number of components in every sequent.

It turns out that the resulting calculus is analytic, and it deals only with formulas of the logic. Indeed, all manipulations involved in any proof are syntactic and they never involve any algebraic or geometrical computation.

2 Preliminaries

To make the paper more self-contained we present here background material on Łukasiewicz logics, McNaughton representation Theorem and convex geometry. For more details we refer to [7,15,8].

Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. A propositional logic is a triple $\mathcal{L} = (S, D, F)$ where: $\emptyset \subset D \subset S$ and $F = \{f_1, \ldots, f_k\}$ is a finite nonempty set of functions $f_i : S^{\nu(i)} \to S$ for a suitably chosen $\nu : \{1, \ldots, k\} \to \mathbf{N}$.

For each integer $m > 0$, let S_m be the set $\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\}$. Then $(m+1)$-*valued Łukasiewicz propositional logic* is defined by $\mathcal{L}_{m+1} = (S_m, \{1\}, \{f_\neg, f_\oplus\})$, where $f_\neg(x) = 1 - x$ and $f_\oplus(x, y) = \min(1, x + y)$. The *infinite-valued propositional logic of Łukasiewicz* is defined by $\mathcal{L}_\infty = ([0, 1], \{1\}, \{f_\neg, f_\oplus\})$. Classical propositional logic coincides with $\mathcal{L}_2$.

Let us denote by L the set of formulas built on the set of connectives $\{\neg, \oplus\}$.

An *interpretation* for $\mathcal{L}_{m+1}$ (resp., $\mathcal{L}_\infty$) is a function $v : L \to S_m$ (resp., $v : L \to [0, 1]$) such that for every formula φ, $v(\neg\varphi) = 1 - v(\varphi)$ and $v(\varphi \oplus \psi) = \min(1, v(\varphi) + v(\psi))$.

A formula φ is *valid* in $\mathcal{L}$ (in symbols $\models_\mathcal{L} \varphi$) if $v(\varphi) \in D$ for each interpretation v for $\mathcal{L}$. Given two sets Γ and Δ of formulas we say that Δ is *a logical consequence* of Γ *in* $\mathcal{L}$ (in symbols $\Gamma \models_\mathcal{L} \Delta$) if for all interpretations v for $\mathcal{L}$ there is $\gamma \in \Gamma$ such that either $v(\gamma) \notin D$, or there is $\delta \in \Delta$ such that $v(\delta) \in D$. Two formulas $\varphi, \psi \in \mathcal{L}$ are said to be $\mathcal{L}$-*equivalent*, in symbols $\varphi \equiv_\mathcal{L} \psi$, if $v(\varphi) = v(\psi)$ for each interpretation v for $\mathcal{L}$.

In every logic $\mathcal{L}_m$ as well as in $\mathcal{L}_\infty$, the following derived connectives are introduced: *strong conjunction*: $\varphi \odot \psi := \neg(\neg\varphi \oplus \neg\psi)$, *implication*: $\varphi \to \psi := \neg\varphi \oplus \psi$, *weak disjunction*: $\varphi \vee \psi := (\varphi \to \psi) \to \psi$, *weak conjunction*: $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$. For all interpretations v we have $v(\varphi \odot \psi) = \max(0, v(\varphi) + v(\psi) - 1)$, $v(\varphi \to \psi) = \min(1, 1 - v(\varphi) + v(\psi))$, $v(\varphi \vee \psi) = \max(v(\varphi), v(\psi))$ and $v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi))$.

Infinite-valued Łukasiewicz logic can be axiomatized by the following formulas: $(A1)$ $\varphi \to (\psi \to \varphi)$, $(A2)$ $(\varphi \to \psi) \to ((\psi \to \vartheta) \to (\varphi \to \vartheta))$, $(A3)$ $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$, $(A4)$ $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

As shown in [10] a Hilbert style axiomatization for $\mathcal{L}_m$ is obtained by adding to $A1$-$A4$ the following axioms: $(A5)$ $(m-1)\varphi \leftrightarrow m\varphi$, $(A6)$ $(p\varphi^{p-1})^m \leftrightarrow m\varphi^p$ for every integer $p = 2, \ldots, m-2$ that does not divide $m-1$. Here $\varphi \leftrightarrow \psi$ stands for $(\varphi \to \psi) \odot (\psi \to \varphi)$, $m\varphi$ for $\varphi \oplus \ldots \oplus \varphi$ (m times) and φ^m for $\varphi \odot \ldots \odot \varphi$ (m times).

A *proof* in $\mathcal{L}_\infty$ (resp., $\mathcal{L}_m$) of φ from a set Γ of formulas is a finite string of formulas $\varphi_1, \ldots, \varphi_t$ with $t \geq 1$ such that $\varphi_t = \varphi$, and for each $1 \leq i \leq t$,

- either $\varphi_i \in \Gamma \cup \{A1, \ldots, A4\}$ (resp., $\in \Gamma \cup \{A1, \ldots, A6\}$),
- or there are $j, k \in \{1, \ldots, i-1\}$ such that $\varphi_k = \varphi_j \to \varphi_i$. Then we say that φ_i follows from φ_j and φ_k by *modus ponens.*

A set of formulas Δ is *provable* from Γ in $\mathcal{L}_\infty$ (resp., $\mathcal{L}_m$), in symbols $\Gamma \vdash_{\mathcal{L}_\infty} \Delta$, (resp., $\Gamma \vdash_{\mathcal{L}_m} \Delta$) if at least one of the formulas in Δ is provable from Γ.

Each formula φ in the variables $X_1, \ldots, X_n$ *determines a function* $f_\varphi : [0,1]^n \to [0,1]$, as follows:

- $f_{X_i}(x_1, \ldots, x_n) = x_i$, the projection along the ith coordinate axis.
- $f_{\neg\varphi} = 1 - f_\varphi$.
- $f_{\psi\oplus\vartheta} = \min(1, f_\psi + f_\vartheta)$.

The class of *McNaughton functions* is defined as the class of functions $g : [0,1]^n \to [0,1]$, for $0 < n \in \mathbf{N}$, which are continuous and piecewise linear, and where each piece has integer coefficients: that is, there exist finitely many polynomials $p_1, \ldots, p_{m_g}$, each p_i of the form $p_i(x_1, \ldots, x_n) = a_{i,1}x_1 + \cdots a_{i,n}x_n + b_i$, with $a_{i,1}, \ldots, a_{i,n}, b_i$ integers, such that for all $\mathbf{x} \in [0,1]^n$ there exists $j \in \{1, \ldots, m_g\}$ for which $f_\varphi(\mathbf{x}) = p_j(\mathbf{x})$.

McNaughton [15] showed that McNaughton functions coincides with functions determined by formulas of infinite-valued Łukasiewicz logic.

By $\mathrm{Sub}(\varphi)$ we denote the set of all subformulas of φ. If $\psi \in \mathrm{Sub}(\varphi)$ then we write $\psi \preceq \varphi$. Let Γ be a set of formulas, and let $\mathrm{Var}(\Gamma)$ denote the set of all variables occurring in the formulas of Γ.

Let n be the cardinality of $\mathrm{Var}(\{\psi\})$. For sake of simplicity, we may tacitly and safely regard the McNaughton function f_ψ to be defined on $[0,1]^{n'}$ also for $n' > n$. In general n' will be the cardinality of $\mathrm{Var}(\{\varphi\})$ for a formula $\psi \preceq \varphi$ which is clear from the context. Of course, such $f_\psi(x_1, \ldots, x_{n'})$ will only depend on the components x_j corresponding to variables $X_j \in \mathrm{Var}(\{\psi\})$.

For each variable X the number of occurrences $\#(X, \varphi)$ of X in φ is inductively defined as follows:

- If $\varphi = X$ then $\#(X, \varphi) = 1$. If $\varphi = Y$, for some variable $Y \neq X$, then $\#(X, \varphi) = 0$.
- $\#(X, \neg\psi) = \#(X, \psi)$.
- $\#(X, \psi \oplus \vartheta) = \#(X, \psi) + \#(X, \vartheta)$.

The total number of occurrences of variables in φ is given by

$$\#(\varphi) = \sum_{X \in \mathrm{Var}(\{\varphi\})} \#(X, \varphi).$$

For any set Γ of formulas, we let $\#(\Gamma) = \sum_{\gamma \in \Gamma} \#(\gamma)$.

Given the euclidean space $\mathbf{R}^n$, and a subset $S \subseteq \mathbf{R}^n$, $\dim(S)$ denotes the *dimension* of S. By definition $\dim(\emptyset) = -1$.

For any two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$, the *scalar product* $\mathbf{x} \cdot \mathbf{y}$ is, as usual, the real number $x_1 y_1 + \cdots + x_n y_n$.

The set H^- of solutions $\mathbf{x} = (x_1, \ldots, x_n)$ of an inequality of the form $\mathbf{a} \cdot \mathbf{x} \leq b$, for $\mathbf{a} \in \mathbf{R}^n$, $b \in \mathbf{R}$, is called an *(affine) halfspace* of $\mathbf{R}^n$. The solutions of the corresponding equation $\mathbf{a} \cdot \mathbf{x} = b$ form the *supporting hyperplane* H of H^-.

A *polyhedron* P is the set-theoretic intersection $P = \bigcap_{i \in I} H_i^-$ of a finite number of halfspaces. Each supporting hyperplane H_j $(j \in I)$ such that $H_j \cap P \neq \emptyset$ is called a *bounding hyperplane* of P. Note that $P = \bigcap \{H_i^- : \dim(H_i \cap P) = \dim(P) - 1\}$.

By a *facet* of P we mean any $(\dim(P) - 1)$-dimensional polyhedron F, arising as the intersection of P with one of its bounding hyperplanes. The set of *faces* of a polyhedron P is defined as follows:

- $\emptyset$ and P are faces of P.
- Each facet of a face of P is a face of P.

Set-theoretic inclusion makes the set of faces of P into a lattice.
A *polyhedral complex* C is a set of polyhedra such that

- Whenever A is in C, all faces of A are in C.
- If A, B are in C then $A \cap B$ is a common face of both A and B.

Each polyhedron in C is called a *cell* of C. Let $0 \leq k \leq n$ be integers such that n is the maximum dimension of cells in C. The set $C^{(k)}$ of all k-dimensional cells of C is called the *k-skeleton* of C. Let φ be a formula and $f_\varphi : [0,1]^n \to [0,1]$ its associated McNaughton function, where n is the cardinality of the set $\mathrm{Var}(\{\varphi\})$. By McNaughton Theorem [15, 7] there is a (not uniquely determined) polyhedral complex C, an index set I, and a set of polynomials $Q = \{q_1, \ldots, q_k\}$, such that

- $\bigcup C = [0,1]^n$,
- there is a bijective map cell from I onto the n-skeleton $C^{(n)}$ of C, and a map q from I onto Q, such that, for all indexes $j \in I = \{1, \ldots, m\}$, f_φ coincides with $q(j)$ over $\mathrm{cell}(j)$.

