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Compactness of composition operators on a Hilbert space of Dirichlet series

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Abstract

In this paper new $L^p_\alpha \to L^q_\beta$ estimates are proved for translation-invariant Radon transforms along curves for $\alpha \leqslant \beta$ and $p < q$. For a fixed α and β, if p is sufficiently close to 2 the best possible q is obtained, up to ε. The method is related to that of [5].

0. Introduction

Let $\mathscr{H}$ be the Hilbert space of Dirichlet series with square-summable coefficients, equipped with the scalar product $\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$ if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ belong to $\mathscr{H}$. By the Cauchy–Schwarz inequality, $\sum_{n=1}^{\infty} |a_n n^{-s}| < \infty$ if $f \in \mathscr{H}$ and $\Re s > \frac{1}{2}$, so that, denoting by $\mathbb{C}_\theta$ the half-plane $\Re s > \theta$, $\mathscr{H}$ appears as a Hilbert space of analytic functions on the half-plane $\mathbb{C}_{1/2}$, with reproducing kernel K_a $(a \in \mathbb{C}_{1/2})$, i.e. $f(a) = \langle f, K_a \rangle$ and $K_a(s) = \zeta(s + \bar{a})$, where ζ denotes the Riemann zêta-function (cf. [HLS]). And the functions $f_n(s) = n^{-s}$ $(n = 1, 2, \ldots)$ are a natural orthonormal basis of $\mathscr{H}$, which can be viewed as a Dirichlet series analog of the Hardy space H^2 of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic

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in the open unit disk D, and such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. For the Hardy space, it follows from the well-known Littlewood subordination principle [Sh2] that any analytic self-map ϕ: $D \to D$ induces a bounded "composition operator" on H^2 by the formula: $C_\phi(f) = f \circ \phi$. Such operators have been intensively studied during the two last decades [Sh2]; in particular, necessary and sufficient conditions for their compactness, in terms of the symbol ϕ, have been obtained by Shapiro [Sh1]. For the space $\mathscr{H}$, mainly due to the fact that not any analytic function in a half-plane can be represented as a Dirichlet series, the situation is different. Yet, Gordon and Hedenmalm [GH] have obtained the following characterization.

Theorem 1. *An analytic self-map ϕ: $\mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded composition operator C_ϕ: $f \mapsto f \circ \phi$ on $\mathscr{H}$ if and only if*

1. *ϕ is "representable" i.e. $\phi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer, and where the analytic function φ can be written as a convergent Dirichlet series $\sum_1^{\infty} c_n n^{-s}$ for $\Re s$ large enough: $\Re s > \theta$ (in short $\varphi \in \mathscr{D}$).*
2. *ϕ is "extendable" with "controlled range", namely ϕ has an analytic extension to $\mathbb{C}_0$, still denoted by ϕ, and such that*
   (a) *$\phi(\mathbb{C}_0) \subset \mathbb{C}_0$ if $c_0 \geqslant 1$.*
   (b) *$\phi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$ if $c_0 = 0$.*

Let us emphasize that the parameter c_0 (called the characteristic of ϕ in [GH]) is very important for the properties of C_ϕ; in particular, the cases $c_0 \geqslant 1$ and $c_0 = 0$ are significantly different; we will mainly concentrate on the case $c_0 = 0$ in this work, and often refer to [Ba2] for the case $c_0 \geqslant 1$.

In view of Theorem 1, it is natural to ask for a characterization of the compact operators C_ϕ: $\mathscr{H} \to \mathscr{H}$. For the Hardy space H^2, such a characterization was obtained in 1985 by J. Shapiro, in terms of the Nevanlinna counting function

$$N_\phi(z) = \sum \log \frac{1}{|w|} \quad \text{if } z \in \phi(D), \ 0 \text{ otherwise},$$

where the sum is extended to those w in D such that $\phi(w) = z$, and where the "weight" $\log \frac{1}{|w|}$ should be viewed essentially as $1 - |w|$, the distance of w to the boundary of D; and Shapiro's condition is a follows:

$$C_\phi\colon H^2 \to H^2 \text{ is compact if and only if } N_\phi(z) = o\left(\log \frac{1}{|z|}\right) \quad \text{as } |z| \overset{<}{\to} 1. \tag{1}$$

If the symbol ϕ is injective on D, (1) reduces to

$$C_\phi\colon H^2 \to H^2 \text{ is compact if and only if : } \lim_{|z| \overset{<}{\to} 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty. \tag{2}$$

The natural analog for the space $\mathscr{H}$ is

$$N_\phi(s) = \sum \mathscr{R}w \text{ if } s \in \phi(\mathbb{C}_0),\ 0 \text{ otherwise},$$

where the sum is extended to those w in $\mathbb{C}_0$ such that $\phi(w) = s$, and Bayart [Ba2] proved the following:

$$\text{If } c_0 \geqslant 1, \text{ then } N_\phi(s) = o(\mathscr{R}s) \text{ as } \mathscr{R}s \overset{>}{\to} 0 \text{ implies (essentially) } C_\phi \text{ compact.} \quad (3)$$

When $c_0 = 0$, the situation appears to be more intricate: first of all, ϕ is never injective in this case, and we will never have a sufficient condition as (2) at our disposal; second, the condition (3) is not easy (this is the same for (1) in the disk) to check on a given symbol ϕ; we will therefore follow a different route, and it will turn out that the "coefficients" point of view, i.e. the study of the matrix of C_ϕ on the canonical basis of $\mathscr{H}$, which gives nothing for H^2, is more tractable here.

