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UNIFORMLY STARLIKE MAPPINGS AND UNIFORMLY CONVEX MAPPINGS ON THE UNIT BALL B^n*

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Abstract In this article, we extend the definition of uniformly starlike functions and uniformly convex functions on the unit disk to the unit ball in $\mathbb{C}^n$, give the discriminant criterions for them, and get some inequalities for them.

Key words Uniformly starlike mappings; uniformly convex mappings; the unit ball

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1 Introduction

In 1936, M.S.Robertson proved that a convex function which is defined on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ can map each small disk contained in D to a convex domain ([1]). Then, is this result true for the starlike function? In 1991, A.W. Goodman introduced the definition of uniformly starlike functions on D, and by using it, he gave a counter example to answer "Negative" for the above question ([2]). Similarly, he introduced the definition of uniformly convex functions on D also ([3]).

Definition 1.1 Suppose that $f : D \to \mathbb{C}$ is a normalized locally biholomorphic function, and that $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

(1) ([2]) If for any $z_0 \in D$ and any arc $\gamma = \partial D(z_0, r) \cap D$, $0 < r < 1 + |z_0|$, $f(\gamma)$ is always starlike relative to $f(z_0)$, that is, $\arg(f(z) - f(z_0))$ is non decreasing when z moving along the positive direction of γ, then, f is said to be a uniformly starlike function on D;

(2) ([3]) If for any $z_0 \in D$ and any arc $\gamma = \partial D(z_0, r) \cap D$, $0 < r < 1 + |z_0|$, $f(\gamma)$ is always convex, that is, $\arg(\frac{\mathrm{d}}{\mathrm{d}t} f(z))$ is non decreasing when z moving along the positive direction of γ, then, f is said to be a uniformly convex function on D.

These two functions have discriminant criterions as follows.

Theorem 1.2 Let f be a normalized locally biholomorphic function on D. Then,

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(1) ([2]) f is a uniformly starlike function if and only if

$$\mathrm{Re}\frac{f(z)-f(z_0)}{(z-z_0)f'(z)} \geq 0 \tag{1.1}$$

holds for $\forall z, z_0 \in D$;

(2) ([3]) f is a uniformly convex function if and only if

$$\mathrm{Re}\left[1+\frac{(z-z_0)f''(z)}{f'(z)}\right] \geq 0 \tag{1.2}$$

holds for $\forall z, z_0 \in D$.

After that, S.Kanas, G.Kohr, and M.Kohr extended the above definitions to higher dimensions, defined uniformly starlike mappings and uniformly convex mappings on the unit ball $B^n = \left\{z = (z_1, \cdots, z_n)' \in \mathbb{C}^n : \|z\|^2 = \sum\limits_{i=1}^{n} |z_i|^2 < 1\right\}$, and gave their discriminant criterions respectively [4]. The two functions defined above have many good results, which we can see in [2, 3, 5, 6], etc. But for the mappings in several complex variables, the result is too little expect [4]. In this article, we will study the properties of these two mappings, give the discriminant criterions for them, and get some inequalities for them.

Suppose that $\Omega \subset \mathbb{C}^n$ is a domain and S is a family of normalized biholomorphic mappings on Ω. S is said to be a linear invariant family, if $f \in S$, then, $\Lambda_\varphi(f) \in S$ for any $\varphi \in \mathrm{Aut}(\Omega)$, where $\Lambda_\varphi(f) = (J_\varphi(0))^{-1}(J_f(\varphi(0)))^{-1}[f(\varphi(z)) - f(\varphi(0))]$, and $\mathrm{Aut}(\Omega)$ is the set of automorphisms of Ω.

It is well known that the family of normalized biholomorphic mappings, the family of normalized biholomorphic convex mappings, and the family of normalized locally biholomorphic mappings are linear invariant families.