3 Introducing a geometric bound

In this section we determine an integer $m > 1$ such that a formula φ is valid in $\mathcal{L}_\infty$ if and only if φ is valid in $\mathcal{L}_m$.

Strengthening our analysis, for any two finite sets of formulas Γ and Δ we shall determine an integer $k > 1$ such that $\Gamma \models_\infty \Delta$ if and only if $\Gamma \models_i \Delta$ for every $i \in \{2, \ldots, k\}$.

Let φ be a formula of $\mathcal{L}_\infty$ having n variables and $f_\varphi : [0,1]^n \to [0,1]$ its associated McNaughton function. Let $\mathbf{x} = (h_1/k_1, \ldots, h_n/k_n)$ be a point in $[0,1]^n$ where f_φ attains its minimum value. Trivially φ is valid if and only if $f_\varphi(\mathbf{x}) = 1$. We shall give an upper bound to the common denominator $\mathrm{den}(\mathbf{x})$ of $h_1/k_1, \ldots, h_n/k_n$, only depending on n and on $\#(\varphi)$. This bound determines the number of truth-values of the finite-valued Łukasiewicz logic in which to check the validity of φ.

Mundici [16] first found a bound of this kind in order to prove the NP-completeness of the satisfiability problem for formulas of $\mathcal{L}_\infty$. He proved that $\mathrm{den}(\mathbf{x}) < 2^{(2\#(\varphi))^2} + 1$.

By refining his analysis we shall show that $\mathrm{den}(\mathbf{x}) \leq 2^{\#(\varphi)}$.

We first *canonically* associate to each φ a polyhedral complex $C(\varphi)$ and a set of polynomials $Q(\varphi)$.

Definition 1. Let $\mathrm{Var}(\{\varphi\}) = \{X_1, \ldots, X_n\}$. By induction on the complexity of subformulas ψ of φ, we define the set $I(\varphi) = \{1, \ldots, m_\varphi\}$ of indices, the set $C^{(n)}(\varphi)$ of polyhedra indexed by elements of $I(\varphi)$, the set $Q(\varphi)$ of linear polynomials and maps $q_\varphi : I(\varphi) \to Q(\varphi)$ and $\mathrm{cell}_\varphi : I(\varphi) \to C^{(n)}(\varphi)$ such that f_φ coincides with $q_\varphi(j)$ over $\mathrm{cell}_\varphi(j)$, as follows:

If $\psi = X_i$ then $C^{(n)}(\psi) = \{[0,1]^n\}$ and $Q(\psi) = \{x_i\}$. Further, $I(\psi) = \{1\}$, $q_\psi(1) = x_i$ and $\mathrm{cell}_\psi(1) = [0,1]^n$.

If $\psi = \neg\vartheta$ then $I(\neg\vartheta) = I(\vartheta)$, $C^{(n)}(\neg\vartheta) = C^{(n)}(\vartheta)$ and $\mathrm{cell}_{\neg\vartheta} = \mathrm{cell}_\vartheta$. $Q(\neg\vartheta) = \{q_{\neg\vartheta}(i) = 1 - q_\vartheta(i) : q_\vartheta(i) \in Q(\vartheta)\}$.

Finally, assume $\psi = \vartheta_1 \oplus \vartheta_2$. For any two indexes $j \in I(\vartheta_1)$ and $k \in I(\vartheta_2)$, the restriction $g_{j,k}$ of f_ψ to $\mathrm{cell}_{\vartheta_1}(j) \cap \mathrm{cell}_{\vartheta_2}(k)$ is given by

$$g_{j,k}(\mathbf{x}) = \begin{cases} (q_{\vartheta_1}(j))(\mathbf{x}) + (q_{\vartheta_2}(k))(\mathbf{x}) & \text{if} \quad (q_{\vartheta_1}(j))(\mathbf{x}) + (q_{\vartheta_2}(k))(\mathbf{x}) \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

We build the index set $I(\psi)$, the maps q_ψ and cell_ψ as described below. Consider the set of pairs $P = \{(j,k) : j \in I(\vartheta_1),\ k \in I(\vartheta_2)\}$ equipped with the lexicographic order. Starting from the pair $(1,1) \in P$ and initializing $C^{(n)}(\psi)$ to $\emptyset$, for each pair (j,k) we proceed as follows:

- Let $S = \mathrm{cell}_{\vartheta_1}(j) \cap \mathrm{cell}_{\vartheta_2}(k) \cap \{\mathbf{x} : (q_{\vartheta_1}(j))(\mathbf{x}) + (q_{\vartheta_2}(k))(\mathbf{x}) \leq 1\}$. If $\dim(S) = n$, then we add S to $C^{(n)}(\psi)$; letting i be the current cardinality of $C^{(n)}(\psi)$, we put $q_\psi(i) = q_{\vartheta_1}(j) + q_{\vartheta_2}(k)$ and $\mathrm{cell}_\psi(i) = S$.
- Let $T = \mathrm{cell}_{\vartheta_1}(j) \cap \mathrm{cell}_{\vartheta_2}(k) \cap \{\mathbf{x} : (q_{\vartheta_1}(j))(\mathbf{x}) + (q_{\vartheta_2}(k))(\mathbf{x}) \geq 1\}$. If $\dim(T) = n$, then we add T to $C^{(n)}(\psi)$; letting i be the current cardinality of $C^{(n)}(\psi)$, we put $q_\psi(i) = 1$ and $\mathrm{cell}_\psi(i) = T$.

The procedure terminates when all pairs in P have been considered.

Note that f_φ is linear over each polyhedron of $C^{(n)}(\varphi)$.

Lemma 1. *The set of polyhedra $C^{(n)}(\varphi)$ is the n-skeleton of a polyhedral complex denoted by $C(\varphi)$.*

Proof: Let us suppose that $\varphi = \psi_1 \oplus \psi_2$. By induction hypothesis $C(\psi_1)$ and $C(\psi_2)$ are polyhedral complexes. Note that the set of all n-dimensional cells H arising as an intersection $H = \mathrm{cell}_{\psi_1}(j_1) \cap \mathrm{cell}_{\psi_2}(j_2)$, for $j_1 \in I(\psi_1)$ and $j_2 \in I(\psi_2)$ is the n-skeleton of a polyhedral complex. Then we only have to consider the case when a polyhedron $\mathrm{cell}_{\psi_1}(j_1) \cap \mathrm{cell}_{\psi_2}(j_2)$ is split into two n-dimensional regions H_1, H_2, both of which being bounded by the hyperplane of equation $(q_{\psi_1}(j_1))(\mathbf{x}) + (q_{\psi_2}(j_2))(\mathbf{x}) = 1$. Then f_φ coincides with a linear polynomial p_1 over H_1 and with a different linear polynomial p_2 over H_2.

Take facets F_1 of H_1 and F_2 of H_2 such that $F_1 \cup F_2$ is a facet of $\mathrm{cell}_{\psi_1}(j_1) \cap \mathrm{cell}_{\psi_2}(j_2)$. Then f_φ is not linear over $F_1 \cup F_2$. By way of contradiction, suppose that $C(\varphi)$ is not a polyhedral complex: without loss of generality we may assume that there is a cell $L \in C^{(n)}(\varphi)$ such that $\dim(L \cap F_1) = \dim(L \cap F_2) = \dim(F_1)$. Then there is a facet G of L such that $\dim(F_1 \cap G) = \dim(F_2 \cap G) = \dim(F_1)$. Clearly, there is a convex $(n-1)$-dimensional region included in G containing the nonempty set $F_1 \cap F_2 \cap G$.

Since f_φ is linear over G, we have a contradiction as required. □

Corollary 1. *The cardinality of $I(\varphi)$ is smaller than $2^{\#(\varphi)-1}$.* □

Example 1. Consider the formula

$$\varphi = \neg(X \oplus Y) \oplus \neg(X \oplus Z).$$

Starting with the subformula $X \oplus Y$ we have: $I(X \oplus Y) = \{1, 2\}$,

$$q_{X\oplus Y}(1) = x + y \qquad \text{and} \qquad q_{X\oplus Y}(2) = 1,$$

while

$$\mathrm{cell}_{X\oplus Y}(1) = \{(x,y,z) \in [0,1]^3 : x + y \le 1\},$$
$$\mathrm{cell}_{X\oplus Y}(2) = \{(x,y,z) \in [0,1]^3 : x + y \ge 1\}.$$

Then, considering the subformula $\neg(X \oplus Y)$, we have

$$C^{(3)}(\neg(X \oplus Y)) = C^{(3)}(X \oplus Y),$$

$$\mathrm{cell}_{\neg(X\oplus Y)} = \mathrm{cell}_{X\oplus Y}$$

and, moreover,

$$q_{\neg(X\oplus Y)}(1) = 1 - x - y \qquad \text{and} \qquad q_{\neg(X\oplus Y)}(2) = 0.$$

The analysis of $X \oplus Z$ and $\neg(X \oplus Z)$ being analogous, we consider now the whole formula φ.

Starting with the first pair of indexes, we have

$$\mathrm{cell}_{\neg(X\oplus Y)}(1)\cap\mathrm{cell}_{\neg(X\oplus Z)}(1)=\{x+y\leq 1,\, x+z\leq 1\},$$

Then

$$\mathrm{cell}_{\varphi}(1)=\{x+y\leq 1,\, x+z\leq 1,\, 2x+y+z\geq 1\}\quad\text{and}\quad q_{\varphi}(1)=2-2x-y-z,$$

while

$$\mathrm{cell}_{\varphi}(2)=\{x+y\leq 1,\, x+z\leq 1,\, 2x+y+z\leq 1\}\qquad\text{and}\qquad q_{\varphi}(2)=1.$$

Listing now all pairs of indexes we have

$$\mathrm{cell}_{\varphi}(3)=\{x+y\leq 1,\, x+z\geq 1\}\quad\text{and}\quad q_{\varphi}(3)=1-x-y,$$

while $\mathrm{cell}_{\neg(X\oplus Y)}(1)\cap\mathrm{cell}_{\neg(X\oplus Z)}(2)\cap\{(x,y,z):1-x-y+0\geq 1\}$ is not 3-dimensional and it is discarded. Furthermore,

$$\mathrm{cell}_{\varphi}(4)=\{x+y\geq 1,\, x+z\leq 1\}\quad\text{and}\quad q_{\varphi}(4)=1-x-z,$$

and, as in the previous case, $\mathrm{cell}_{\neg(X\oplus Y)}(2)\cap\mathrm{cell}_{\neg(X\oplus Z)}(1)\cap\{(x,y,z):0+1-x-z\geq 1\}$ is not full-dimensional. Finally,

$$\mathrm{cell}_{\varphi}(5)=\{x+y\geq 1,\, x+z\geq 1\}\quad\text{and}\quad q_{\varphi}(5)=0,$$

while $\mathrm{cell}_{\neg(X\oplus Y)}(2)\cap\mathrm{cell}_{\neg(X\oplus Z)}(2)\cap\{(x,y,z):0+0\geq 1\}$ is not 3-dimensional. There are no other cases to consider.

