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Asymptotic approximations of orthogonal polynomials

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Abstract

It is well known that some orthogonal polynomials can be expressed in terms of Hermite polynomials taking limits in some of the parameters. Two of the most remarkable examples are the limit relations between the Jacobi and Laguerre polynomials and the Hermite polynomials. In this paper, we advance in this sense and establish a more general method which allows us to obtain asymptotic expansions of classical orthogonal polynomials in terms of the Hermite ones. By means of these asymptotic expansions, we can easily obtain well known limits between the polynomials of the Askey table and to obtain some near ones. In this paper we express the asymptotic representation of the Charlier polynomials in terms of Hermite polynomials and show some numerical experiments about the approximation convergence.

Keywords: Asymptotic expansions; Hermite polyomials; Charlier polynomials; Orthogonal polynomials; Askey scheme

AMS Classification: 33C25, 41A60, 30C15, 41A10

1 Introduction

It is well known that the Hermite polynomials

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}(2x)^{n-2k}, \tag{1}$$

play a crucial role in certain limits of the classical orthogonal polynomials. For example, the Laguerre polynomials, $L_n(x,\alpha)$, which are defined by the generating function

$$(1-w)^{-\alpha-1}e^{-wx/(1-w)} = \sum_{n=0}^{\infty} L_n(x,\alpha)w^n, \qquad |w|<1, \tag{2}$$

with $\alpha, x \in \mathbf{C}$, have the well-known limit

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n\left(x\sqrt{\alpha}+\alpha, \alpha\right) = \frac{(-1)^n 2^{-n/2}}{n!} H_n\left(\frac{x}{\sqrt{2}}\right). \tag{3}$$

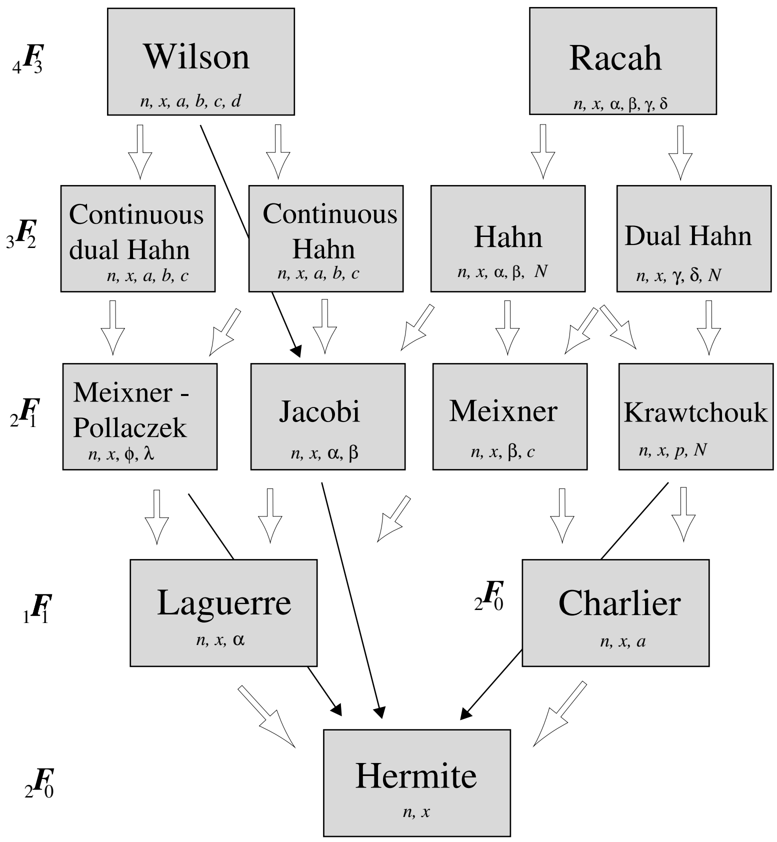

Figure 1: The Askey scheme for hypergeometric orthogonal polynomials, with limit relations between the polynomials.

For the Charlier polynomials, $C_n(x,a)$, which are defined by the generating function

$$e^w\left(1-\frac{w}{a}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x,a)}{n!} w^n, \quad x, a, w \in \mathbf{C}, \tag{4}$$

the corresponding limit reads

$$\lim_{a\to\infty} (2a)^{n/2} C_n\left((2a)^{1/2}x + a, a\right) = (-1)^n H_n(x). \tag{5}$$

These limits give insight into the location of the zeros for large values of the limit parameter and the asymptotic relation with the Hermite polynomials, if the parameters α or a are large enough and x is properly scaled.

Following the technique developed in [4], for some orthogonal polynomials, in this paper we describe the asymptotic relation that governs the limit of the Charlier polynomials. Here, we consider large values of the parameter a and then we obtain an asymptotic representation of the polynomials $C_n(x,a)$, from which the above limit can be derived, as a special case.

2 Expansions in terms of Hermite polynomials

Many special functions can be represented by means of generating series of the form

$$F(x,w) = \sum_{n=0}^{\infty} p_n(x) w^n, \tag{6}$$

where $F(x,w)$ is a given analytic function with respect to w, in a domain which contains the origin, and the functions p_n $(n = 0, 1, \ldots)$ do not depend on the variable w.