In the second section, we will get an inequality for uniformly starlike mappings on B^n, which agrees with the result of starlike mappings in [7]. In the third section, we will give a discriminant criterion for uniformly convex mappings on B^n, and by using it, we will get an inequality similar to the result of Section 2. In Section 4, we will give two remarks and some examples to see the relations of families of some mappings. At the same time, we can see that the families of uniformly starlike mappings and uniformly convex mappings are not linear invariant families. In the last section, we will prove that a special convex mapping on B^n can map each small ball which is contained in B^n to a convex domain. But we do not know whether this result is true for general convex mappings.

2 Some Results for Uniformly Starlike Mappings

Firstly, we give the definition of uniformly starlike mappings on the unit ball B^n.

Definition 2.1 Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping, and denote $B^n(z_0, r) = \{z \in \mathbb{C}^n : \|z - z_0\| < r\}$. If for any $z_0 \in B^n, 0 < r < 1 + \|z_0\|$, the manifold $f(\partial B^n(z_0, r) \cap B^n)$ is starlike relative to $f(z_0)$, that is, the ray which is starting from $f(z_0)$ intersect with $f(\partial B^n(z_0, r) \cap B^n)$ at most one point, then, f is said to be a uniformly starlike mapping on B^n. The set of all of the uniformly starlike mappings on B^n is denoted to be $UST(B^n)$.

If let $z_0 = 0$ in Definition 2.1, then, it is seen that a uniformly starlike mapping is a starlike mapping.

In 2000, S. Kanas, G. Kohr, and M. Kohr gave a discriminant criterion for it on the unit ball B^n.

Theorem 2.2 ([4]) Suppose that $f : B^n \to \mathbb{C}^n$ is a normalized locally biholomorphic mapping, then, f is a uniformly starlike mapping on B^n if and only if

$$\mathrm{Re}[\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0))] \geq 0 \tag{2.1}$$

holds for any $z, z_0 \in B^n$.

When $z_0 = 0$, it becomes the discriminant criterion of starlike mapping exactly ([8]).

In this section, we will prove the following result.

Theorem 2.3 Suppose that f is a uniformly starlike mapping on B^n, then,

$$\begin{aligned} \|z-z_0\|^2 \frac{1-\max(\|z\|,\|z_0\|)}{1+\max(\|z\|,\|z_0\|)} &\leq \mathrm{Re}[\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0))] \\ &\leq |\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0))| \\ &\leq \|z-z_0\|^2 \frac{1+\max(\|z\|,\|z_0\|)}{1-\max(\|z\|,\|z_0\|)} \end{aligned} \tag{2.2}$$

holds for any $z, z_0 \in B^n$.

When $z_0 = 0$, it becomes the relative result of starlike mappings exactly ([7]).

In order to prove Theorem 2.3, we need a well known result in complex analysis.

Lemma 2.4 ([9]) If $g(\zeta)$ is a holomorphic function on the unit disk D, and it satisfies $g(0) = 1$, $\mathrm{Re}g(\zeta) \geq 0$, $\forall \zeta \in D$, then, we have

$$\frac{1-|\zeta|}{1+|\zeta|} \leq \mathrm{Re}g(\zeta) \leq |g(\zeta)| \leq \frac{1+|\zeta|}{1-|\zeta|}, \qquad \forall \zeta \in D. \tag{2.3}$$

Proof of Theorem 2.3 We only need to prove (2.2) for two cases $\|z_0\| < \|z\| < 1$ and $\|z\| < \|z_0\| < 1$.

For fixed $z, z_0 \in B^n$, we define

$$g_{z,z_0}(\zeta) = \frac{\overline{(z-z_0)}' J_f^{-1}(\zeta z)[f(\zeta z) - f(\zeta z_0)]}{\zeta \|z-z_0\|^2}.$$

It is a holomorphic function on a neighborhood of $\overline{D}$, and satisfies $g_{z,z_0}(0) = 1$. From the definition of uniformly starlike mappings, we may obtain the result that

$$\mathrm{Re}g_{z,z_0}(e^{\mathrm{i}\theta}) = \mathrm{Re}\frac{\overline{(e^{\mathrm{i}\theta}z - e^{\mathrm{i}\theta}z_0)}' J_f^{-1}(e^{\mathrm{i}\theta}z)[f(e^{\mathrm{i}\theta}z) - f(e^{\mathrm{i}\theta}z_0)]}{\|z-z_0\|^2} \geq 0$$

holds for any $\theta \in \mathbb{R}$.