For each formula φ, the McNaughton function f_{φ} attains its minimum value on a vertex $\mathbf{p}$ of some cell C of its associated polyhedral complex $C(\varphi)$. As the intersection of finitely many bounding hyperplanes of C, $\mathbf{p}$ can be determined by a system of linear equations. We shall preliminarily give an upper bound to the absolute value of the determinant of the matrix M of this system.

We need some technical lemmas enabling us to build up a matrix $M_{\mathbf{p}}$ such that $|\det(M_{\mathbf{p}})|=|\det(M)|$ and whose entries are in a one to one correspondence with suitably chosen occurrences of variables in formulas.

Given a polynomial $p(\mathbf{x})=\mathbf{a}\cdot\mathbf{x}+b$, we denote the vector $\mathbf{a}$ by $\pi(p)$. For each formula φ we also introduce the set of vectors

$$\Pi(\varphi)=\{\pi(q_{\varphi}(i)),-\pi(q_{\varphi}(i)):q_{\varphi}(i)\in Q(\varphi)\}.$$

We stress explicitly that different occurrences of the same subformula of φ shall be regarded as different subformulas.

As we shall see, the rows of the matrix $M_{\mathbf{p}}$ are taken from the sets $\Pi(\psi)$, for suitably chosen $\psi\preceq\varphi$.

As usual, $\gcd(a_1,\ldots,a_u)$ denotes the *greatest common divisor of* $a_1,\ldots,a_u$.

Definition 2. Let φ be a formula and assume f_φ is defined on $[0,1]^n$, for $n > 0$. For each $i \in I(\varphi)$, $D(\varphi, i)$ denotes the set of all vectors $(\mathbf{a}, c) = (a_1, \ldots, a_n, c) \in \mathbf{Z}^{n+1}$ having the following two properties:

- $\gcd(a_1, \ldots, a_n, c) = 1$.
- There exists a bounding hyperplane of $\mathrm{cell}_\varphi(i)$, of equation $\mathbf{a} \cdot \mathbf{x} = c$, such that $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = c\} \cap \mathrm{cell}_\varphi(i)$ is a facet of $\mathrm{cell}_\varphi(i)$.

The dependence of $D(\varphi, i)$ on n is understood. From now on, we shall freely deal with sets $D(\psi, j)$ for subformulas ψ of some fixed formula φ which is clear from the context. Then we shall fix n as the cardinality of $\mathrm{Var}(\{\varphi\})$.

Let φ be a formula and $\psi \preceq \varphi$. For each $i \in I(\varphi)$ there is a uniquely determined index $j \in I(\psi)$ such that $\mathrm{cell}_\varphi(i) \subseteq \mathrm{cell}_\psi(j)$.

For every integer $n > 0$, let us denote by E_n the set of *coordinate unit vectors* $E_n = \{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\}$, where $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, …, $\mathbf{e}_n = (0, \ldots, 0, 1)$.

Example 2. Let φ, q_φ and cell_φ be as in Example 1. Then

$$D(\varphi, 1) = \{\pm(\mathbf{e}_i, 0) : i \in \{1,2,3\}\} \cup \{\pm((1,1,0),1), \pm((1,0,1),1), \pm((2,1,1),1)\},$$

$$D(\varphi, 2) = \{\pm(\mathbf{e}_i, 0) : i \in \{1,2,3\}\} \cup \{\pm((2,1,1),1)\},$$

$$D(\varphi, 3) = \{\pm((0,1,0),0), \pm((0,0,1),1)\} \cup \{\pm((1,1,0),1), \pm((1,0,1),1)\},$$

$$D(\varphi, 4) = \{\pm((0,1,0),1), \pm((0,0,1),0)\} \cup \{\pm((1,1,0),1), \pm((1,0,1),1)\},$$

$$D(\varphi, 5) = \{\pm(\mathbf{e}_i, 1) : i \in \{1,2,3\}\} \cup \{\pm((1,1,0),1), \pm((1,0,1),1)\}.$$

Definition 3. Let $(\mathbf{a}, c) \in D(\varphi, i)$, $\psi \preceq \varphi$ and $j \in I(\psi)$. We say that (ψ, j) *introduces the bound* $(\mathbf{a}, c)$ *for* (φ, i) if the following conditions are satisfied:

- There exists an integer $d \neq 0$ such that $d\mathbf{a} = \pi(q_\psi(j))$.
- The set $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = c\} \cap \mathrm{cell}_\varphi(i)$ is included in a facet of $\mathrm{cell}_\psi(j)$ and, either $\mathrm{cell}_\varphi(i) \subseteq \mathrm{cell}_\psi(j)$ or $\mathrm{cell}_\varphi(i) \subseteq \mathrm{cell}_\psi(j+1)$ and $q_\psi(j+1) = 1$.
- There is no proper subformula ϑ of ψ such that the set $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = c\} \cap \mathrm{cell}_\varphi(i)$ is included in a facet of some cell of $C^{(n)}(\vartheta)$.

Lemma 2. *Let φ be a formula in the variables $X_1, \ldots, X_n$, and let i be any arbitrary element in $I(\varphi)$. Then for each $(\mathbf{a}, c) \in D(\varphi, i)$ there is an integer $d \neq 0$ such that*

$$d\mathbf{a} \in E_n \cup \bigcup_{\psi \in \mathrm{Sub}(\varphi)} \Pi(\psi).$$

If, in addition, $\mathbf{a} \notin E_n$ there exists $\eta \preceq \varphi$, together with an index $l \in I(\eta)$ such that (η, l) introduces $(\mathbf{a}, c)$ for (φ, i).

Proof: By induction on the complexity of subformulas ψ of φ.

If $\psi = X_i \in \mathrm{Var}(\{\varphi\})$ then $\Pi(\psi) = \{\pi(q_\psi(1)), -\pi(q_\psi(1))\}$, where $q_\psi(1) = x_i$, and $\mathrm{cell}_\psi(1) = [0,1]^n$. For each $(\mathbf{a}, c) \in D(\psi, 1)$ we have $\mathbf{a} \in E_n$.

The case $\psi = \neg\vartheta$ follows by induction hypothesis, since by construction $C^{(n)}(\psi) = C^{(n)}(\vartheta)$ and this implies $D(\psi, j) = D(\vartheta, j)$ for all indexes j.

There remains to consider the case $\psi = \vartheta_1 \oplus \vartheta_2$. Pick $j \in I(\psi)$ and let $(\mathbf{a}, c)$ be any vector of $D(\psi, j)$. By construction of $C^{(n)}(\psi)$ we have to consider two cases:

(i) $(\mathbf{a}, c) \in D(\vartheta_1, h)$ for some $h \in I(\vartheta_1)$, or $(\mathbf{a}, c) \in D(\vartheta_2, k)$ for some $k \in I(\vartheta_2)$, and h and k can be so chosen that $\mathrm{cell}_\psi(j) \subseteq \mathrm{cell}_{\vartheta_1}(h)$ and $\mathrm{cell}_\psi(j) \subseteq \mathrm{cell}_{\vartheta_2}(k)$. Without loss of generality, let us suppose that $(\mathbf{a}, c) \in D(\vartheta_1, h)$. Then, by induction hypothesis, there is an integer $d \neq 0$ such that $d\mathbf{a} \in E_n \cup \bigcup_{\eta \in \mathrm{Sub}(\vartheta_1)} \Pi(\eta) \subseteq E_n \cup \bigcup_{\eta \in \mathrm{Sub}(\psi)} \Pi(\eta)$. Furthermore, if $\mathbf{a} \notin E_n$, there exists $\eta \preceq \vartheta_1$, together with an index $l \in I(\eta)$, such that (η, l) introduces $(\mathbf{a}, c)$ for (ϑ_1, h). Since $(\mathbf{a}, c) \in D(\psi, j)$ we conclude that (η, l) introduces $(\mathbf{a}, c)$ for (ψ, j).

(ii) There is an integer $d \neq 0$ such that $d\mathbf{a} = \pi(q_{\vartheta_1}(h) + q_{\vartheta_2}(k))$ for suitable indexes $h \in I(\vartheta_1)$ and $k \in I(\vartheta_2)$. Thus, either $d\mathbf{a} = \pi(q_\psi(j))$ and (ψ, j) introduces $(\mathbf{a}, c)$ for (ψ, j), or $d\mathbf{a} = \pi(q_\psi(j-1))$ and $(\psi, j-1)$ introduces $(\mathbf{a}, c)$ for (ψ, j). In both cases, $d\mathbf{a} \in \Pi(\psi)$.

□

Definition 4. Formulas $\varphi_1, \ldots, \varphi_n$ are said to be *disjoint* if there do not exist distinct indexes $j, k \in \{1, \ldots, n\}$ such that $\varphi_j \preceq \varphi_k$.

A formula φ is said to be *negated* if it is of the form $\neg\psi$.

In the following lemma we show that for any two formulas $\psi \preceq \varphi$ and index i in $I(\varphi)$, the vector $\pi(q_\varphi(i))$ can be expressed as a suitable $\pm$-combination of vectors $\pi(q_{\vartheta_h}(k_h))$ for non-negated disjoint formulas $\vartheta_1, \ldots, \vartheta_h$, possibly together with ψ.

Lemma 3. *Let φ be a formula in the variables $X_1, \ldots, X_n$ and ψ be a subformula of φ. Pick $i \in I(\varphi)$ such that $q_\varphi(i)$ is not a constant function. Then there exists a (possibly empty) set of non-negated subformulas $\{\vartheta_1, \ldots, \vartheta_m\}$ of φ, and a set of indexes $\{k_h : k_h \in I(\vartheta_h),\ 1 \leq h \leq m\}$ such that:*

1. *The formulas $\psi, \vartheta_1, \ldots \vartheta_m$ are disjoint. For any formula $\vartheta_h \preceq \zeta \preceq \varphi$ we either have $\psi \preceq \zeta$ or $\zeta = \underbrace{\neg \cdots \neg}_{p} \vartheta_h$ for some $p \geq 0$.*
2. *No function $q_{\vartheta_h}(k_h)$ is a constant. Moreover we either have*
   (a) $\pi(q_\varphi(i)) = \pm\pi(q_{\vartheta_1}(k_1)) \pm \cdots \pm \pi(q_{\vartheta_m}(k_m))$, or
   (b) There exists an index $j \in I(\psi)$ such that $q_\psi(j)$ is not a constant function, $\mathrm{cell}_\varphi(i) \subseteq \mathrm{cell}_\psi(j)$ and

$$\pi(q_\varphi(i)) = \pm\pi(q_{\vartheta_1}(k_1)) \pm \cdots \pm \pi(q_{\vartheta_m}(k_m)) \pm \pi(q_\psi(j)).$$

Proof: By induction on the complexity of φ. When φ is a variable X_i then the unique subformula of X_i is X_i itself, $I(X_i) = \{1\}$, and $q_\varphi(1) = q_{X_i}(1)$.