1. Compactness of composition operators on $\mathscr{H}$

For an analytic self-map ϕ: $D \to D$, automatically inducing a bounded composition operator C_ϕ: $H^2 \to H^2$, the following is known [Sh2]:

$$\text{If } \phi \text{ has restricted range (i.e. if } ||\phi||_\infty < 1), \text{ then } C_\phi \text{ is compact.} \quad (4)$$

In fact [Sh2], the approximation numbers $a_n(C_\phi)$ are $O(r^n)$ for some $r<1$, and C_ϕ belongs to any Schatten class $S_p(H^2)$, $p>0$.

$$\text{If } C_\phi \text{ is compact, then } \lim_{|z| \overset{<}{\to} 1} \frac{1-|\phi(z)|}{1-|z|} = \infty. \quad (5)$$

The converse of (5) is true if ϕ is injective. (6)

The converse of (5) is false in general; there exists a necessary and sufficient condition in terms of the Nevanlinna counting function of ϕ, see (1) of the introduction. (7)

The operator C_ϕ: $H^2 \to H^2$ is Hilbert–Schmidt if and only if :

$$\int_0^{2\pi} \frac{d\theta}{1-|\phi(e^{i\theta})|} < \infty \quad (8)$$

Now, suppose that $||\phi||_\infty = 1$. Then C_ϕ is not compact in general, but (5) and (8) allow to produce non-trivial examples:

1. C_ϕ is compact, because (5) holds and ϕ is injective.
2. C_ϕ is compact and non-Hilbert–Schmidt because (5) holds and not (8), ϕ being still injective.

We would like to produce similar non-trivial examples for composition operators on $\mathscr{H}$. The analog of (5) is that if $C_\phi\colon \mathscr{H} \to \mathscr{H}$ is compact, then

$$\lim_{\mathscr{R}a \overset{>}{\to} \frac{1}{2}} \frac{\zeta(2\mathscr{R}a)}{\zeta(2\mathscr{R}\phi(a))} = \infty. \tag{9}$$

But, as is easily seen, (9) says nothing more than

$$\text{If } C_\phi \text{ is compact on } \mathscr{H}, \text{ then } \mathscr{R}\phi(a) > \tfrac{1}{2} \text{ for } \mathscr{R}a \geqslant \tfrac{1}{2}. \tag{10}$$

And, as it is clearly explained in [Ba2], (10) gives no extra-information on ϕ, contrarily to the case of the disk. To produce our examples, we will therefore have to follow a different route.

By analogy with (4), and in view of the Gordon–Hedenmalm Theorem, we shall say that $\phi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$, giving rise to a bounded composition operator, has *restricted range* if:

(a) $c_0 \geqslant 1$ and $\phi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$ for some $\varepsilon > 0$.
(b) $c_0 = 0$ and $\phi(\mathbb{C}_0) \subset \mathbb{C}_{\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$.

The following simple fact was observed by Bayart [Ba1]:

$$\text{If } \phi \text{ has restricted range, then } C_\phi\colon \mathscr{H} \to \mathscr{H} \text{ is compact.} \tag{11}$$

We shall now study the converse of (11) for symbols ϕ of the form

$$\phi(s) = c_0 s + c_1 + \sum_{j=1}^{\infty} c_{q_j} q_j^{-s}, \tag{12}$$

where $2 \leqslant q_1 < q_2 < \cdots$ are integers.

It will be convenient to give the following definition: integers $2 \leqslant q_1 < \cdots < q_d$ will be said to be *multiplicatively independent* if

Any integer $n \geqslant 2$ can be written as $n = q_1^{\alpha_1} \ldots q_d^{\alpha_d}$, $\alpha_j \in \mathbb{N}$, in at most one way; equivalently, $\log q_1, \ldots, \log q_d$ are rationally independent on the real line ($\bigstar$).

(Example: $q_1 = 2$, $q_2 = 3$, $q_3 = 30$.)

We will first study the "polynomial" (although non-trivial!) case:

$$\phi(s) = c_0 s + c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s},$$

where d is an integer, and $c_{q_1}, \dots, c_{q_d}$ are *non-zero* complex numbers, the integers q_j being multiplicatively independent. Kronecker's Theorem [HST] then implies that, for each $\sigma > 0$, one has

$$\inf_{t \in \mathbb{R}} \mathscr{R}\phi(\sigma + it) = c_0\sigma + \mathscr{R}c_1 - \sum_{j=1}^{d} |c_{q_j}| q_j^{-\sigma},$$

so that Gordon–Hedenmalm condition in Theorem 1 reads

$$\mathscr{R}c_1 \geqslant \frac{1}{2} + \sum_{j=1}^{d} |c_{q_j}| \text{ or } \mathscr{R}c_1 \geqslant \sum_{j=1}^{d} |c_{q_j}|, \tag{13}$$

according to whether $c_0 = 0$ or $c_0 \geqslant 1$.

ϕ has restricted range if and only if

$$\mathscr{R}c_1 > \frac{1}{2} + \sum_{j=1}^{d} |c_{q_j}| \text{ or } \mathscr{R}c_1 > \sum_{j=1}^{d} |c_{q_j}|, \tag{14}$$

according to whether $c_0 = 0$ or $c_0 \geqslant 1$.

If $c_0 = 0$ in (12), ϕ is never injective: in fact, according to a well-known result ([F], p. 131) of the theory of analytic, almost-periodic functions, a Dirichlet series $\varphi(s) = \sum_1^\infty c_n n^{-s}$, absolutely convergent in a half-plane $\mathscr{R}s > \theta$, will be injective on no vertical strip $\alpha < \mathscr{R}s < \beta$ of this half-plane. We will therefore make a direct study of the converse of (11): this study will reveal a striking analogy with the well-known Theorem of Polya [R] on the behavior of the standard random walk on the lattice $\mathbb{Z}^d$ of $\mathbb{R}^d$: recurrent for $d = 1, 2$, and transient for $d \geqslant 3$. But for the moment, the analogy is only technical: we ignore if there are conceptual reasons for it. We shall prove the three following theorems (Theorem 2 for the case $d = 1$; Theorem 6 for the case $d = 2$; Theorem 7 for the case $d \geqslant 3$).