By the minimum principle of real harmonic function, we know that $\mathrm{Re}g_{z,z_0}(\zeta) \geq 0$, $\forall \zeta \in D$. Then, $g_{z,z_0}(\zeta)$ satisfies the condition of Lemma 2.4, so,

$$\frac{1-|\zeta|}{1+|\zeta|} \leq \mathrm{Re}g_{z,z_0}(\zeta) \leq |g_{z,z_0}(\zeta)| \leq \frac{1+|\zeta|}{1-|\zeta|}, \qquad \forall \zeta \in D. \tag{2.4}$$

When $\|z_0\| < \|z\| < 1$, we can replace z in (2.4) by $\frac{z}{\|z\|}$, z_0 by $\frac{z_0}{\|z\|}$, and then the inequality (2.4) is also valid. So, we have

$$\frac{1-|\zeta|}{1+|\zeta|} \leq \mathrm{Re}\frac{\overline{(z-z_0)}' J_f^{-1}(\zeta\frac{z}{\|z\|})[f(\zeta\frac{z}{\|z\|}) - f(\zeta\frac{z_0}{\|z\|})]}{\frac{\zeta}{\|z\|}\|z-z_0\|^2}$$

$$\leq \left| \frac{\overline{(z-z_0)}' J_f^{-1}(\zeta \frac{z}{\|z\|})[f(\zeta \frac{z}{\|z\|}) - f(\zeta \frac{z_0}{\|z\|})]}{\frac{\zeta}{\|z\|}\|z-z_0\|^2} \right|$$
$$\leq \frac{1+|\zeta|}{1-|\zeta|}, \quad \forall \zeta \in D.$$

Taking $\zeta = \|z\|$ in the above inequality, we can get (2.2).

When $\|z\| < \|z_0\| < 1$, we can replace z in (2.4) by $\frac{z}{\|z_0\|}$, z_0 by $\frac{z_0}{\|z_0\|}$, and then the inequality (2.4) is also valid. So, we have

$$\frac{1-|\zeta|}{1+|\zeta|} \leq \mathrm{Re}\frac{\overline{(z-z_0)}' J_f^{-1}(\zeta \frac{z}{\|z_0\|})[f(\zeta \frac{z}{\|z_0\|}) - f(\zeta \frac{z_0}{\|z_0\|})]}{\frac{\zeta}{\|z_0\|}\|z-z_0\|^2}$$
$$\leq \left| \frac{\overline{(z-z_0)}' J_f^{-1}(\zeta \frac{z}{\|z_0\|})[f(\zeta \frac{z}{\|z_0\|}) - f(\zeta \frac{z_0}{\|z_0\|})]}{\frac{\zeta}{\|z_0\|}\|z-z_0\|^2} \right|$$
$$\leq \frac{1+|\zeta|}{1-|\zeta|}, \quad \forall \zeta \in D.$$

Taking $\zeta = \|z_0\|$ in the above inequality, we can get (2.2) also.

3 Some Results for Uniformly Convex Mappings

Definition 3.1 Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping. If for any $z_0 \in B^n, 0 < r < 1 + \|z_0\|$, the manifold $f(\partial B^n(z_0, r) \cap B^n)$ is convex relative to $f(z_0)$, that is, $f(\partial B^n(z_0, r) \cap B^n)$ and $f(z_0)$ are located on the same side of the tangent space of the manifold at the point $f(z)$, then, f is said to be a uniformly convex mapping on B^n. The set of all of the uniformly convex mappings on B^n is denoted to be $UCV(B^n)$.