If $\varphi = \neg\eta$ then $\pi(q_\varphi(i)) = -\pi(q_\eta(i))$. If $\psi = \eta$ the case is settled, otherwise we apply the induction hypothesis.

If $\varphi = \eta_1 \oplus \eta_2$ then there are three cases to consider:

(i) $\psi = \eta_1$ (or symmetrically $\psi = \eta_2$). Then $\pi(q_\varphi(i)) = \pi(q_{\eta_1}(r)) + \pi(q_{\eta_2}(s))$ for suitable indexes $r \in I(\eta_1)$ and $s \in I(\eta_2)$. Since, by Definition 1, we can choose r and s such that $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_1}(r)$ and $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_2}(s)$, the case is settled.

(ii) $q_\varphi(i) = q_{\eta_1}(r)+0$ for some $r \in I(\eta_1)$, or $q_\varphi(i) = 0+q_{\eta_2}(s)$ for some $s \in I(\eta_2)$, with r and s such that $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_1}(r)$ and $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_2}(s)$. We can settle the case by induction hypothesis. Note that if either $q_\varphi(i) = q_{\eta_1}(r)$ and $\psi \preceq \eta_2$, or $q_\varphi(i) = q_{\eta_2}(s)$ and $\psi \preceq \eta_1$, then Condition 2(a) must hold.

(iii) $q_\varphi(i) = q_{\eta_1}(r) + q_{\eta_2}(s)$ for suitable indexes $r \in I(\eta_1)$ and $s \in I(\eta_2)$ such that $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_1}(r)$ and $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_2}(s)$. Then either $\psi \in \text{Sub}(\eta_1)$ or $\psi \in \text{Sub}(\eta_2)$. Suppose, without loss of generality, $\psi \preceq \eta_1$. Then $q_{\eta_1}(r)$ is not a constant function. We can apply the induction hypothesis to η_1 and find suitable sets $\{\vartheta_1, \ldots, \vartheta_m\}$ and $\{k_1, \ldots, k_m\}$ such that, either

$$(\star) \qquad \pi(q_\varphi(i)) = \pm\pi(q_{\vartheta_1}(k_1)) \pm \cdots \pm \pi(q_{\vartheta_m}(k_m)) \pm \pi(q_{\eta_2}(s))$$

or there is $j \in I(\psi)$ such that $q_\psi(j)$ is not a constant function, $\text{cell}_\varphi(i) \subseteq \text{cell}_{\eta_1}(r) \subseteq \text{cell}_\psi(j)$ and

$$(\star\star) \qquad \pi(q_\varphi(i)) = \pm\pi(q_{\vartheta_1}(k_1)) \pm \cdots \pm \pi(q_{\vartheta_m}(k_m)) \pm \pi(q_{\eta_2}(s)) \pm \pi(q_\psi(j)).$$

For each $h \in \{1, \ldots, m\}$, η_2 and ϑ_h are disjoint. Further, φ is the only subformula of φ such that $\eta_2 \neq \varphi$ and $\eta_2 \preceq \varphi$. Moreover, $q_{\eta_2}(s)$ is not a constant function. Finally, if $\eta_2 = \neg \cdots \neg \eta'$ is a negated formula, and η' is the greatest non-negated subformula of η_2, then we replace $\pi(q_{\eta_2}(s))$ by $\pi(q_{\eta'}(s'))$ in $(\star)$ and $(\star\star)$. Note that $s = s' \in I(\eta')$ by the construction of Definition 1.

□

Remark 1. The proof of Lemma 3 uniquely determines the sets $\{\vartheta_1, \ldots, \vartheta_m\}$ and $\{k_1, \ldots, k_m\}$.

Definition 5. Let φ, i and ψ be as in Lemma 3. If Condition 3.2(a) does not hold, while there is $j \in I(\psi)$ such that Condition 3.2(b) holds, then we say that $q_\varphi(i)$ is *built on* $q_\psi(j)$.

Example 3. Let φ be as in Example 1. Let ψ be the occurrence of X in the subformula $\neg(X \oplus Y)$ (Note that we consider this occurrence of X as determining a different subformula from the occurrence of X in $\neg(X \oplus Z)$. Thus, ψ and $\neg(X \oplus Z)$ are disjoint). Then $q_\varphi(1) = 2 - 2x - y - z$ and

$$\pi(q_\varphi(1)) = (-2, -1, -1) = (-1, -1, 0) + (-1, 0, -1),$$

whence

$$\pi(q_\varphi(1)) = \pi(q_{\neg(X\oplus Y)}(1)) + \pi(q_{\neg(X\oplus Z)}(1)).$$

On the other hand,

$$\pi(q_{\neg(X\oplus Y)}(1)) = (-1,-1,0) = -\pi(q_X(1)) - \pi(q_Y(1)),$$

and

$$\pi(q_\varphi(1)) = -\pi(q_X(1)) - \pi(q_Y(1)) + \pi(q_{\neg(X\oplus Z)}(1)),$$
$$\pi(q_\varphi(1)) = -\pi(q_X(1)) - \pi(q_Y(1)) - \pi(q_{X\oplus Z}(1)).$$

Consider now the subformula $\psi = X \oplus Y$. Then

$$\pi(q_\varphi(1)) = -\pi(q_{X\oplus Y}(1)) + \pi(q_{\neg(X\oplus Z)}(1)),$$
$$\pi(q_\varphi(1)) = -\pi(q_{X\oplus Y}(1)) - \pi(q_{X\oplus Z}(1)).$$

For another example, consider $q_\varphi(3) = 1 - x - y$ and $\psi = X \preceq \neg(X \oplus Y)$. Then

$$\pi(q_\varphi(3)) = \pi(q_{\neg(X\oplus Y)}(1)) = -\pi(q_X(1)) - \pi(q_Y(1)).$$

If $\psi = X \oplus Y$ then

$$\pi(q_\varphi(3)) = -\pi(q_{X\oplus Y}(1)).$$

The analysis of $q_\varphi(4)$ is similar.

The following two technical lemmas are the key tools in the proof of Lemma 7.

Lemma 4. *Let α be a formula and $t \in I(\alpha)$. Let $(\mathbf{a}_1, b)$ and $(\mathbf{a}_2, c)$ be elements of $D(\alpha, t)$ respectively introduced by (φ, i) and (ψ, j) (according to Definition 3) for suitable indexes $i \in I(\varphi)$ and $j \in I(\psi)$. Assume $\psi \preceq \varphi$ and suppose that there is an index $j' \in I(\psi)$ such that $q_\varphi(i)$ is built on $q_\psi(j')$. Then $j = j'$.*

Proof: Note that $q_\varphi(i)$ and $q_\psi(j)$ are not constant functions. By our hypotheses together with Definition 3, either $\mathrm{cell}_\alpha(t) \subseteq \mathrm{cell}_\varphi(i)$ or $\mathrm{cell}_\alpha(t) \subseteq \mathrm{cell}_\varphi(i+1)$. Similarly, either $\mathrm{cell}_\alpha(t) \subseteq \mathrm{cell}_\psi(j)$ or $\mathrm{cell}_\alpha(t) \subseteq \mathrm{cell}_\psi(j+1)$. Then $\psi \preceq \varphi$ implies $\mathrm{cell}_\varphi(i) \cup \mathrm{cell}_\varphi(i+1) \subseteq \mathrm{cell}_\psi(j)$ or $\mathrm{cell}_\varphi(i) \cup \mathrm{cell}_\varphi(i+1) \subseteq \mathrm{cell}_\psi(j+1)$.

If $\mathrm{cell}_\varphi(i) \cup \mathrm{cell}_\varphi(i+1) \subseteq \mathrm{cell}_\psi(j+1)$ then Lemma 3.2(a) would hold, since $q_\psi(j+1) = 1$, thus contradicting the assumption that $q_\varphi(i)$ is built on $q_\psi(j')$. Hence, $\mathrm{cell}_\varphi(i) \subseteq \mathrm{cell}_\psi(j)$ and $j = j'$. □

Lemma 5. *Let α be a formula and $(\mathbf{a}_0, c_0), (\mathbf{a}_1, c_1), \ldots, (\mathbf{a}_u, c_u)$ be vectors of $D(\alpha, t)$ for some $t \in I(\alpha)$. Take disjoint formulas $\psi_1, \ldots, \psi_u$ and a formula φ such that $\psi_1, \ldots, \psi_u \preceq \varphi \preceq \alpha$, and let the indexes $j_l \in I(\psi_l)$ ($l \in \{1, \ldots, u\}$), $i \in I(\varphi)$ be such that each (ψ_l, j_l) introduces $(\mathbf{a}_l, c_l)$ for (α, t) and (φ, i) introduces $(\mathbf{a}_0, c_0)$ for (α, t). Suppose there are indexes $j'_1 \in I(\psi_1), \ldots, j'_u \in I(\psi_u)$ such that, for each $l \in \{1, \ldots, u\}$, $q_\varphi(i)$ is built on $q_{\psi_l}(j'_l)$. Then there exists a set of formulas $\{\eta_1, \ldots, \eta_v\}$ together with a set of indexes $\{h_r : 1 \le r \le v,\ h_r \in I(\eta_r)\}$ such that*

$$\pi(q_\varphi(i)) = \sum_{l=1}^{u} \pm\pi(q_{\psi_l}(j_l)) + \sum_{r=1}^{v} \pm\pi(q_{\eta_r}(h_r)).$$

Further, the formulas $\psi_1, \ldots, \psi_u, \eta_1, \ldots, \eta_v$ are disjoint.

Proof: Pick sets of formulas $\Theta_1, \dots, \Theta_u$ and of indexes $K_1, \dots, K_u$ such that, for each $l \in \{1, \dots, u\}$, $\Theta_l = \{\vartheta_{l,1}, \dots, \vartheta_{l,m(l)}\}$ and $K_l = \{k_{l,1}, \dots, k_{l,m(l)}\}$ satisfy Condition 3.1 and Condition 3.2(b) with respect to φ, i, ψ_l and j'_l.

By Lemma 4, $\pi(q_\varphi(i)) = \pm\pi(q_{\psi_1}(j_1)) + \sum_{h=1}^{m(1)} \pm\pi(q_{\vartheta_{1,h}}(k_{1,h}))$.