Theorem 2. *Suppose that* $d = 1$, *i.e.* $\phi(s) = c_0 s + c_1 + c_{q_1} q_1^{-s}$, *and that* (13) *holds. Then*:

1. *If* $c_0 \geqslant 1$, *then we have the following*:
   (a) C_ϕ *is compact if and only if* ϕ *has restricted range* (*i.e.* $\mathscr{R}c_1 > |c_{q_1}|$).
   (b) C_ϕ *is Hilbert–Schmidt if and only if* $\mathscr{R}c_1 > |c_{q_1}| + \frac{1}{2}$.
2. *If* $c_0 = 0$, *then the following are equivalent*:
   (a) ϕ *has restricted range*.
   (b) C_ϕ *is compact*.
   (c) C_ϕ *is Hilbert–Schmidt*.

The proof will rely on the following simple lemmas.

Lemma 3. *Let $b > 1$ be fixed, and let $a > 0$ tend to infinity. Then one has*

$$\sum_{k \geqslant 1} \frac{(\log k)^a}{k^b} \sim \int_1^{\infty} \frac{(\log t)^a}{t^b} dt = \frac{\Gamma(a+1)}{(b-1)^{a+1}} \tag{15}$$

(Γ *denoting the Euler gamma function*).

$$\sum_{n \geqslant 0} \frac{a^{2n}}{(n!)^2} \sim \frac{1}{2\sqrt{\pi}} \frac{e^{2a}}{\sqrt{a}}. \tag{16}$$

(*See for example* [D], p. 195.)

Lemma 4. *Let (u_{ij}), (v_{ij}) be the matrices, on a fixed orthonormal basis $(e_i)_{i \geqslant 0}$ of a Hilbert space H, of two continuous operators U and V. Assume that $|u_{ij}| \leqslant v_{ij}$ for all i, j. Then ($\|\,\|$ denoting the operator norm) one has: $\|U\| \leqslant \|V\|$. Moreover, if V is compact, so is U.*

This is the so-called minorant property for the Schatten class $S_\infty(H) = K(H)$ (see [DPQ] or [Si]).

Lemma 5. *Let i be a positive integer, and $|w| < 1$. Then*

$$(1-w)^{-i-1} = \sum_{j=0}^{\infty} \binom{i+j}{i} w^j. \tag{17}$$

This is the negative binomial expansion.

Let us go back to the proof of Theorem 2.

For 1(a), we refer to [Ba2].

For 1(b), see the proof of 2 below.

Proof of Theorem 2. (a) $\Rightarrow$ (c). Set $\gamma_1 = \mathcal{R} c_1$, $\delta_1 = |c_{q_1}|$. For $j \in \mathbb{N}^\star$, one has

$$C_\phi(j^{-s}) = j^{-\phi(s)} = j^{-c_0 s} j^{-c_1} \exp(-c_{q_1} q_1^{-s} \log j)$$

$$= j^{-c_1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} c_{q_1}^i (\log j)^i (q_1^i j^{c_0})^{-s},$$

whence

$$\|C_\phi(j^{-s})\|^2 = j^{-2\gamma_1} \sum_{i=0}^{\infty} \frac{\delta_1^{2i}}{(i!)^2} (\log j)^{2i}.$$

Now, (16) of Lemma 3 implies:

$$\|C_\phi(j^{-s})\|^2 \sim \frac{1}{2\sqrt{\pi}\sqrt{\log j}} j^{-2(\gamma_1 - \delta_1)}.$$

Since $2(\gamma_1 - \delta_1) > 1$ from our assumption or from (14), we see that C_ϕ is Hilbert–Schmidt. Using (18) below, where S_p denotes the Schatten p-class, we could be slightly more precise

$$\text{If } 0 < p \leqslant 2,\ T \in \mathscr{L}(H) \quad \text{and} \quad \sum_{i=1}^{\infty} \|T(i^{-s})\|^p < \infty, \ \text{then } T \in S_p(\mathscr{H}) \tag{18}$$

(cf. [Si]). Here, we see that $\Sigma \|C_\phi(j^{-s})\|^p < \infty$ if and only if $p(\gamma_1 - \delta_1) > 1$. Therefore,

$$\text{If } \gamma_1 - \delta_1 > \frac{1}{p} \geqslant \frac{1}{2}, \quad \text{then } C_\phi \in S_p(\mathscr{H}). \tag{19}$$

(c) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (a). This is slightly more delicate. We shall denote by ℓ^2 (resp. ℓ_0^2) the Hilbert space of square-summable sequences $(a_n)_{n\geqslant 0}$ (resp. $(a_n)_{n\geqslant 1}$), with their respective canonical bases $(e_n)_{n\geqslant 0}$, $(\varepsilon_n)_{n\geqslant 1}$, and we shall denote by $(\varphi_i)_{i\geqslant 1}$ the canonical basis (i^{-s}) of $\mathscr{H}$. As we saw in (a) $\Rightarrow$ (c), the matrix $(\alpha_{pj}) = (\langle C_\phi(\varphi_j), \varphi_p \rangle)$ of C_ϕ on (φ_i) is

$$\alpha_{pj} = \frac{(-1)^i c_{q_1}^i (\log j)^i}{i! j^{c_1}} =: \alpha'_{ij} \quad \text{if } p = q_1^i,$$
$$\alpha_{pj} = 0 \qquad\qquad \text{otherwise.}$$

It is plain that C_ϕ is unitarily equivalent to the operator A': $\ell_0^2 \to \ell^2$ whose matrix on the bases (e_n) and (ε_n) is $(\alpha'_{ij}) = \langle A'(\varepsilon_j), e_i \rangle$. Moreover, one can write $\alpha'_{ij} = u_i v_j a_{ij}$, where $|u_i| = |v_j| = 1$, and where

$$a_{ij} = \frac{\delta_1^i (\log j)^i}{i! j^{\gamma_1}} =: \langle A(\varepsilon_j), e_i \rangle, \quad i \geqslant 0,\ j \geqslant 1.$$