The relation (6) gives for p_n the following Cauchy-type integral

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} F(x,w) w^{-(n+1)} dw, \tag{7}$$

where $\mathcal{C}$ is the boundary of the circle around the origin inside the domain where F is analytic (as a function of w).

In particular, the Hermite polynomials follow from the generating function

$$e^{2xw-w^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} w^n, \quad x, w \in \mathbf{C}, \tag{8}$$

which gives the Cauchy-type integral

$$H_n(x) = \frac{n!}{2\pi i} \int_{C} e^{2xw-w^2} w^{-(n+1)} dw, \tag{9}$$

where $\mathcal{C}$ is defined as in (7).

Since the function F is analytic, then we can write

$$F(x,w) = e^{Aw-Bw^2} f(x,w),$$

where the coefficients A, B do not depend on w. This gives

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{Aw-Bw^2} f(x,w) w^{-(n+1)} dw, \tag{10}$$

and, taking into account that the function f is also analytic, we can expand

$$f(x,w) = e^{-Aw+Bw^2} F(x,w) = \sum_{k=0}^{\infty} c_k w^k, \tag{11}$$

that is,

$$f(x,w) = 1 + [p_1(x) - A]w + [p_2(x) - Ap_1(x) + B + \frac{1}{2}A^2]w^2 + \cdots,$$

taking $p_0(x) = 1$ (and then $a_0 = 1$).

Let us now substitute (11) in (10). Taking into account that the generating function of the Hermite polynomials is given in (8) and that all terms corresponding to indices $k > 0$ do not contribute in the integral of (7), we obtain the finite expansion

$$p_n(x) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\xi)}{(n-k)!}, \qquad z = \sqrt{B}, \quad \xi = \frac{A}{2\sqrt{B}}. \tag{12}$$

In order to obtain an asymptotic property of (12), let us take A and B such that $c_1 = c_2 = 0$. Then, we have,

$$A = p_1(x), \qquad B = \frac{1}{2}p_1(x)^2 - p_2(x). \tag{13}$$

As we will show in the next section, the asymptotic property of $p_n(x)$ can be deduced from the behavior of the coefficients c_k when the corresponding parameter of this function tends to infinity. As an example, we shall study what happens in the particular case of the Charlier polynomials.

2.1 Expansion of the Charlier polynomials

The Charlier polynomials are defined by

$$C_n(x, a) = {}_2F_0\left(\begin{array}{c} -n, \ -x \\ - \end{array} \middle| -\frac{1}{a} \right). \tag{14}$$

In this case, the generating function is

$$e^w \left(1 - \frac{w}{a}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x, a)}{n!} w^n, \quad x, a, w \in \mathbf{C}, \tag{15}$$

which gives the Cauchy-type integral

$$C_n(x, a) = \frac{n!}{2\pi i} \int_{\mathcal{C}} e^w \left(1 - \frac{w}{a}\right)^x w^{-(n+1)} dw, \tag{16}$$

where $\mathcal{C}$ is a circle around the origin and the integration is in the positive direction.

In order to obtain the asymptotic approximation of these polynomials, we have expressed the generating function of the Charlier polynomials in a similar way to the generating function of the Hermite polynomials. We have written

$$e^w \left(1 - \frac{w}{a}\right)^x = e^{Aw - Bw^2} f(x, w), \tag{17}$$

where A and B do not depend on w. This gives

$$C_n(x, a) = \frac{n!}{2\pi i} \int_{\mathcal{C}} e^{Aw - Bw^2} f(x, w) w^{-(n+1)} dw. \tag{18}$$

Since f is analytic, as a function of w, we can write

$$f(x, w) = e^{w - Aw + Bw^2} \left(1 - \frac{w}{a}\right)^x = \sum_{i=0}^{\infty} c_k w^k, \tag{19}$$

that is

$$f(x, w) = 1 + (C_1(x, a) - A)\, w + \Big(C_2(x, a) - AC_1(x, a) + B + A^2/2\Big)\, w^2 + \cdots,$$

taking $C_0(x, a) = 1$.

We substitute (19) into (16) and, taking into account the integral representation (9) of the Hermite polynomials, we get the finite expansion

$$C_n(x,a) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\xi)}{(n-k)!}, \qquad z = \sqrt{B}, \quad \xi = \frac{A}{2\sqrt{B}}. \tag{20}$$

In order to obtain an asymptotic property of (20), we take A and B such that $c_1 = 0$ and $c_2 = 0$. Then

$$A := C_1(x,a) = \frac{1}{a}(a-x), \quad B := \frac{1}{2}(C_1^2(x,a) - C_2(x,a)) = \frac{x}{2a^2}.$$

Finally, (20) becomes

$$C_n(x,a) = \left(\frac{x}{2a^2}\right)^{n/2} \sum_{k=0}^{n} \frac{c_k}{\left(\frac{x}{2a^2}\right)^{k/2}} \frac{H_{n-k}\left(\frac{a-x}{\sqrt{2x}}\right)}{(n-k)!}. \tag{21}$$