If we let $z_0 = 0$ in Definition 3.1, then, it is seen that a uniformly convex mapping is a convex mapping.

In [4], S.Kanas, G.Kohr, and M.Kohr gave a discriminant criterion for uniformly convex mappings on B^n also.

Theorem 3.2 ([4]) Suppose that $f : B^n \to \mathbb{C}^n$ is a normalized locally biholomorphic mapping, then, f is a uniformly convex mapping on B^n if and only if

$$1 - \mathrm{Re}[\overline{(z-z_0)}' J_f^{-1}(z)\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}(z)(v, v)] \geq 0 \tag{3.1}$$

holds for any $z, z_0 \in B^n, v \in \partial B^n$, $\mathrm{Re}[\overline{(z-z_0)}' v] = 0$, where $\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}(z)(v, v) = \sum\limits_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial z_k}(z) v_j v_k$ is a column vector.

When $z_0 = 0$, it becomes one of a discriminant criterion of starlike mapping exactly ([7]).

As we can not judge if a uniformly convex mapping is a uniformly starlike mapping from Theorem 3.2 and this criterion is not convenient for application, we will give another discriminant criterion. From it, we can see easily that a uniformly convex mapping must be a uniformly starlike mapping.

Theorem 3.3 Suppose that $f : B^n \to \mathbb{C}^n$ is a normalized locally biholomorphic mapping, then, f is a uniformly convex mapping on B^n if and only if

$$\mathrm{Re}[\overline{(z-z_0)}' J_f^{-1}(z)(f(z) - f(w))] \geq 0 \tag{3.2}$$

holds for $z, z_0, w \in B^n$, $\|w - z_0\| \le \|z - z_0\|$.

When we take $z_0 = 0$ in (3.2), then, it becomes the discriminant criterions of convex mappings ([10]); when we take $w = z_0$, then, it becomes the discriminant criterions of uniformly starlike mappings. So, uniformly convex mappings must be convex mappings and uniformly starlike mappings.

Proof of Theorem 3.3 (Necessity) Let f be a uniformly convex mapping on the unit ball B^n, $z, z_0, w \in B^n$, $\|w - z_0\| \le \|z - z_0\|$. When $z = z_0 = w$, it is easy to see that (3.2) holds. So, we only need to suppose $r = \|z - z_0\| > 0$. Note that $r < 1 + \|z_0\|$. As $f(B^n)$ is a convex domain, and $f(\partial B^n(z_0, r) \cap B^n)$ is convex relative to $f(z_0)$, $f(B^n(z_0, r) \cap B^n)$ is a convex domain too. Then, for any $t \in (0, 1)$, and $z, w \in (\overline{B^n}(z_0, r) \cap B^n)$, we have $f^{-1}[(1-t)f(z) + tf(w)] \in (\overline{B^n}(z_0, r) \cap B^n)$. So,

$$\|f^{-1}[(1-t)f(z) + tf(w)] - z_0\|^2 \le r^2 = \|z - z_0\|^2,$$

or

$$\|(z - z_0) - J_f^{-1}(z)(f(z) - f(w))t + O(t^2)\|^2 \le \|z - z_0\|^2,$$

that is,

$$\|z - z_0\|^2 - 2\mathrm{Re}[\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(w))t] + O(t^2) \le \|z - z_0\|^2,$$

and it equivalents to

$$Re[\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(w))] + O(t) \ge 0.$$

Let $t \to 0^+$, we can get (3.2).

(Sufficiency) If (3.2) holds for any $z, z_0, w \in B^n$, $\|w - z_0\| \le \|z - z_0\|$, then, when we take $z_0 = 0$, it becomes the discriminant criterion of convex mappings, so f is a convex mapping on B^n. For any $z_0 \in B^n, 0 < r < 1 + \|z_0\|$, $z \in (\partial B^n(z_0, r) \cap B^n)$, we will prove that $f(\partial B^n(z_0, r) \cap B^n)$ and $f(z_0)$ are located on the same side of the tangent space of the manifold at the point $f(z)$.