Therefore, $A' = UAV$, where V: $\ell_0^2 \to \ell_0^2$ and U: $\ell^2 \to \ell^2$ are the diagonal operators defined by (v_j) and (u_i), respectively. This shows that C_ϕ is unitarily congruent to A, and we may as well study the compactness of A. Alternatively, we might have assumed $c_1 > 0$, $c_{q_1} > 0$ without loss of generality. Now, the compactness of A is equivalent to that of $B = AA^\star$: $\ell^2 \to \ell^2$, whose matrix on the basis (e_n) is

$$b_{ij} = \langle B(e_j), e_i \rangle = \sum_{h=1}^{\infty} a_{ih} a_{jh} = \frac{\delta_1^{i+j}}{i! j!} \sum_{h=1}^{\infty} \frac{(\log h)^{i+j}}{h^{2\gamma_1}}. \tag{20}$$

Suppose now that the image of ϕ is not restricted, i.e. that $\gamma_1 = \delta_1 + \frac{1}{2}$. It follows from (20), and from (15) in Lemma 3, that $0 \leqslant M_0^{-1} c_{ij} \leqslant b_{ij} \leqslant M_0 c_{ij}$, where $M_0 > 0$ is a constant, and where

$$c_{ij} = \frac{(i+j)!}{i!j!}\left(\frac{\delta_1}{2\gamma_1 - 1}\right)^{i+j} = \frac{(i+j)!}{i!j!}2^{-i-j}. \tag{21}$$

By Lemma 4, the compactness of B is equivalent to that of $C = (c_{ij})$. We shall show that C is not compact by studying it on the canonical basis (z^i) of $H^2 = H^2(D)$, which is a model for the canonical basis (e_i) of ℓ^2. We have by definition

$$C(z^i) = \sum_{j=0}^{\infty}\binom{i+j}{i}2^{-i-j}z^j = 2^{-i}\left(1-\frac{z}{2}\right)^{-i-1} = \frac{1}{1-z/2}\left(\frac{1}{2-z}\right)^i,$$

where we have used Lemma 5. By linearity, we have $C[f(z)] = (1-\frac{z}{2})^{-1}f(\frac{1}{2-z})$ for any $f \in H^2$, which amounts to say that

$$C = MC_h, \tag{22}$$

where M is the invertible multiplication operator (acting on H^2) by the H^∞-function $(1-\frac{z}{2})^{-1}$, and where C_h is the composition operator on H^2 (back to composition operators !) induced by the linear fractional transformation $h(z) = \frac{1}{2-z}$: $D \to D$. Now, it is plain that C_h is not compact on H^2; for example: $\lim_{r \overset{<}{\to} 1} \frac{1-h(r)}{1-r} = \lim_{r \overset{<}{\to} 1} \frac{1}{2-r} = 1$, therefore the necessary condition (5) for compactness is violated. From (22), we see that C is not compact either, since M is invertible; and then B, A, C_ϕ are not compact, which finishes the proof of Theorem 2 by contradiction. □

Turning to the cases $d = 2$ and $d \geqslant 3$, we shall now prove the two following theorems.

Conclusion (c) of Theorem 6 and conclusion (b) of Theorem 7 will in particular allow us to produce the non-trivial examples which we had in mind: see 1 and 2 following relation (8).

Theorem 6. *Suppose that* $d = 2$, *i.e.* $\phi(s) = c_0 s + c_1 + c_{q_1}q_1^{-s} + c_{q_2}q_2^{-s}$, *and that* (13) *holds* (*recall that* q_1, q_2 *are multiplicatively independent*). *Then*:

1. *If* $c_0 \geqslant 1$, *we have the following*:
   (a) C_ϕ *is compact if and only if* ϕ *has restricted range*, *i.e.* $\mathcal{R}c_1 > |c_{q_1}| + |c_{q_2}|$.
   (b) C_ϕ *is Hilbert–Schmidt if and only if* $\mathcal{R}c_1 > \frac{1}{2} + |c_{q_1}| + |c_{q_2}|$.
2. *If* $c_0 = 0$, *we have*:
   (a) C_ϕ *is always compact*.
   (b) C_ϕ *is Hilbert–Schmidt if and only if* ϕ *has restricted range*, *i.e.* $\mathcal{R}c_1 > \frac{1}{2} + |c_{q_1}| + |c_{q_2}|$.
   (c) *There exist composition operators on* $\mathcal{H}$, *with non-restricted range*, *which are compact and not Hilbert–Schmidt*.

Proof. (1) For the equivalence (a), we refer to [Ba2].

The equivalence (b) will follow from 2: the presence of $c_0 s$ just shifts $n^{-\phi(s)}$ by a multiplicative term $(n^{c_0})^{-s}$ and does not affect its norm.

(2)

(a) is more difficult and will be postponed to the end of this section.
(b) will be an obvious consequence of the forthcoming Lemma 8.
(c) is clearly a consequence of (a) and (b): take for example $\phi(s) = \frac{3}{2} + \frac{2^{-s}+3^{-s}}{2}$. If we do not insist that ϕ has non-restricted range, and if we allow $c_0 \neq 0$, there are very simple examples: if $\phi(s) = s + \varepsilon$, C_ϕ is compact for any $\varepsilon > 0$, and is Hilbert–Schmidt if and only if $\varepsilon > \frac{1}{2}$. □

Turning to the case $d \geqslant 3$, we have the following:

Theorem 7. *Suppose that $d \geqslant 3$, and that* (13) *holds.*

1. *If $c_0 \geqslant 1$, C_ϕ is Hilbert–Schmidt if and only if $\Re c_1 \geqslant \frac{1}{2} + \sum_{j=1}^{d} |c_{q_j}|$.*
2. *If $c_0 = 0$, we have*:
   (a) *C_ϕ is always Hilbert–Schmidt.*
   (b) *There exist composition operators C_ϕ on $\mathscr{H}$, with $c_0 = 0$, with non-restricted range, which are Hilbert–Schmidt.*

Proof. The proof is based on the following lemma. □

Lemma 8. *Let d be an integer $\geqslant 2$, $\delta_1, \ldots, \delta_d > 0$, $n \geqslant 1$, and*

$$S_n = \sum_{i_1+\cdots+i_d=n} \left(\frac{n!}{i_1!\ldots i_d!}\right)^2 \delta_1^{2i_1} \ldots \delta_d^{2i_d}$$

be the sum of squares of the multinomial coefficients. Then one has, as $n \to \infty$:

$$S_n \sim \lambda n^{-\frac{(d-1)}{2}} (\delta_1 + \cdots + \delta_d)^{2n}, \tag{23}$$

where $\lambda > 0$ is a constant independent from n.