By Lemma 3.1 there exists a unique $\vartheta_{1,\tilde{h}}$ such that $\psi_2 \preceq \vartheta_{1,\tilde{h}}$. Applying Lemma 3 to $\vartheta_{1,\tilde{h}}$ and $k_{1,\tilde{h}}$ we have

$$\pi(q_{\vartheta_{1,\tilde{h}}}(k_{1,\tilde{h}})) = \pm\pi(q_{\psi_2}(j''_2)) + \sum_{s=1}^{w} \pm\pi(q_{\zeta_s}(p_s))$$

for uniquely determined formulas $\zeta_1, \dots, \zeta_w$ and indexes $p_1, \dots, p_w$. Since $q_\varphi(i)$ is built on $q_{\psi_2}(j_2)$, by Lemma 4, $j''_2 = j'_2 = j_2$. Hence we can write

$$\pi(q_\varphi(i)) = \pm\pi(q_{\psi_1}(j_1)) \pm \pi(q_{\psi_2}(j_2)) + \sum_{s=1}^{w} \pm\pi(q_{\zeta_s}(p_s)) + \sum_{h \in J} \pm\pi(q_{\vartheta_{1,h}}(k_{1,h})),$$

where $J = \{1, \dots, m(1)\} \setminus \{\tilde{h}\}$.

Repeating this procedure for all formulas $\psi_3, \dots, \psi_u$, we obtain the desired conclusion. □

Given a vector $\mathbf{v} \in \mathbf{R}^n$, and a function $g : \mathbf{R}^n \to \mathbf{R}$, let $dg/d\mathbf{v}$ denote the *derivative of g in direction* $\mathbf{v}$. Note that every McNaughton function f is derivable in all directions (with trivial restrictions on the border of $[0,1]^n$).

Lemma 6. *For each formula φ in the variables $X_1, \dots, X_n$, and each point $\mathbf{x} \in [0,1]^n$, let $\mathbf{v} \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ for some coordinate unit vector $\mathbf{e}_i$. For each $q_\varphi(j) = a_{j,1}x_1 + \cdots + a_{j,n}x_n + b_j \in Q(\varphi)$ we have $|a_{j,i}| \leq \#(X_i, \varphi)$. Hence*

$$\left|\frac{df_\varphi}{d\mathbf{v}}(\mathbf{x})\right| \leq \#(X_i, \varphi).$$

Proof: Let K be the set of all $q_\varphi(j) \in Q(\varphi)$, $j \in I(\varphi)$, such that $\mathbf{x} \in \mathrm{cell}_\varphi(j)$ and $\{\mathbf{x} + \lambda\mathbf{v} : 0 < \lambda \in \mathbf{R}\} \cap \mathrm{cell}_\varphi(j) \neq \emptyset$. Then there exists $\eta > 0$, only depending on φ, $\mathbf{x}$ and $\mathbf{v}$, such that f_φ coincides with every function $q_\varphi(j)$ of K over $\{\mathbf{x} + \lambda\mathbf{v} : 0 \leq \lambda \leq \eta\}$. So $(df_\varphi/d\mathbf{v})(\mathbf{x}) = (dq_\varphi(j)/d\mathbf{v})(\mathbf{x}) = a_{j,i}$. The conclusion now follows by an easy induction on the complexity of φ. □

For each $\pi(q_\varphi(j)) = (a_{j,1}, \dots, a_{j,n}) \in \Pi(\varphi)$ we have $\sum_{i=1}^n |a_{j,i}| \leq \#(\varphi)$.

For any point $\mathbf{x} = (x_1, \dots, x_n) \in ([0,1] \cap \mathbf{Q})^n$, we let $\mathrm{den}(\mathbf{x})$ denote the least common denominator of $x_1, \dots, x_n$.

Lemma 7. *Let φ be a formula in the variables $X_1, \dots, X_n$, and $\mathbf{p}$ be a vertex of some n-dimensional cell of $C^{(n)}(\varphi)$. Then $\mathrm{den}(\mathbf{p}) \leq |\det(M_\mathbf{p})|$, for a suitable $n \times n$ matrix $M_\mathbf{p}$ whose entries $a_{i,j}$ are integers, and $\sum_{i,j=1}^n |a_{i,j}| \leq \#(\varphi)$.*

Proof: For some vector $\mathbf{b} \in \mathbf{Z}^n$ and some $s \in I(\varphi)$ the vertex $\mathbf{p}$ is the solution of a linear system $M\mathbf{x} = \mathbf{b}$ where the rows $\mathbf{l}_1, \ldots, \mathbf{l}_n$ of M are such that every $(\mathbf{l}_i, b_i)$ is an element of $D(\varphi, s)$. Hence

$$\mathbf{p} = M^{-1}\mathbf{b} = \frac{1}{\det(M)}\tilde{M}\mathbf{b},$$

where each entry of $\tilde{M}$ is an integer. Then $\text{den}(\mathbf{p}) \le |\det(M)|$.

Since each $(\mathbf{l}_i, b_i)$ is in $D(\varphi, s)$, we know, from Lemma 2, that either $\mathbf{l}_i = \pm\mathbf{e}_t$ for some coordinate unit vector $\mathbf{e}_t$, or $\mathbf{l}_i = (1/d_i)\pi(q_{\psi_i}(j_i))$ for some $\psi_i \in \text{Sub}(\varphi)$, integer $d_i \neq 0$ and $j_i \in I(\psi_i)$. Moreover $q_{\psi_i}(j_i)$ is not a constant function and we may assume that (ψ_i, j_i) introduces $(\mathbf{l}_i, b_i)$ for (φ, s). Since we are going to build a matrix $M_\mathbf{p}$ with the same determinant as M, for our current purposes of finding an upper bound to $\text{den}(\mathbf{p})$ we may safely assume that each d_i is equal to ± 1.

If there do not exist two distinct rows $\mathbf{l}_h, \mathbf{l}_k$ such that $\psi_h \preceq \psi_k$, and there is no row of the form $\mathbf{l}_i = \mathbf{e}_t$ for any $\mathbf{e}_t \in E_n$ then, by Lemma 6, we immediately get $M_\mathbf{p} = M$, and $\sum_{i,j=1}^{n} |a_{i,j}| \le \sum_{i=1}^{n} \#(\psi_i) \le \#(\varphi)$. Otherwise, let $P \subseteq \{1, \ldots, n\}$ be the index set such that $p \in P$ if and only if $\mathbf{l}_p \notin E_n$. Pick $h \in P$ such that there is no index $h' \in P$ for which $\psi_h \preceq \psi_{h'}$. Let $\psi_{k_1}, \ldots, \psi_{k_u}$ be the formulas from $\{\psi_p : p \in P\}$ such that $\psi_{k_l} \preceq \psi_h$ for all $l \in \{1, \ldots, u\}$ and there is no other formula $\psi_{k'} \in \{\psi_p : p \in P\}$ for which $\psi_{k_l} \preceq \psi_{k'} \preceq \psi_h$. We shall refer to $\mathbf{l}_h = \pi(q_{\psi_h}(j_h))$ as the *pivot* of M.

By Lemmas 3 and 4, for each $l \in \{1, \ldots, u\}$ there exists a uniquely determined set of formulas $\Theta_l = \{\vartheta_{l,1}, \ldots, \vartheta_{l,m(l)}\}$ and a set of indexes $K_l = \{g_{l,1}, \ldots, g_{l,m(l)}\}$ such that either

$$\pi(q_{\psi_h}(j_h)) = \sum_{t=1}^{m(l)} \pm\pi(q_{\vartheta_{l,t}}(g_{l,t}))$$

or

$$\pi(q_{\psi_h}(j_h)) = \pm\pi(q_{\psi_{k_l}}(j_{k_l})) + \sum_{t=1}^{m(l)} \pm\pi(q_{\vartheta_{l,t}}(g_{l,t})).$$

By Lemma 5 there exists a set of formulas $\{\eta_1, \ldots, \eta_{v_h}\}$ and a set of indexes $\{p_1, \ldots, p_{v_h}\}$ such that

$$\pi(q_{\psi_h}(j_h)) = \sum_{l \in J} \pm\pi(q_{\psi_{k_l}}(j_{k_l})) + \sum_{r=1}^{v_h} \pm\pi(q_{\eta_r}(p_r)), \text{ where } J \subseteq \{1, \ldots, u\}.$$

Replacing in M the row $\mathbf{l}_h$ by $\sum_{r=1}^{v_h} \pm\pi(q_{\eta_r}(p_r))$ we obtain a matrix M' such that $|\det(M')| = |\det(M)|$.

Now we choose a row $\mathbf{l}_{h''} \neq \mathbf{l}_h$ $(h'' \in P)$ from M' as the new pivot for M'. Repeating the above procedure finitely many times, we obtain a matrix $\bar{M}_\mathbf{p}$ such that each row $\mathbf{l}_a$, with $a \in P$, has the form $\sum_{r_a=1}^{v_a} \pm\pi(q_{\eta_{r_a}}(p_{r_a}))$ as given by Lemma 5. By our assumptions on the pivot formula, for any two distinct rows

$\mathbf{l}_a, \mathbf{l}_b$ of $\bar{M}_{\mathbf{p}}$ such that $a, b \in P$, and for each pair of indexes $r_a \in \{1, \ldots, v_a\}$, $r_b \in \{1, \ldots, v_b\}$, we have $\eta_{r_a} \not\preceq \eta_{r_b}$.

Consider now any row $\mathbf{l}_i$ of $\bar{M}_{\mathbf{p}}$ such that $\mathbf{l}_i = \mathbf{e}_t \in E_n$. By subtracting suitable multiples of $\mathbf{l}_i$ from the other rows, we get a matrix $\bar{M}'_{\mathbf{p}}$ such that all entries in the tth column are 0 except for the (i,t)th entry which is 1, while all the other columns and the determinant are left unchanged. Repeating this procedure for all rows $\mathbf{l}_i \in E_n$, by Lemma 6 we finally obtain a matrix $M_{\mathbf{p}} = \{a_{i,j} \in \mathbf{Z} : 1 \leq i, j \leq n\}$ such that $\sum_{i,j=1}^{n} |a_{i,j}| \leq \sum_{c=1}^{n} \sum_{r_c=1}^{v_c} \#(\eta_{r_c}) \leq \#(\varphi)$.

Moreover, $|\det(M_{\mathbf{p}})| = |\det(M)|$. □

Theorem 1. *Let φ be a formula in the variables $X_1, \ldots, X_n$, and let $\mathbf{p}$ be a vertex of a n-dimensional cell of $C^{(n)}(\varphi)$. Then*

$$\operatorname{den}(\mathbf{p}) \leq \left(\frac{\#(\varphi)}{n}\right)^n.$$

Proof: Let the matrix $M_{\mathbf{p}}$ be defined as in Lemma 7. By Hadamard's inequality, $\operatorname{den}(\mathbf{p}) \leq |\det(M_{\mathbf{p}})| \leq \prod_{i=1}^{n} \|\mathbf{l}_i\|$, where $\|\mathbf{l}_i\|$ is the euclidean norm $\sqrt{\mathbf{l}_i \cdot \mathbf{l}_i}$.