Suppose that there exists a point $w \in (\partial B^n(z_0, r) \cap B^n)$ such that $f(z_0)$ and $f(w)$ are located on different side of this tangent space. As $V = \{v \in \mathbb{C}^n : \mathrm{Re}[\overline{(z - z_0)}' v] = 0\}$ is the tangent space of $\partial B^n(z_0, r) \cap B^n$ at the point z, $J_f(z)V$ is the tangent space of the manifold $f(\partial B^n(z_0, r) \cap B^n)$ at the point $f(z)$. Note that the column vector $\overline{(J_f^{-1}(z))}'(z - z_0)$ which is nonzero satisfies $\mathrm{Re}[\overline{(z - z_0)}' J_f^{-1}(z)u] = 0$, $\forall u \in J_f(z)V$, then, $\overline{(J_f^{-1}(z))}'(z - z_0)$ is a normal vector of the tangent space of the manifold $f(\partial B^n(z_0, r) \cap B^n)$ at the point $f(z)$. So, in this two numbers $\mathrm{Re}[\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(w))]$ and $\mathrm{Re}[\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(z_0))]$, there must be a negative. This is contradict with (3.2).

Using the above Theorem, we can get a result for uniformly convex mappings on B^n, which is similar to the result for uniformly starlike mappings in Section 2.

Theorem 3.4 Suppose that $f : B^n \to \mathbb{C}^n$ is a uniformly convex mapping, then,

$$\begin{aligned}\|z - z_0\|^2(1 - \max(\|z\|, \|z_0\|)) &\le \mathrm{Re}[\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(z_0))] \\ &\le |\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(z_0))| \\ &\le \|z - z_0\|^2(1 + \max(\|z\|, \|z_0\|)) \qquad (3.3)\end{aligned}$$

holds for any $z, z_0 \in B^n$, where z and z_0 have the same direction.

When $z_0 = 0$, (3.3) becomes the corresponding result of convex mappings in [11]. When z and z_0 have the same direction, it is reasonable that (3.3) is better than (2.2), for a uniformly convex mapping is also a uniformly starlike mapping.

To prove Theorem 3.4, we give the following lemma firstly.

Lemma 3.5 ([9]) If $g : D \to D$ is a holomorphic function, 1 is not the singular point of function g, and $g(0) = 0$, $g(1) = 1$, then $g'(1) \geq 1$.

Proof of Theorem 3.4 We only need to prove (3.3) for two cases $\|z_0\| < \|z\| < 1$ and $\|z\| < \|z_0\| < 1$.

Suppose that $z, z_0 \in B^n$, and define

$$h_{z,z_0}(\zeta) = \overline{(z - z_0)}' J_f^{-1}(z)[f(z) - f(z_0 + \zeta(z - z_0))].$$

When $\zeta \in D$, then $w = z_0 + \zeta(z - z_0) \in B^n$, and $\|w - z_0\| < \|z - z_0\|$, so, it is seen that $h_{z,z_0}(\zeta)$ is holomorphic on D, 1 is not a singular point of it, and $\mathrm{Re}h_{z,z_0}(\zeta) \geq 0, \forall \zeta \in D$. Because f is a biholomorphic mapping, $\mathrm{Re}h_{z,z_0}(\zeta)$ is not constant. We can get $\mathrm{Re}h_{z,z_0}(0) > 0$ by the minimum principle of real harmonic functions. Note that $h_{z,z_0}(0)$ and $-\overline{h_{z,z_0}(0)}$ are symmetric about imaginary axis, and $\mathrm{Re}h_{z,z_0}(\zeta) \geq 0$, $\forall \zeta \in D$, so,

$$\left| \frac{h_{z,z_0}(\zeta) - h_{z,z_0}(0)}{h_{z,z_0}(\zeta) - (-\overline{h_{z,z_0}(0)})} \right| \leq 1.$$