Proof. This is well-known (Polya's theorem) when the δ_j's are equal, and the general case is similar: one has

$$(\bigstar) \quad S_n = \int_{Q_d} |\delta_1 e(\theta_1) + \cdots + \delta_d e(\theta_d)|^{2n} d\theta_1 \ldots d\theta_d,$$

where Q_d is the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ of $\mathbb{R}^d$, and where $e(\theta) = e^{2i\pi\theta}$. Now, one just estimates the right-hand side of $(\bigstar)$.

It is now easy to complete the proof of Theorem 7; let $n \in \mathbb{N}^\star$. Then

$$C_\phi(n^{-s}) = n^{-\phi(s)} = n^{-c_1}[\exp(-c_{q_1} q_1^{-s}\log n)\ldots\exp(-c_{q_d} q_d^{-s}\log n) n^{-c_0 s}]$$

$$= n^{-c_1} \sum_{i_1,\ldots,i_d \geqslant 0} \frac{(-1)^{i_1+\cdots+i_d}}{i_1!\ldots i_d!} c_{q_1}^{i_1}\ldots c_{q_d}^{i_d} (\log n)^{i_1+\cdots+i_d} (q_1^{i_1}\ldots q_d^{i_d} n^{c_0})^{-s}.$$

Due to the hypothesis, the integers $q_1^{i_1}\ldots q_d^{i_d} n^{c_0}$ are all distinct, so that

$$||C_\phi(n^{-s})||^2 = n^{-2\gamma_1} \sum_{i_1,\ldots,i_d \geqslant 0} \frac{\delta_1^{2i_1}\ldots\delta_d^{2i_d}}{(i_1!\ldots i_d!)^2} (\log n)^{2(i_1+\cdots+i_d)}.$$

(Recall that $\gamma_1 = \mathscr{R}c_1$ and $\delta_j = |c_{q_j}|$.) Summing in n and permuting, we get

$$\sum_{n=1}^{\infty} ||C_\phi(n^{-s})||^2 = \sum_{i_1,\ldots,i_d \geqslant 0} \frac{\delta_1^{2i_1}\ldots\delta_d^{2i_d}}{(i_1!\ldots i_d!)^2} \sum_{n=1}^{\infty} \frac{(\log n)^{2(i_1+\cdots+i_d)}}{n^{2\gamma_1}}.$$

Now use Lemma 3 (observe that $\gamma_1 \geqslant \frac{1}{2} + \delta_1 + \cdots + \delta_d$, so that $\gamma_1 > \frac{1}{2}$) to get

$$\sum_{n=1}^{\infty} ||C_\phi(n^{-s})||^2 \approx \sum_{i_1,\ldots,i_d \geqslant 0} \frac{\delta_1^{2i_1}\ldots\delta_d^{2i_d}}{(i_1!\ldots i_d!)^2} \frac{(2i_1+\cdots+2i_d)!}{(2\gamma_1-1)^{2(i_1+\cdots+i_d)}}$$

(where $A \approx B$ means that $\alpha^{-1}B \leqslant A \leqslant \alpha B$,

$\alpha > 0$ being a constant)

$$= \sum_{\ell=0}^{\infty} \frac{(2\ell)!}{(2\gamma_1-1)^{2\ell}(\ell!)^2} \sum_{i_1+\cdots+i_d=\ell} \left(\frac{\ell!}{i_1!\ldots i_d!}\right)^2 \delta_1^{2i_1}\ldots\delta_d^{2i_d}$$

$$= \sum_{\ell=0}^{\infty} \binom{2\ell}{\ell} (2\gamma_1-1)^{-2\ell} S_\ell,$$

with the notation of Lemma 8. Now, use Lemma 8 and the estimate $\binom{2\ell}{\ell} \sim \frac{4^\ell}{\sqrt{\pi\ell}}$ (coming from Stirling's formula, but which can be viewed as a special case of Lemma 8, when $d=2$, $\delta_1 = \delta_2 = 1/2$) to get:

$$\sum_{n=1}^{\infty} ||C_\phi(n^{-s})||^2 \approx \sum_{\ell=1}^{\infty} \frac{4^\ell}{\sqrt{\ell}(2\gamma_1-1)^{2\ell}} \frac{1}{\ell^{\frac{d-1}{2}}} (\delta_1+\cdots+\delta_d)^{2\ell}$$

$$= \sum_{\ell=1}^{\infty} \ell^{-d/2} \left(\frac{2(\delta_1+\cdots+\delta_d)}{2\gamma_1-1}\right)^{2\ell} =: \sum_{\ell=1}^{\infty} \omega_\ell(d).$$

Now, if $\gamma_1 > \frac{1}{2} + \delta_1 + \cdots + \delta_d$, i.e. if the image of ϕ is restricted, we trivially have $\sum_{\ell=1}^{\infty} \omega_\ell(d) < \infty$ for any value of d, and C_ϕ is Hilbert–Schmidt. And if $\gamma_1 = \frac{1}{2} + \delta_1 + \cdots + \delta_d$, i.e. if the image of ϕ is non-restricted, we have $\omega_\ell(d) = \ell^{-d/2}$, so that:

(a) If $d = 2$, C_ϕ is not Hilbert–Schmidt, which proves 2(b) of Theorem 6.
(b) If $d \geqslant 3$, C_ϕ is Hilbert–Schmidt, proving 2 of Theorem 7 which we are discussing.