The remaining coefficients c_k, $k > 2$, can be obtained from the following recurrence relation

$$a^3(k+1)c_{k+1} = a^2 k c_k - x c_{k-2}. \tag{22}$$

This relation follows from substituting the Maclaurin series of f into the differential equation

$$x w^2 f = (w-a) a^2 \frac{df}{dw}. \tag{23}$$

To verify the asymptotic character of (21), we observe that the sequence c_k has the following asymptotic structure

$$c_k = \mathcal{O}\left(a^{-k}\right), \quad a \to \infty.$$

Moreover, the Hermite polynomials $H_n\left((a-x)/\sqrt{2x}\right)$ have degree n with respect to a. This gives the asymptotic nature of the terms in (21), for large values of a, with n and x fixed

$$\lim_{a\to\infty} c_k \left(\frac{x}{2a^2}\right)^{(n-k)/2} H_{n-k}\left(\frac{a-x}{\sqrt{2x}}\right) = 0, \quad \forall k > 0. \tag{24}$$

In order to express the Charlier polynomials in terms of Hermite ones, we substitute $x \mapsto (2a)^{1/2}x + a$. Then we have

$$\lim_{a\to\infty} \xi = \lim_{a\to\infty} \frac{a - \sqrt{2a}x - a}{\sqrt{2(\sqrt{2a}x + a)}} = -x, \tag{25}$$

and $z = \sqrt{2(\sqrt{2a}x + a)}/2a$ satisfies

$$\lim_{a\to\infty} \sqrt{2a} z = 1. \tag{26}$$

Taking limits on (21) and using (24), (25) and (26) we obtain

$$\lim_{a\to\infty} (2a)^{n/2} C_n((2a)^{1/2}x + a, a) = H_n(-x) = (-1)^n H_n(x), \tag{27}$$

which corresponds to the limit of the Askey scheme.

In Figure 2, we have plotted the approximation (27) for different values of a and degrees $n = 6$ and $n = 7$.

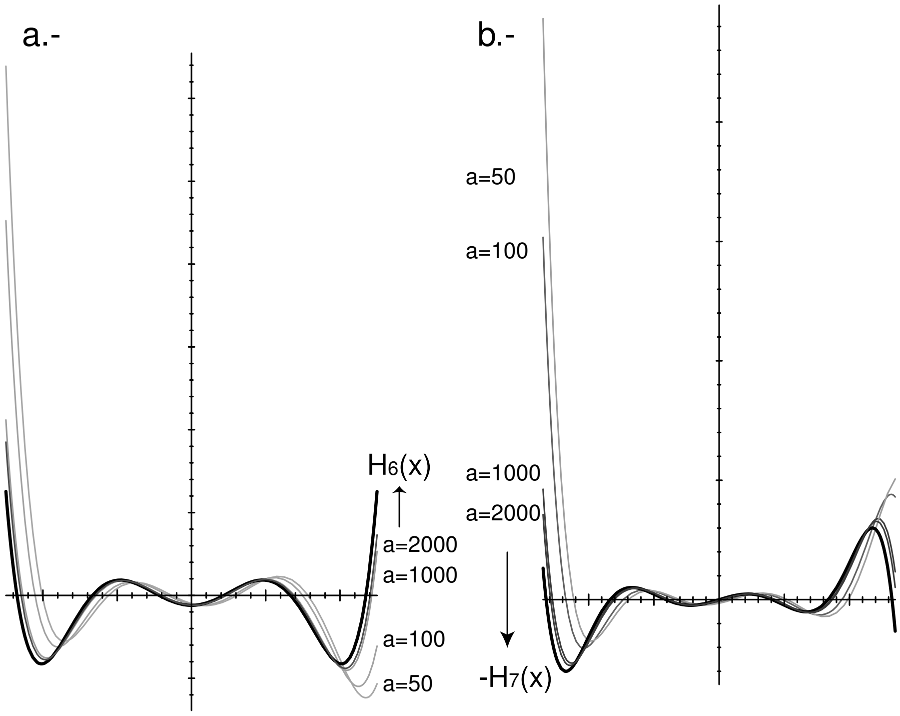

Figure 2: a.- Degree 6 approximation (27). b.- Degree 7 approximation (27).

References

[1] R. A. Askey, Orthogonal polynomials and special functions, *S.I.A.M., Philadelphia*, 1975.

[2] T. S. Chihara, An introduction to orthogonal polynomials, *Gordon and Breach, New York*, 1978.

[3] R. Koekoek and R. F. Swarttouw, Askey scheme of hypergeometric orthogohal polynomials, *http://aw.twi.tudelft.nl/koekoek/askey.*, (1999)

[4] J.L. López and N.M. Temme, Approximations of orthogonal polynomials in terms of Hermite polynomials, *Meth. Appl. Anal.* **6**, (1999) 131-146.

[5] N.M. Temme and J. L. López, The Askey scheme for hypergeometric orthogonal polynomials viewed from asymptotic analysis, *J. Comput. Appl. Math* **133**, (2001) 623-633.

[6] R. Wong, Asymptotic approximations of integrals, *Academic Press, New York*, 1989.