Then,

$$g_{z,z_0}(\zeta) = \frac{\overline{h_{z,z_0}(0)}}{h_{z,z_0}(0)} \frac{h_{z,z_0}(0) - h_{z,z_0}(\zeta)}{\overline{h_{z,z_0}(0)} + h_{z,z_0}(\zeta)}$$

satisfies the condition of Lemma 3.5, hence,

$$1 \leq g'_{z,z_0}(1) = \frac{\overline{h_{z,z_0}(0)}\|z - z_0\|^2 + h_{z,z_0}(0)\|z - z_0\|^2}{|h_{z,z_0}(0)|^2},$$

then,

$$|h_{z,z_0}(0)|^2 \leq 2\|z - z_0\|^2 \mathrm{Re}h_{z,z_0}(0),$$

that is,

$$|h_{z,z_0}(0) - \|z - z_0\|^2| \leq \|z - z_0\|^2,$$

or

$$|\overline{(z - z_0)}' J_f^{-1}(z)(f(z) - f(z_0)) - \|z - z_0\|^2| \leq \|z - z_0\|^2.$$

So,

$$\left| \frac{\overline{(z - z_0)}' J_f^{-1}(\zeta z)(f(\zeta z) - f(\zeta z_0))}{\zeta \|z - z_0\|^2} - 1 \right| \leq 1, \qquad \forall \zeta = e^{\mathrm{i}\theta}, \theta \in \mathbb{R}.$$

Then, by Schwarz' Lemma, we have

$$\left| \frac{\overline{(z - z_0)}' J_f^{-1}(\zeta z)(f(\zeta z) - f(\zeta z_0))}{\zeta \|z - z_0\|^2} - 1 \right| \leq |\zeta|, \qquad \forall \zeta \in D. \tag{3.4}$$

When $\|z_0\| < \|z\| < 1$, we replace z in (3.4) by $\frac{z}{\|z\|}$, z_0 by $\frac{z_0}{\|z\|}$, and inequality (3.4) is also valid. So, we have

$$\left| \frac{\overline{(z - z_0)}' J_f^{-1}(\zeta \frac{z}{\|z\|})(f(\zeta \frac{z}{\|z\|}) - f(\zeta \frac{z_0}{\|z\|}))}{\frac{\zeta}{\|z\|} \|z - z_0\|^2} - 1 \right| \leq |\zeta|, \qquad \forall \zeta \in D.$$

Taking $\zeta = \|z\|$ in the above inequality, we have

$$|\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0)) - \|z-z_0\|^2| \le \|z-z_0\|^2\|z\|, \tag{3.5}$$

so,

$$|\mathrm{Re}[\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0))] - \|z-z_0\|^2| \le \|z-z_0\|^2\|z\|. \tag{3.6}$$

Then, we can obtain the right part of (3.3) by (3.5), and obtain the left part of (3.3) by (3.6).

When $\|z\| < \|z_0\| < 1$, we replace z in (3.4) by $\frac{z}{\|z_0\|}$, z_0 by $\frac{z_0}{\|z_0\|}$, and the inequality (3.4) is also valid. So, we have

$$\left| \frac{\overline{(z-z_0)}' J_f^{-1}(\zeta\frac{z}{\|z_0\|})(f(\zeta\frac{z}{\|z_0\|}) - f(\zeta\frac{z_0}{\|z_0\|}))}{\frac{\zeta}{\|z_0\|}\|z-z_0\|^2} - 1 \right| \le |\zeta|, \qquad \forall \zeta \in D.$$

Taking $\zeta = \|z_0\|$ in the above inequality, we have

$$|\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0)) - \|z-z_0\|^2| \le \|z-z_0\|^2\|z_0\|, \tag{3.7}$$

so,

$$|\mathrm{Re}[\overline{(z-z_0)}' J_f^{-1}(z)(f(z)-f(z_0))] - \|z-z_0\|^2| \le \|z-z_0\|^2\|z_0\|. \tag{3.8}$$

we can obtain (3.3) by (3.7) and (3.8).