End of Proof of Theorem 6. In the proof of Theorem 2, we had reduced the problem to the study of a composition operator on the disk. Here, the same method will not lead to a composition operator on the bidisk; moreover, such operators are not automatically bounded (think of $\varphi(z, w) = (z, z)$). On the other hand, Sarason [Sa] observed that integral operators are more general than composition ones, and we will be led here to such an integral operator, with help of the following simple lemma.

Lemma 9. *Let* $u, v \in \mathbb{C}$ *with* $|u| + |v| < 1$, *and let* $i, j \in \mathbb{N}$. *Then*:

$$S := \sum_{k,\ell \geqslant 0} \frac{(i + j + k + \ell)!}{i!j!k!} u^k v^\ell = \binom{i + j}{i} (1 - u - v)^{-i-j-1}.$$

Proof. This is easily checked by using Lemma 5 twice.

In Theorem 6, it remains to prove that, if $\phi(s) = c_1 + c_{q_1} q_1^{-s} + c_{q_2} q_2^{-s}$, with $\mathscr{R} c_1 = 1/2 + |c_{q_1}| + |c_{q_2}|$, then C_ϕ: $\mathscr{H} \to \mathscr{H}$ is compact.

To ease notation, we will assume that $q_1 = 2$, $q_2 = 3$, $c_1, c_2, c_3 > 0$, which does not lose any generality: the general case is the same as this one, up to unitary equivalence. We then have

$$\begin{aligned} C_\phi(n^{-s}) &= n^{-c_1} \exp(-c_2 2^{-s} \log n) \exp(-c_3 3^{-s} \log n) \\ &= n^{-c_1} \sum_{i,j \geqslant 0} \frac{(-1)^{i+j} c_2^i c_3^j}{i!j!} (\log n)^{i+j} (2^i 3^j)^{-s}, \end{aligned}$$

that is, with the notations of this section: $C_\phi(\varphi_n) = \sum_p^{\geqslant 1} a_{pn} \varphi_p$, with $a_{pn} = \langle C_\phi(\varphi_n), \varphi_p \rangle$ such that

$$a_{pn} = \begin{cases} n^{-c_1} \dfrac{(-1)^{i+j} c_2^i c_3^j}{i!j!} (\log n)^{i+j} & \text{if } p = 2^i 3^j, \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 2, we see that C_ϕ is unitarily congruent to the operator A: $\ell_0^2 \to H^2(D^2)$ defined by the matrix $a_{(i,j),n} = \langle A(\varepsilon_n), e_i \otimes e_j \rangle$, on the canonical basis $e_i \otimes e_j$ of $H^2(D^2)$, given by

$$a_{(i,j),n} = n^{-c_1} \frac{c_2^i c_3^j}{i!j!} (\log n)^{i+j}, \quad n \geqslant 1, \ \ i,j \geqslant 0.$$

Let $B = AA^\star$: $H^2(D^2) \to H^2(D^2)$. The matrix of B on $e_i \otimes e_j$ is given by

$$\begin{aligned} b_{ijk\ell} &= \langle B(e_k \otimes e_\ell), e_i \otimes e_j \rangle = \sum_{n=1}^{\infty} a_{(i,j),n} a_{(k,\ell),n} \\ &= \sum_{n=1}^{\infty} n^{-2c_1} \frac{c_2^i c_3^j}{i!j!} \frac{c_2^k c_3^\ell}{k!} (\log n)^{i+j+k+\ell} \approx \frac{c_2^i c_3^j c_2^k c_3^\ell}{i!j!k!\ell!} \frac{(i+j+k+\ell)!}{(2c_1-1)^{i+j+k+\ell}}, \end{aligned}$$

where we used again Lemmas 3 and 4. The compactness of C_ϕ: $\mathscr{H} \to \mathscr{H}$ is therefore equivalent (since $AA^\star$ compact $\Leftrightarrow$ A compact) to that of C: $H^2(D^2) \to H^2(D^2)$ given by

$$C(z^i w^j) = \sum_{k,\ell \geqslant 0} \frac{c_2^i c_3^j c_2^k c_3^\ell}{i!j!k!\ell!} \frac{(i+j+k+\ell)!}{(2c_1-1)^{i+j+k+\ell}} z^k w^\ell.$$

Equivalently, by Lemma 9 for $u = \frac{c_2 z}{2c_1 - 1}$, $v = \frac{c_3 w}{2c_1-1}$:

$$C(z^i w^j) = \binom{i+j}{i} \frac{c_2^i c_3^j}{(2c_1-1)^{i+j}} \left(1 - \frac{c_2 z}{2c_1-1} - \frac{c_3 w}{2c_1-1}\right)^{-i-j-1}. \tag{24}$$

Now, set $\alpha = \frac{c_2}{2(c_2+c_3)}$, $\beta = \frac{c_3}{2(c_2+c_3)}$, $\varphi(z,w) = \varphi = 1 - \alpha z - \beta w$; since $2c_1 - 1 = 2(c_2 + c_3)$ from our assumptions, we have as well

$$C(z^i w^j) = \binom{i+j}{i} \alpha^i \beta^j \varphi^{-i-j-1}.$$

Let $f(z,w) = \Sigma a_{ij} z^i w^j$ be a polynomial in H^2; set temporarily $\omega = \sqrt{-1}$, $F(u,v) = f(e^{\omega u}, e^{\omega v})$, to see that