We have $\sqrt{a_{i,1}^2 + \cdots + a_{i,n}^2} \leq |a_{i,1}| + \cdots + |a_{i,n}|$, and hence

$$\operatorname{den}(\mathbf{p}) \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|.$$

Let k be an arbitrary real number > 0. Among all choices of n real numbers $c_1, \ldots, c_n \geq 0$ satisfying $\sum_{i=1}^{n} c_i \leq k$, the maximum of $\prod_{i=1}^{n} c_i$ is attained when $c_1 = c_2 = \cdots = c_n = k/n$. Therefore, by Lemma 7,

$$\operatorname{den}(\mathbf{p}) \leq \prod_{i=1}^{n} \frac{\#(\varphi)}{n} = \left(\frac{\#(\varphi)}{n}\right)^n.$$

□

Remark 2. Note that

$$\left(\frac{\#(\varphi)}{n}\right)^n = 2^{n \log(\#(\varphi)/n)} < 2^{\#(\varphi)},$$

where the logarithm is taken to the basis 2.

Lemma 8. *Let φ be a formula in the variables $X_1, \ldots, X_n$. If there is a point $\mathbf{q} \in [0,1]^n$ such that $f_\varphi(\mathbf{q}) < 1$, then there exists a point $\mathbf{p} \in ([0,1] \cap \mathbf{Q})^n$ such that* $\operatorname{den}(\mathbf{p})$ *divides* $2^{\#(\varphi)}$ *and* $f_\varphi(\mathbf{p}) < 1$.

Proof: Since f_φ attains its minimum value at a vertex of some n-dimensional cell of $C^{(n)}(\varphi)$, we can safely assume $\mathbf{q}$ to be such a vertex.

Among the points of the form $\mathbf{x} = (x_1/2^{\#(\varphi)}, \ldots, x_n/2^{\#(\varphi)})$ let $\mathbf{p}$ be the nearest point to $\mathbf{q}$, with respect to the euclidean distance $\|\mathbf{p} - \mathbf{q}\|$, By way of contradiction suppose that $f_\varphi(\mathbf{p}) = 1$. Then, by Theorem 1, $f_\varphi(\mathbf{p}) - f_\varphi(\mathbf{q}) \geq 1/\mathrm{den}(\mathbf{q}) \geq (n/\#(\varphi))^n$. On the other hand, by Lemma 6,

$$f_\varphi(\mathbf{p}) - f_\varphi(\mathbf{q}) \leq \sum_{i=1}^{n} (|p_i - q_i| \, \#(X_i, \varphi)).$$

It follows that

$$\left(\frac{n}{\#(\varphi)}\right)^n \leq \sum_{i=1}^{n} \left(\frac{1}{2}\frac{1}{2^{\#(\varphi)}}\#(X_i, \varphi)\right) \leq \frac{1}{2^{\#(\varphi)}}\frac{\#(\varphi)}{2}.$$

Therefore,

$$\frac{\#(\varphi)}{2}\left(\frac{\#(\varphi)}{n}\right)^n \geq 2^{\#(\varphi)}.$$

A straightforward calculation shows that the above inequality cannot be satisfied. Indeed, letting $g = \#(\varphi)/n$, we have

$$\frac{\#(\varphi)}{2} g^{(\#(\varphi)/g)} \geq 2^{\#(\varphi)} \quad \text{if and only if} \quad \frac{\log \#(\varphi) - 1}{\#(\varphi)} \geq \frac{g - \log g}{g}.$$

Let $\ln x$ denote the logarithm of x to the base e. The real-valued function $f_1(x) = (\log x - 1)/x$ reaches its maximum value $1/(2\mathrm{e}\ln 2)$ for $x = 2\mathrm{e}$. The function $f_2(x) = (x - \log x)/x$ attains its minimum value $1 - 1/(\mathrm{e}\ln 2)$ for $x = \mathrm{e}$.

Noticing that $1/(2\mathrm{e}\ln 2) < 1 - 1/(\mathrm{e}\ln 2)$ we have obtained a contradiction, and the proof is completed. □

Theorem 2. *For any formula* φ *we have*

$$\models_{\mathcal{L}_\infty} \varphi \qquad \textit{if and only if} \qquad \models_{\mathcal{L}_{2^{\#(\varphi)}+1}} \varphi.$$

Proof: As is well known, $\models_{\mathcal{L}_\infty} \varphi$ implies $\models_{\mathcal{L}_m} \varphi$, for all integers $m > 1$.

For the converse direction, suppose $\not\models_{\mathcal{L}_\infty} \varphi$. Then there exists a point $\mathbf{q} \in [0,1]^n$, where n is the cardinality of $\mathrm{Var}(\{\varphi\})$, such that $f_\varphi(\mathbf{q}) < 1$. By Lemma 8, there exists a point $\mathbf{p} \in ([0,1] \cap \mathbf{Q})^n$ such that $f_\varphi(\mathbf{p}) < 1$ and $\mathrm{den}(\mathbf{p})$ divides $2^{\#(\varphi)}$. Hence $\not\models_{\mathcal{L}_{2^{\#(\varphi)}+1}} \varphi$. □

Remark 3. The previous result can be immediately applied to formulas containing the derived connectives $\odot$ and $\rightarrow$, since $\#(\phi \odot \psi) = \#(\varphi \rightarrow \psi) = \#(\varphi \oplus \psi)$. So, we may consider $\odot$ and $\rightarrow$ as primitive connectives in formulas when counting the number of occurrences of variables.

When dealing with the lattice connectives $\wedge$ and $\vee$, some extra care must be taken, since $\#(\varphi \wedge \psi) = \#(\varphi \vee \psi) > \#(\varphi \oplus \psi)$.

Let A be the set $\{\pm\pi(q_\varphi(i)) \pm \pi(q_\psi(j)) : i \in I(\varphi), j \in I(\psi)\}$. Since $\varphi \vee \psi = \neg(\neg\varphi \oplus \psi) \oplus \psi$ we can write

$$\bigcup_{\mu \in \mathrm{Sub}(\varphi\vee\psi)} \Pi(\mu) \subseteq A \cup \bigcup_{\mu \in \mathrm{Sub}(\varphi)} \Pi(\mu) \cup \bigcup_{\mu \in \mathrm{Sub}(\psi)} \Pi(\mu).$$

The case for $\wedge$ is similar. By Lemma 2 we can safely assume $\#(\varphi \wedge \psi) = \#(\varphi \vee \psi) = \#(\varphi \oplus \psi)$.

The same trick of Lemma 8 cannot be used to reduce logical consequence in $\mathcal{L}_\infty$ to logical consequence in *exactly one* finite-valued logic $\mathcal{L}_{\tilde{m}+1}$ unless we take for $\tilde{m}$ a value that cannot be smaller than the factorial of $2^{\#(\Gamma)+\#(\Delta)}$.

Indeed, let us consider two finite sets Γ and Δ and a point $\mathbf{p} \in [0,1]^n$, where n is the cardinality of $\mathrm{Var}(\Gamma \cup \Delta)$, such that $f_{\bigwedge \Gamma}(\mathbf{p}) = 1$, while $f_{\bigwedge \Gamma}(\mathbf{q}) < 1$ for any other point $\mathbf{q} \in [0,1]^n$. For each $\mathbf{p} \in ([0,1] \cap \mathbf{Q})^n$ such Γ exists by McNaughton Theorem. A suitable choice is $\Gamma = \{\gamma\}$, where γ is the formula determining a Schauder hat with maximum value in $\mathbf{p}$ taken with multiplicity $\mathrm{den}(\mathbf{p})$ (see [7] for Schauder hats and related topics).

Suppose $f_{\bigvee \Delta}(\mathbf{p}) < 1$, thus falsifying $\Gamma \models_{\mathcal{L}_\infty} \Delta$. Although the argument in Lemma 8 still works for Δ, whenever $\mathrm{den}(\mathbf{p})$ does not divide $\tilde{m}$ we are not able to fix a point $\mathbf{p}'$ such that $\mathrm{den}(\mathbf{p}')$ divides $\tilde{m}$ and $f_{\bigwedge \Gamma}(\mathbf{p}') = 1$.

Theorem 3. *Let Γ and Δ be finite sets of formulas. Then we have*

$$\Gamma \models_{\mathcal{L}_\infty} \Delta \text{ if and only if } \Gamma \models_{\mathcal{L}_{m+1}} \Delta \text{ for all integers } 1 \leq m \leq 2^{\#(\Gamma)+\#(\Delta)}.$$

Proof: Trivially, $\Gamma \models_{\mathcal{L}_\infty} \Delta$ implies $\Gamma \models_{\mathcal{L}_m} \Delta$ for all integers $m > 1$. For the converse direction, suppose that $\Gamma \not\models_{\mathcal{L}_\infty} \Delta$. Then there exists a point $\mathbf{p} \in [0,1]^n$, where n is the cardinality of $\mathrm{Var}(\Gamma \cup \Delta)$, such that $f_{\bigwedge \Gamma}(\mathbf{p}) = 1$, while $f_{\bigvee \Delta}(\mathbf{p}) < 1$.

Let the polyhedral complex $C(\Gamma, \Delta)$ be defined as the common refinement of the complexes $C(\bigwedge \Gamma)$ and $C(\bigvee \Delta)$ (assumed to be n-dimensional, see Definition 1). That is, $C^{(n)}(\Gamma, \Delta)$ contains precisely the n-dimensional polyhedra arising as intersections $\mathrm{cell}_{\bigwedge \Gamma}(j) \cap \mathrm{cell}_{\bigvee \Delta}(k)$ for all possible choices of indexes $j \in I(\bigwedge \Gamma)$ and $k \in I(\bigvee \Delta)$. Note that each vertex $\mathbf{v}$ of $\mathrm{cell}_{\bigwedge \Gamma}(j) \cap \mathrm{cell}_{\bigvee \Delta}(k)$ is the solution of a system $M\mathbf{x} = \mathbf{b}$, where each row $\mathbf{l}_i$ of M is such that $(\mathbf{l}_i, b_i) \in D(\bigwedge \Gamma, j) \cup D(\bigvee \Delta, k)$. Then, arguing as in the proof of Lemma 7 and recalling Theorem 1, we get $\mathrm{den}(\mathbf{v}) \leq 2^{\#(\Gamma)+\#(\Delta)}$.