4 Some Remarks

Denote by $K(\Omega)$ the set of all of the convex mappings on the domain Ω, and by $S^*(\Omega)$ the set of all of the starlike mappings on Ω.

I Firstly, it is easy to see that

$$UCV(B^n) \subset K(B^n) \subset S^*(B^n)$$

and

$$UCV(B^n) \subset UST(B^n) \subset S^*(B^n),$$

from the discriminant criterions of starlike mappings, convex mappings, uniformly starlike mappings, and uniformly convex mappings. But we do not know the relation of $K(B^n)$ and $UST(B^n)$. In the following, we will give two examples to explain that either of these two classes of mappings can not be contained in the other.

Example 4.1 Let $f(z) = z + az^2, z \in D$. Then,

$$f \in K(D) \Leftrightarrow |a| \le \frac{1}{4}; \quad f \in UST(D) \Leftrightarrow |a| \le \frac{\sqrt{3}}{4}.$$

So, when $\frac{1}{4} < |a| \le \frac{\sqrt{3}}{4}$, $f \in UST(D)$, but $f \notin K(D)$.

Example 4.2 Let $f(z) = \dfrac{z}{1-az}, z \in D$. Then,

$$f \in K(D) \Leftrightarrow |a| \le 1; \quad f \in UST(D) \Leftrightarrow |a| \le \frac{\sqrt{2}}{2}.$$

So, when $\frac{\sqrt{2}}{2} < |a| \le 1$, we have $f \in K(D)$, but $f \notin UST(D)$.

II We have given the definition of linear invariant families and some examples of it in Section 1, such as $K(B^n)$, etc. Then, a natural question is: are $UST(B^n)$ and $UCV(B^n)$ linear invariant families? We can answer this question by the following example.

Example 4.3 Let $f(z)=\frac{z}{1-az}, z\in D$. Then $f\in UST(D)\Leftrightarrow |a|\le\frac{\sqrt{2}}{2}$; $f\in UCV(D)\Leftrightarrow |a|\le\frac{1}{3}$. Taking $\phi(z)=\frac{z+c}{1+\overline{c}z}$, which is a holomorphic automorphism of D, by simple calculation, we obtain

$$\Lambda_\phi(f)(z)=\frac{f(\phi(z))-f(\phi(0))}{\phi'(0)f'(\phi(0))}=\frac{z}{1-bz},$$

where $b=\frac{a-\overline{c}}{1-ca}$. Taking $a=\frac{1}{2}<\frac{\sqrt{2}}{2},\ \ c=-\frac{1}{2}$, then, we have $b=\frac{4}{5}>\frac{\sqrt{2}}{2}$, so $\Lambda_\phi(f)\notin UST(D)$. If we take $a=\frac{1}{4}<\frac{1}{3},\ \ c=-\frac{1}{2}$, then, $b=\frac{2}{3}>\frac{1}{3}$, so $\Lambda_\phi(f)\notin UCV(D)$.

From the above two remarks, we can see that the classes of uniformly starlike mappings and uniformly convex mappings are independent with each other, and they have some different properties from starlike mappings and convex mappings. But the result about them is too small, so we can do more studies on them.

5 Image of a Small Ball Which Lies in B^n Under a Class of Convex Mappings

From the definition of uniformly convex function on the unit disk D, we know that it can map any convex arc which lies in D to a convex curve. Then, for any small disk $D(z_0,r)=\{z\in\mathbb{C}:|z-z_0|<r, r\le 1-|z_0|\}\subseteq D$, the function can map it to a convex domain. In [1, 12], M.S. Robertson and J.E. Brown have proved that convex function f on D has the same property, that is, $f(D(z_0,r))$ is a convex domain in $\mathbb{C}$. So, we want to ask if a convex mapping on B^n has the similar property.