$$\begin{aligned} C(f)(z,w) &= \sum_{i,j} a_{ij} \binom{i+j}{i} \alpha^i \beta^j \varphi^{-i-j-1} \\ &= \frac{1}{4\pi^2} \int \int_{-\pi \leqslant u,v \leqslant \pi} F(u,v) \left(\sum_{i,j \geqslant 0} e^{-\omega(iu+jv)} \binom{i+j}{i} \alpha^i \beta^j \varphi^{-i-j-1} \right) du\, dv \end{aligned}$$

$$= (2\pi)^{-2} \int\int_{-\pi \leqslant u,v \leqslant \pi} F(u,v) \left(\sum_{\ell \geqslant 0} \varphi^{-\ell-1} \sum_{i+j=\ell} \binom{\ell}{i} (\alpha e^{-\omega u})^i (\beta e^{-\omega v})^j \right) du\, dv$$

$$= (2\pi)^{-2} \int\int_{-\pi \leqslant u,v \leqslant \pi} F(u,v) \sum_{\ell \geqslant 0} \frac{1}{\varphi} \left(\frac{\alpha e^{-\omega u} + \beta e^{-\omega v}}{\varphi} \right)^{\ell} du\, dv$$

$$= (2\pi)^{-2} \int\int_{-\pi \leqslant u,v \leqslant \pi} F(u,v) \frac{1}{\varphi - (\alpha e^{-\omega u} + \beta e^{-\omega v})} du\, dv.$$

Ignoring the constant factor $(2\pi)^{-2}$, and setting $a = 4\alpha$, $b = 4\beta$, we see that C: $H^2(T^2) \to H^2(T^2)$ is the operator given by the kernel (denoted as the operator itself)

$$C(u,v;x,y) = \frac{1}{4 - a(e^{ix} + e^{-iu}) - b(e^{iy} + e^{-iv})}, \quad \text{with } a + b = 2. \tag{25}$$

Here, we have set $z = e^{ix}$, $w = e^{iy}$, replaced the symbol ω by i, and taken $a = 4\alpha$, $b = 4\beta$. In fact,

$$C(f)(z,w) = \int\int_{-\pi \leqslant u,v \leqslant \pi} F(u,v) \frac{1}{1 - \alpha(e^{ix} + e^{-iu}) - \beta(e^{iy} + e^{-iv})} du\, dv,$$

with $\alpha + \beta = \frac{1}{2}$, and we multiply by the constant factor 4, without affecting the nature of the operator C. If we denote f and F in the same way, we can as well write

$$C(f)(x,y) = \int\int_{-\pi \leqslant u,v,v \leqslant \pi} C(u,v;x,y) f(u,v)\, du\, dv,$$

where $C(u,v;x,y)$ is as in (25). The end of the proof is based on the following lemma:

Lemma 10. *The kernel C defines a Hilbert–Schmidt operator on $L^2([-\pi,\pi]^2)$. Equivalently, we have*

$$C \in L^2(\mathbb{T}^4), \tag{26}$$

where $\mathbb{T}$ denotes the unit circle with its Haar measure.

Proof. Let K, the kernel defined by

$$K(u,v;x,y) = \frac{1}{-i[a(x-u) + b(y-v)] + \frac{a}{2}(u^2 + x^2) + \frac{b}{2}(v^2 + y^2)}. \tag{27}$$

A simple computation shows that one has, for $-\pi \leqslant u,v,x,y \leqslant \pi$:

$$(C - K)(u,v;x,y) = O\left(\frac{1}{r + \rho}\right),$$

where $r = |x + iy|$, $\rho = |u + iv|$. Integration in polar coordinates therefore gives, since $4\rho r \leqslant (\rho + r)^2$:

$$C - K \in L^2(\mathbb{T}^4). \tag{28}$$

To finish the proof of Lemma 10, it remains to prove that

$$K \in L^2(\mathbb{T}^4). \tag{29}$$

Clearly, $|K(u,v;x,y)| \leqslant \frac{M}{|a(x-u)+b(y-v)|+u^2+v^2+x^2+y^2}$, where the constant M only depends on a and b. Now:

$$\begin{aligned}
&\int_{\mathbb{T}^4} |K(u,v;x,y)|^2 \, du\, dv\, dx\, dy \\
&\quad \leqslant M^2 \int\int\int\int_{-\pi \leqslant u,v,x,y \leqslant \pi} \frac{du\, dv\, dx\, dy}{[|a(x-u)+b(y-v)| + u^2 + v^2 + x^2]^2} \\
&\quad = M^2 \int\int\int_{-\pi \leqslant u,v,x \leqslant \pi} du\, dv\, dx \left[\int_{-\pi \leqslant y \leqslant \pi} \frac{dy}{[|a(x-u)+b(y-v)| + u^2 + v^2 + x^2]^2} \right].
\end{aligned}$$

Now, for fixed u, v, x, the change of variable $a(x-u) + b(y-v) = z$ in the inner integral shows that this inner integral is less than $b^{-1} \int_{|z| \leqslant 4\pi} \frac{dz}{(u^2+v^2+x^2+|z|)^2}$. Therefore,

$$\int_{\mathbb{T}^4} |K(u,v;x,y)|^2 \, du\, dv\, dx\, dy \leqslant M^2 b^{-1} \int_{|z| \leqslant 4\pi} \left[\int_{\mathbb{R}^3} \frac{du\, dv\, dx}{(u^2+v^2+x^2+|z|)^2} \right] dz.$$