Among the cells of $C(\Gamma, \Delta)$ containing the point $\mathbf{p}$ let H be such that the restriction of $f_{\bigwedge \Gamma}$ to H is the constant function 1. Note that H need not be n-dimensional. By linearity there is at least a vertex $\mathbf{q}$ of H such that $f_{\bigvee \Delta}(\mathbf{q}) < 1$, otherwise $f_{\bigvee \Delta}$ would evaluate to 1 all over H. Then, $\mathrm{den}(\mathbf{q}) \leq 2^{\#(\Gamma)+\#(\Delta)}$, and $\Gamma \not\models_{\mathcal{L}_{\mathrm{den}(\mathbf{q})+1}} \Delta$. □

As is well known, in Łukasiewicz logics the Deduction Theorem does not hold. Nevertheless we have

Theorem 4. [7] *Let Γ and Δ be finite sets of formulas, then*

(i) For any integer $m > 1$, $\Gamma \models_{\mathcal{L}_{m+1}} \Delta$ if and only if $\models_{\mathcal{L}_{m+1}} (\bigwedge \Gamma)^m \to \bigvee \Delta$
(ii) $\Gamma \models_{\mathcal{L}_\infty} \Delta$ if and only if there exists k such that $\models_{\mathcal{L}_\infty} (\bigwedge \Gamma)^k \to \bigvee \Delta$. □

As an easy corollary of Theorem 3 we obtain an alternative version of Theorem 4(ii).

Corollary 2. *Let Γ and Δ be finite sets of formulas of $\mathcal{L}_\infty$. Then $\Gamma \models_{\mathcal{L}_\infty} \Delta$ if and only if, for every $m \leq 2^{\#(\Gamma)+\#(\Delta)}$, we have $\models_{\mathcal{L}_{m+1}} (\bigwedge \Gamma)^m \to \bigvee \Delta$.* □

4 A Sequent Calculus for infinite-valued Łukasiewicz Logic

At present, $\mathcal{L}_\infty$ lacks a well established proof theory despite the fact that cut-free calculi (or tableau systems), for finite-valued Łukasiewicz logics do exist, see e.g., [19, 20, 4, 9, 3, 5, 11].

Despite some calculi for infinite-valued Łukasiewicz logic have been defined in the literature, nevertheless these calculi either give no hints on the construction of proofs in $\mathcal{L}_\infty$ [21, 18, 6], or they are not internal, requiring spurious computations such as solving integer programs [12], intersecting hyperplanes [22], or determining θ-supports of formulas [17].

In this section we use our geometrical results to define an effective and internal sequent calculus for infinite-valued Łukasiewicz logic.

First we introduce the sequent calculus T_∞ to prove the tautologies of $\mathcal{L}_\infty$. This calculus is built up on SC_m calculi for finite-valued Łukasiewicz logics, that we briefly recall (see [3]).

The SC_{m+1} sequent calculus for $(m+1)$-valued Łukasiewicz logic is based on a representation of each formula of $\mathcal{L}_{m+1}$ by m many $\{0,1\}$-valued formulas of $\mathcal{L}_{m+1}$ as follows:

For all integers i, m with $0 < i \leq m$ and variable X, let $R_i^m(X)$ be the set of formulas of $\mathcal{L}_{m+1}$ such that, for any interpretation v for $\mathcal{L}_{m+1}$ and $\varphi \in R_i^m(X)$, $v(\varphi) = 0$ if $v(X) \leq \frac{m-i}{m}$ and $v(\varphi) = 1$ otherwise (see e.g. [1]).

Let $\beta_i^m(X)$ denote some fixed formula in $R_i^m(X)$.

Definition 6. For any integer $m > 0$ we define the map $B^m : L \to L^m$ by the following inductive stipulation:

- For any variable X, $B^m(X) = (\beta_1^m(X), \ldots, \beta_m^m(X))$.
- $B^m(\neg\varphi) = (\neg B_m^m(\varphi), \ldots, \neg B_1^m(\varphi))$ where, for all $i \in \{1, \ldots, m\}$, B_i^m is the projection of B^m along the ith component.
- For each $i \in \{1, \ldots, m\}$: $B_i^m(\psi \oplus \vartheta) = \bigwedge_{j+k=m+i} (B_j^m(\psi) \vee B_k^m(\vartheta))$.

$B^m(\varphi)$ is called the vector of *boolean components* of φ.

Note that each $B_i^m(\varphi)$ is a formula of $\mathcal{L}_{m+1}$. In [3] it is proved that $B^m(\varphi)$ is an increasing m-tuple in $\{0,1\}^m$, under any interpretation v for $\mathcal{L}_{m+1}$. More precisely, $v(B_i^m(\varphi)) = 0$ if and only if $v(\varphi) \leq (m-i)/m$.

In SC_{m+1}, sequents consist of m parts; formulas in each part respectively correspond to their components in the vector of boolean components. More precisely, *sequents* Υ have the form

$$\Gamma_1 \vdash \Delta_1 \,|\, \Gamma_2 \vdash \Delta_2 \,|\, \ldots \,|\, \Gamma_m \vdash \Delta_m,$$

where, for every $i = 1, \ldots, m$, Γ_i and Δ_i are finite sets of formulas. $\Gamma_i \vdash \Delta_i$ is called the ith *component* of Υ.

Sequents are interpreted in the following way:

Definition 7. A sequent $\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ldots \mid \Gamma_m \vdash \Delta_m$ is said to be *valid* in $\mathcal{L}_{m+1}$ if for all interpretations v for $\mathcal{L}_{m+1}$, there exists $h \in \{1, \ldots, m\}$ such that

$$(\star) \qquad v\Big(\bigwedge_{\gamma \in \Gamma_h} B_h^m(\gamma)\Big) \leq v\Big(\bigvee_{\delta \in \Delta_h} B_h^m(\delta)\Big).$$

Remark 4. The previous definition can be alternatively formulated as follows:

$$(\star) \qquad v\Big(\bigwedge_{\gamma \in \Gamma_h} \gamma\Big) \leq \frac{m-h}{m} \quad \text{or} \quad v\Big(\bigvee_{\delta \in \Delta_h} \delta\Big) \geq \frac{m-h+1}{m}.$$

Rules can be logical or structural. For every connective there are m rules, each rule dealing with a fixed component of the sequent. As in standard sequent calculi, each rule comes in a left and in a right version, according to which side of the sequent is being modified. Structural rules are either the usual weakening rules or the so called *migration rules* $(m, l)_j$ and $(m, r)_j$. Intuitively, logical rules simulate the construction of Definition 6, while axioms are the analogues of the classical axiom $A \vdash A$.

Notation: In order to give rules of SC_m calculi in a uniform way for every finite-valued Łukasiewicz logic, we adopt the following notation. $[\Gamma \vdash \Delta]_i^m$ denotes the m-component sequent whose ith component is $\Gamma \vdash \Delta$ and the remaining ones are empty. Whenever the number of components of a sequent is clear from the context, we write $[\Gamma \vdash \Delta]_i$.

Given m-component sequents $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_h$ we let $[\Upsilon_1 \Upsilon_2 \cdots \Upsilon_h]$ denote the sequent Υ obtained by componentwise *merging* of $\Upsilon_1, \ldots, \Upsilon_h$: that is, for any $i \in \{1, \ldots, m\}$, the ith component of Υ will contain one copy of the ith component of the sequent Υ_j for each $j \in \{1, \ldots, h\}$ and nothing else.

For each integer $m > 0$ the sequent calculus SC_{m+1} is given by:

Axioms. For every $0 < j \leq m$,

$$[A \vdash A]_j.$$

Logical Rules. For each $i \in \{1, \ldots, m\}$,

$$(\neg, l)_i = \frac{[\Upsilon\, [\vdash A]_{m+1-i}\,]}{[\Upsilon\, [\neg A \vdash]_i\,]} \qquad (\neg, r)_i = \frac{[\Upsilon\, [A \vdash]_{m+1-i}\,]}{[\Upsilon\, [\vdash \neg A]_i\,]}$$

$$(\oplus, l)_i = \frac{[\Upsilon\, [A \vdash]_i\,] \quad [\Upsilon\, [B \vdash]_i\,] \quad [\Upsilon\, [A \vdash]_j\, [B \vdash]_k\,]_{j+k=m+i+1}}{[\Upsilon\, [A \oplus B \vdash]_i\,]}$$

$$(\oplus, r)_i = \frac{[\Upsilon\, [\vdash A]_j\, [\vdash B]_k\,]_{j+k=m+i}}{[\Upsilon\, [\vdash A \oplus B]_i\,]}.$$

Structural Rules. For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m-1\}$,

$$(w,l)_i = \frac{\Upsilon}{[\,\Upsilon\,[A \vdash]_i\,]} \qquad (w,r)_i = \frac{\Upsilon}{[\,\Upsilon\,[\vdash A]_i\,]}$$

$$(m,l)_j = \frac{[\,\Upsilon\,[A \vdash]_{j+1}\,]}{[\,\Upsilon\,[A \vdash]_j\,]} \qquad (m,r)_j = \frac{[\,\Upsilon\,[\vdash A]_j\,]}{[\,\Upsilon\,[\vdash A]_{j+1}\,]}.$$

Cut Rules. For each $i, j \in \{1, \ldots, m\}$ such that $i \leq j$,

$$(cut)_{ij} = \frac{[\,\Upsilon\,[\vdash A]_i\,] \quad [\,\Upsilon\,[A \vdash]_j\,]}{\Upsilon},$$

where the notation for the $(\oplus, l)_i$ rule is to be read as follows:
$[\,\Upsilon\,[A \vdash]_j\,[B \vdash]_k\,]_{j+k=m+i+1}$ stands for the set of sequents $\{[\Upsilon\,\Upsilon_{j,k}] : 1 \leq j, k \leq m,\ j+k = m+i+1\}$ such that the jth and kth components of $\Upsilon_{j,k}$ are respectively given by $A \vdash$ and $B \vdash$ while the other components are empty.

As usual, a sequent Γ is said to be *provable* in the SC_{m+1} calculus, if there is a tree of sequents, rooted in Γ, such that every leaf is an axiom and every internal node is obtained from its parent nodes by an application of a rule.

Rules for some derived connectives are as follows (see [2]):

$$(\wedge, l)_i = \frac{[\,\Upsilon\,[A, B \vdash]_i\,]}{[\,\Upsilon\,[A \wedge B \vdash]_i\,]} \qquad (\wedge, r)_i = \frac{[\,\Upsilon\,[\vdash A]_i\,] \quad [\,\Upsilon\,[\vdash B]_i\,]}{[\,\Upsilon\,[\vdash A \wedge B]_i\,]}$$

$$(\vee, l)_i = \frac{[\,\Upsilon\,[A \vdash]_i\,] \quad [\,\Upsilon\,[B \vdash]_i\,]}{[\,\Upsilon\,[A \vee B \vdash]_i\,]} \qquad (\vee, r)_i = \frac{[\,\Upsilon\,[\vdash A, B]_i\,]}{[\,\Upsilon\,[\vdash A \vee B]_i\,]}.$$

Notice that upon dualizing SC_m calculi one obtains Hähnle's tableau systems [11].