For the general convex mappings on B^n, we have not obtain any answer. But for a special convex mapping, the answer is true.

Theorem 5.1 Suppose that $f(z)=\frac{z}{1-\overline{p}'z}$, $z\in B^n$, $p\in\partial B^n$, then, f is a normalized biholomorphic convex mapping on B^n, and for any $z_0\in B^n$, $r\in(0,1-\|z_0\|]$, $f(B^n(z_0,r))$ is a convex domain in $\mathbb{C}^n$.

Proof It is easy to see that f is a normalized biholomorphic convex mapping on B^n.

In order to get that the domain $f(B^n(z_0,r))$ is convex, we only need to prove that

$$\mathrm{Re}\overline{(z-z_0)}'J_f^{-1}(z)(f(z)-f(w))\ge 0, \qquad \text{and}\ \ \|w-z_0\|\le\|z-z_0\|<1-\|z_0\|.$$

By the definition of f, we have

$$J_f^{-1}(z)=(1-\overline{p}'z)\left(I+\frac{z\overline{p}'}{1-\overline{p}'z}\right)^{-1},$$

then,

$$\begin{aligned}
&\mathrm{Re}\overline{(z-z_0)}'J_f^{-1}(z)(f(z)-f(w))\\
=&\mathrm{Re}\overline{(z-z_0)}'(1-\overline{p}'z)\left(I+\frac{z\overline{p}'}{1-\overline{p}'z}\right)^{-1}\left(\frac{z}{1-\overline{p}'z}-\frac{w}{1-\overline{p}'w}\right)\\
=&\mathrm{Re}\frac{1-\overline{p}'z}{(1-\overline{p}'z)(1-\overline{p}'w)}\overline{(z-z_0)}'\left(I+\frac{z\overline{p}'}{1-\overline{p}'z}\right)^{-1}(z-w-z\overline{p}'w+w\overline{p}'z)\\
=&\mathrm{Re}\frac{1-\overline{p}'z}{1-\overline{p}'w}\overline{(z-z_0)}'((1-\overline{p}'z)I+z\overline{p}')^{-1}((1-\overline{p}'z)I+z\overline{p}')(z-w)\\
=&\mathrm{Re}\frac{1-\overline{p}'z}{1-\overline{p}'w}\overline{(z-z_0)}'(z-w).
\end{aligned}$$

As f is a convex mapping on B^n, then when $z_0 = 0$, we have

$$\mathrm{Re}\overline{z}' J_f^{-1}(z)(f(z) - f(w)) = \mathrm{Re}\frac{1 - \overline{p}' z}{1 - \overline{p}' w}\overline{z}'(z - w) \geq 0, \qquad \|w\| \leq \|z\| < 1.$$

When $\|w - z_0\| \leq \|z - z_0\| < 1 - \|z_0\|$, we have $\|\frac{w-z_0}{1-\overline{p}'z_0}\| \leq \|\frac{z-z_0}{1-\overline{p}'z_0}\| < 1$. Replacing z by $\frac{z-z_0}{1-\overline{p}'z_0}$, and w by $\frac{w-z_0}{1-\overline{p}'z_0}$ in the above inequality, we have

$$\mathrm{Re}\frac{1 - \overline{p}'\dfrac{z - z_0}{1 - \overline{p}'z_0}}{1 - \overline{p}'\dfrac{w - z_0}{1 - \overline{p}'z_0}}\overline{\left(\frac{z - z_0}{1 - \overline{p}'z_0}\right)}'\frac{z - w}{1 - \overline{p}'z_0} = \frac{1}{|1 - \overline{p}'z_0|^2}\mathrm{Re}\frac{1 - \overline{p}'z}{1 - \overline{p}'w}\overline{(z - z_0)}'(z - w) \geq 0,$$

so the result is proved.

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