The new inner integral $I(z)$ is evaluated in spherical coordinates

$$I(z) = 4\pi \int_0^\infty \frac{r^2\, dr}{(r^2+|z|)^2} = 4\pi \int_0^\infty \frac{|z|^{3/2} s^2\, ds}{|z|^2 (s^2+1)^2},$$

making the change of variable $r = |z|^{1/2} s$. Finally, we see that

$$\int_{\mathbb{T}^4} |K(u,v;x,y)|^2 \, du\, dv\, dx\, dy \leqslant M^2 b^{-1} \int_{|z| \leqslant 4\pi} |z|^{-1/2} dz \int_0^\infty \frac{s^2\, ds}{(s^2+1)^2} < \infty$$

and this ends the proof of Lemma 10. □

To summarize, using the notation $T_1 \sim T_2$ to indicate that the operators T_j are simultaneously compact or non-compact, we have shown that

$$C_\phi \sim A, \quad \text{where } A\colon \ell_0^2 \to H^2(D^2);$$

$$AA^\star \sim C\colon H^2(D^2) \to H^2(D^2).$$

C is Hilbert–Schmidt, therefore compact. It follows that, although we know that C_ϕ is not Hilbert–Schmidt for $d = 2$ and $\mathscr{R}c_1 = \frac{1}{2} + |c_{q_1}| + |c_{q_2}|$ (in short, $C_\phi \notin S_2$), we have that $|C_\phi|^2 = C_\phi^\star C_\phi$ is Hilbert–Schmidt, i.e.: C_ϕ belongs to the Schatten class S_4. In particular, C_ϕ is compact, and this ends the proof of Theorem 6.

For more general symbols than "polynomial" ones, we have, as a corollary, the following

(Recall (see ($\bigstar$) page 4) that the positive integers $a, b, \dots$ are said to be multiplicatively independent if $\log a, \log b, \dots$ are rationally independent.)

Theorem 11. *Suppose that* $\phi(s) = c_0 s + c_1 + \sum_{j=1}^{\infty} c_{q_j} q_j^{-s}$, *with*

$$c_{q_j} \neq 0 \quad \textit{and} \quad \mathscr{R}c_1 \geqslant \frac{1}{2} + \sum_{j=1}^{\infty} |c_{q_j}|,$$

three at least of the q_j's *being multiplicatively independent*; *then*, C_ϕ: $\mathscr{H} \to \mathscr{H}$ *is a Hilbert–Schmidt operator*, *and in particular it is compact*.

Proof. Without loss of generality, we can assume that q_1, q_2, q_3 are independent. Set $\varepsilon = \sum_4^\infty |c_{q_j}|$, and write $\phi(s) = \phi_1(s) + R_1(s)$, where $\phi_1(s) = c_0 s + c_1 - \varepsilon + \sum_1^3 c_{q_j} q_j^{-s}$ and $R_1(s) = \varepsilon + \sum_4^\infty c_{q_j} q_j^{-s}$. We can apply Theorem 7 to ϕ_1, since $\mathscr{R}c_1 - \varepsilon - \sum_1^3 |c_{q_j}| \geqslant \frac{1}{2} + \sum_1^\infty |c_{q_j}| - \varepsilon - \sum_1^3 |c_{q_j}| = \frac{1}{2}$.

Moreover, we have $\mathscr{R}R_1(s) \geqslant 0$ if $s \in \mathbb{C}_0$, so that for any $n \geqslant 1$, $\varphi_n(s) = n^{-R_1(s)} \in \mathscr{H}^\infty$, the space of bounded analytic functions in $\mathbb{C}_0$, which are moreover representable by a convergent Dirichlet series for large $\mathscr{R}s$ [HLS]. It is proved in [HLS], that $\mathscr{H}^\infty$ is isometrically the space of multipliers of $\mathscr{H}$, so that:

$$\begin{aligned} ||n^{-\phi(s)}||_{\mathscr{H}} = ||n^{-\phi_1(s)} n^{-R_1(s)}||_{\mathscr{H}} &\leqslant ||n^{-\phi_1(s)}||_{\mathscr{H}} ||n^{-R_1(s)}||_{\mathscr{H}^\infty} \\ &\leqslant ||n^{-\phi_1(s)}||_{\mathscr{H}}, \end{aligned}$$

since $|n^{-R_1(s)}| \leqslant 1$ for $s \in \mathbb{C}_0$. Therefore

$$\sum_1^\infty ||n^{-\phi(s)}||_2^2 \leqslant \sum_1^\infty ||n^{-\phi_1(s)}||_2^2 < \infty,$$

in view of Theorem 7. □

2. Concluding remarks and questions

1. As concerns compactness, another proof of Theorem 2 has been obtained by F. Bayart [Ba2].

2. As we saw in Theorems 2, 6 and 7, the situation is completely different, according to whether $c_0 \geqslant 1$ or $c_0 = 0$. Looking at the statement of the Gordon–Hedenmalm Theorem 1, one could say that it is more difficult for C_ϕ to be bounded on $\mathscr{H}$ if $c_0 = 0$, but once it has succeeded to be continuous, it is more likely to be compact. The precise statement of Bayart's result can be summarized as follows: let $\phi(s) = c_0 s + c_1 + \sum_{j=1}^d c_{q_j} q_j^{-s}$, where the q_j's are multiplicatively independent and $c_0 \geqslant 1$. Then, the following are equivalent:
   (a) ϕ has restricted range,
   (b) $\mathscr{R} c_1 > \sum_{j=1}^d |c_{q_j}|$,
   (c) $C_\phi \colon \mathscr{H} \to \mathscr{H}$ is compact.(Recall that we must have $\phi(\mathbb{C}_0) \subset \mathbb{C}_0$, i.e. $\mathscr{R} c_1 \geqslant \sum_{j=1}^d |c_{q_j}|$.) In particular, the operators C_ϕ with $c_0 \geqslant 1$ will never be able to give examples as those at the end of Theorems 6 and 7.
3. The case of general (i.e. non-multiplicatively independent) q_j's seems to be very difficult to handle, although some results in that direction are obtained in [Ba2]. We hope to devote another work to this more general case.
4. In [LZ], a necessary and sufficient condition for a composition operator C_ϕ on $H^2(D)$, D the unit disk, to be in the Schatten class $S_p = S_p(H^2(D))$, $p > 0$, is given in terms of the symbol ϕ of the operator. The problem is touched here for $\mathscr{H}$ (see (19) in the course of the proof of Theorem 2), but clearly a systematic study remains to be done.

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