Theorem 5. [3]. *A sequent is valid in $\mathcal{L}_{m+1}$ if and only if it is provable in* SC_{m+1}. □

Remark 5. SC_m calculi without cut rules are complete for $\mathcal{L}_m$. Moreover the cut-elimination theorem holds.

The idea behind the T_∞ calculus is to define rules that translate any formula φ of $\mathcal{L}_\infty$ to the corresponding sequent in the proper calculus SC_m, according to Theorem 2.

This is to say, these rules enable one to automatically determine the integer m such that the validity of φ in $\mathcal{L}_\infty$ is equivalent to its validity in $\mathcal{L}_m$. Afterwards, the rules of SC_m calculi are used to complete the proof.

To do this we consider *labelled sequents* of the form

$$(\Sigma) : \Upsilon$$

where Υ is a sequent of the SC_m calculus, for some integer $m > 1$, and Σ is a (possibly empty) multiset of formulas of $\mathcal{L}_\infty$ called *label*. The empty label is denoted by ϵ.

The rules of the T_∞ calculus are divided into *sequent* and *label* rules.

Axioms and sequent rules are the same as for SC_{m+1} calculi, for every m, with the proviso that the label ϵ is added to every sequent. For instance, the $(\neg, l)_i$ rule is given by

$$(\neg, l)_i = \frac{(\epsilon) : [\Upsilon\,[\vdash A]_{m+1-i}\,]^m}{(\epsilon) : [\Upsilon\,[\neg A \vdash]_i\,]^m}.$$

Axioms are as follows:
For every m and $0 < j \leq m$,

$$(\epsilon) : [A \vdash A]_j^m.$$

Let Θ be any multiset of formulas.
Label rules are the following:

$$\frac{(\Theta, A) : [\vdash C]_1^m}{(\Theta, \neg A) : [\vdash C]_1^m} \qquad\qquad \frac{(\Theta) : [\vdash C]_1^m}{(\Theta, A) : [\vdash C]_1^m} \quad \text{if } A \text{ is a variable}$$

$$\frac{(\Theta, A, B) : [\vdash C]_1^{2m}}{(\Theta, A \star B) : [\vdash C]_1^m} \quad \text{where } \star \in \{\oplus, \odot, \to, \wedge, \vee\}.$$

Definition 8. A formula φ is *provable* in the T_∞ calculus if there is a tree of labelled sequents, rooted in $(\varphi) : \vdash \varphi$, such that every leaf is an axiom and every internal node is obtained from its parent nodes by an application of a rule.

Proposition 1. *The* subformula property *holds in* T_∞*, that is, if* φ *is provable in* T_∞ *then there is a proof of* φ *only containing subformulas of* φ*.* □

Sequent rules can be applied only to sequents labelled with (ϵ). As a matter of fact, label rules allow to increase the number of components of sequents in a way depending on the length of formulas to be proved.

Lemma 9. *Let* φ *be a formula of* $\mathcal{L}_\infty$*. Starting from* $(\varphi) : \vdash \varphi$ *and applying label rules in the inverse direction one obtains the sequent* $(\epsilon) : [\vdash \varphi]_1^{2^{\#(\varphi)-1}}$*.*

We show that the T_∞ calculus proves precisely the tautologies of $\mathcal{L}_\infty$. First of all we need to strengthen Lemma 8.

Lemma 10. *Let* φ *be a formula in the variables* $X_1, \ldots, X_n$*. If there is a point* $\mathbf{q} \in [0,1]^n$ *such that* $f_\varphi(\mathbf{q}) < 1$*, then there exists a point* $\mathbf{p} \in ([0,1] \cap \mathbf{Q})^n$ *such that* $\mathrm{den}(\mathbf{p})$ *divides* $2^{\#(\varphi)-1}$ *and* $f_\varphi(\mathbf{p}) < 1$*.*

Proof. For any variable $X_i \in \mathrm{Var}(\{\varphi\})$ and for any arbitrary $(n-1)$-tuple $\mathbf{a} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in [0,1]^{n-1}$ let the function $f_{\mathbf{a},i} : [0,1] \to [0,1]$ be defined as $f_{\mathbf{a},i}(x) = f_\varphi(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$. If the function $f_{\mathbf{a},i}$ is monotonic for every X_i and for every $\mathbf{a} \in [0,1]^{n-1}$ then the claim is trivially true. Otherwise, we can safely assume that for some $X_j \in \mathrm{Var}(\{\varphi\})$ there exist two indexes $h', h'' \in I(\varphi)$ for which the signs of the jth components of the vectors $\pi(q_\varphi(h'))$ and $\pi(q_\varphi(h''))$ are different. Therefore, in each point in $[0,1]^n$ the sum of the slopes of f_φ along the directions corresponding to the coordinate axes is always strictly smaller than $\#(\varphi)$. With reference to the proof of Lemma 8, to conclude the proof we must show that the following inequality is never satisfied:

$$\frac{\#(\varphi)-1}{2}\left(\frac{\#(\varphi)}{n}\right)^n \geq 2^{\#(\varphi)-1}.$$

Let $g = \#(\varphi)/n$. Then

$$\frac{\#(\varphi)-1}{2} g^{(\#(\varphi)/g)} \geq 2^{\#(\varphi)-1} \quad \text{if and only if} \quad \frac{\log(\#(\varphi)-1)}{\#(\varphi)} \geq \frac{g - \log g}{g}.$$

The real-valued function $f(x) = \log(x-1)/x$ attains its maximum value at a point $4 < \tilde{x} < 5$. Since f is strictly increasing for all $x \leq \tilde{x}$ and is strictly decreasing for all $x \geq \tilde{x}$, the restriction of f to the integers attains its maximum value at $x = 4$ or at $x = 5$. Indeed, letting e denote Euler's constant, $(\log 3)/4 < (\log 4)/5 < 1 - 1/(\mathrm{e} \ln 2) = \min\{(g - \log g)/g\}$. □

Remark 6. The proof of Lemma 10 implies that for every cell K of $C^{(n)}(\varphi)$ and for each vertex $\mathbf{p}$ of K:

$$\mathrm{den}(\mathbf{p}) \leq 2^{\#(\varphi)-1}.$$

Theorem 6. *A formula φ is a tautology in $\mathcal{L}_\infty$ if and only if it is provable in T_∞.*

Proof. By the previous lemma, together with Theorems 5 and 2. □

Corollary 3. [16] *The satisfiability problem of $\mathcal{L}_\infty$ is in $\mathcal{NP}$.*

Proof. Since for every formula φ of $\mathcal{L}_\infty$ the length of any branch of a proof as well as the number of nonempty components in every sequent of T_∞ only polinomially depend on the length of φ, the assert follows by Theorem 6. □

See [12] for one more proof.

Here are some examples of proofs in T_∞.

Example 4. We show that $A \oplus \neg A$ is provable in T_∞.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\epsilon) : A \vdash A | \vdash}{(\epsilon) : \vdash A | \vdash \neg A} \qquad \dfrac{(\epsilon) : \vdash |A \vdash A}{(\epsilon) : \vdash \neg A | \vdash A}}{(\epsilon) : \vdash A \oplus \neg A | \vdash}}{(A, A) : \vdash A \oplus \neg A | \vdash}}{(A, \neg A) : \vdash A \oplus \neg A | \vdash}}{(A \oplus \neg A) : \vdash A \oplus \neg A}$$

By contrast, the following example shows that the law of excluded middle $A \vee \neg A$ is not provable in T_∞

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\epsilon) : \vdash A \mid A \vdash}{(\epsilon) : \vdash A, \neg A | \vdash}}{(\epsilon) : \vdash A \vee \neg A | \vdash}}{(A, A) : \vdash A \vee \neg A | \vdash}}{(A, \neg A) : \vdash A \vee \neg A | \vdash}}{(A \vee \neg A) : \vdash A \vee \neg A}$$

By Proposition 1 together with the remark following Theorem 5, there is no way to obtain the sequent $(A \vee \neg A) : \vdash A \vee \neg A$ from axioms.

Remark 7. All manipulations performed in proofs of the T_∞ calculus are purely syntactic and never involve any algebraic or geometrical computation. Thus, our calculus differs from all other existing analytic calculi for $\mathcal{L}_\infty$ quoted in the Introduction.

Moreover, our calculus shows how proofs in infinite-valued Łukasiewicz logic can be subsumed by proofs in finite-valued logics: if we read from the bottom a proof in T_∞, we see that the elimination of a binary connective from a label of a SC_{n+1} sequent essentially amounts to lifting this sequent to a SC_{2n+1} sequent.

By slightly modifying the T_∞ calculus in the light of Theorem 3 we obtain a calculus for $\mathcal{L}_\infty$, as follows: Axioms and sequent rules are those of T_∞.
Let Θ be any multiset of formulas.
Label rules of the SC_∞ calculus are the following:

$$\frac{(\Theta, A) : \ [\Gamma \vdash \Delta]_1^m}{(\Theta, \neg A) : \ [\Gamma \vdash \Delta]_1^m} \qquad \frac{(\Theta) : \ [\Gamma \vdash \Delta]_1^m}{(\Theta, A) : \ [\Gamma \vdash \Delta]_1^m} \quad \text{if } A \text{ is a variable}$$

$$\frac{(\Theta, A, B) : \ [\Gamma \vdash \Delta]_1^{2m-1} \qquad (\Theta, A, B) \ : \ [\Gamma \vdash \Delta]_1^{2m}}{(\Theta, A \star B) : \ [\Gamma \vdash \Delta]_1^m} \quad \text{where } \star \in \{\oplus, \odot, \rightarrow, \wedge, \vee\}.$$

Definition 9. A sequent Υ is *provable* in the SC_∞ calculus if there is a tree of labelled sequents, rooted in Υ, such that every leaf is an axiom and every internal node is obtained from its parent nodes by an application of a rule.

Theorem 7. *Let Γ and Δ be finite sets of formulas of $\mathcal{L}_\infty$. Then $\Gamma \models_{\mathcal{L}_\infty} \Delta$ if and only if the sequent $(\bigwedge \Gamma, \bigvee \Delta) : \Gamma \vdash \Delta$ is provable in* SC_∞.

Proof. An easy induction on k shows that $(\bigwedge \Gamma, \bigvee \Delta) : \Gamma \vdash \Delta$ is provable in SC_∞ if and only if all the sequents $[\Gamma \vdash \Delta]_1^k$ are provable in SC_{k+1}, for every $1 \leq k \leq 2^{\#(\Gamma)+\#(\Delta)-1}$. Now apply Lemma 10 together with Theorems 5 and 3. □

Our results can be straightforwardly dualized to a tableaux framework. Due to the aforementioned correspondence between the SC_m calculi and Hähnle's tableau systems, one then obtains a tableau calculus for $\mathcal{L}_\infty$. By contrast with [12], our dualized calculus does not need integer programming at all.

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