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MULTIVARIATE WEIGHTED BERNSTEIN-TYPE INEQUALITY AND ITS APPLICATIONS*

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Abstract Bernstein inequality played an important role in approximation theory and Fourier analysis. This article first introduces a general system of functions and the so-called multivariate weighted Bernstein, Nikol'skiĭ, and Ul'yanov-type inequalities. Then, the relations among these three inequalities are discussed. Namely, it is proved that a family of functions equipped with Bernstein-type inequality satisfies Nikol'skiĭ-type and Ul'yanov-type inequality. Finally, as applications, some classical inequalities are deduced from the obtained results.

Key words Bernstein-type inequality; Nikol'skiĭ-type inequality; Ul'yanov-type inequality; approximation

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1 Introduction

In this article, we always denote by n a non-negative integer, and by C and C_i $(i = 1, 2, \cdots)$ the absolute positive constants. We also use $C\ (a_1, a_2, a_3, \cdots)$ to denote a positive constant depending only on $a_i,\ i = 1, 2, \cdots$. Let $\mathcal{P}_n$ and $\mathcal{T}_n$ denote the set of all algebraic and trigonometric polynomials of degree at most n with real coefficients, respectively. Let $S \subset \mathbb{R}^d$ be a bounded convex body, and denote by $L^p(S)$ the space of real valued and p-integrable functions on S endowed with the norms

$$\|f\|_\infty := \|f\|_{L^\infty(S)} := \operatorname*{ess\,sup}_{x\in S} |f(x)|$$

and

$$\|f\|_p := \|f\|_{L^p(S)} := \left\{\int_S |f(x)|^p \mathrm{d}x\right\}^{1/p} < \infty, \quad 0 < p < \infty.$$

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As early as 1911, Bernstein in his doctoral dissertation [2] proved so-called Bernstein inequality

$$\|T_n'\|_{C[-\pi,\pi]} \le n\|T_n\|_{C[-\pi,\pi]}, \quad T_n \in \mathcal{T}_n.$$

Three years later, Riesz [18] extended the inequality to the case of L^p-norm. Namely, he proved that

$$\|T_n'\|_{L^p[-\pi,\pi]} \le n\|T_n\|_{L^p[-\pi,\pi]}, \quad 1 \le p < \infty.$$

In 1969, Nikol'skiĭ [16] proved that, if $1 \le q < p \le \infty$, then,

$$\|T_n\|_{L^p[-\pi,\pi]} \le C(p,q)n^{\frac{1}{q}-\frac{1}{p}}\|T_n\|_{L^q[-\pi,\pi]},$$

which is called Nikol'skiĭ inequality.

It is well-known that Bernstein inequality plays an important role in Fourier analysis and approximation theory, for example, in the proofs of inverse and imbedding theorems (see [5, 7, 14, 15]). By now, inequalities of the same type were established for various systems of functions. We refer the readers to Borwein [3], Borwein and Erdélyi [4–6], Baranov [1], Pesenson [17], Jung [12], and Erdélyi [10, 11].

Denote

$$D_k(f) := \frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}, \quad k_1+k_2+\cdots+k_d = k.$$

Let $F := \{f_n\}_{n=0}^{\infty}$ be a linearly independent system of functions defined on S, and $\{\lambda\} := \{\lambda_n \uparrow \infty\}$ be an increasing sequence of positive numbers tending to ∞. For $f_k \in F$, $k = 0,1,\cdots,n$, we define a set of functions:

$$F_n := \left\{P_n = \sum_{k=0}^{n} a_k f_k : a_k \in \mathbb{R}\right\},$$

and call the elements of F_n general polynomials. We say the system F satisfies a general multivariate weighted Bernstein-type inequality in $L^p(S)$ $(0 < p < \infty)$ of order $\{\lambda\}$, with notation $F \in B_{\varphi,\psi}(L^p,\{\lambda\})$, if for every $f_k \in F$, $f_k \in L^p(S)$, and $D_1(f_k) \in L^p(S)$, the inequality

$$\|\varphi D_1(P_n)\|_p \le C(p)\lambda_n\|P_n\psi\|_p \tag{1.1}$$

holds for every $P_n \in F_n$, where $\varphi(x)$ and $\psi(x)$ are continuous weight functions defined on S.

If, for a given pair $0 < q \le p \le \infty$ and for any $f_k \in F$ and $f_k \in L^p(S)$, there hold

$$\|\varphi P_n\|_p \le C(p,q,S)\lambda_n^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\psi P_n\|_q, \quad P_n \in F_n, \quad n = 1,2,\cdots, \tag{1.2}$$

then, F is said to satisfy a general multivariate weighted Nikol'skiĭ-type inequality between $L^q(S)$ and $L^p(S)$ of order $\{\lambda\}$, with notation $F \in N_{\varphi,\psi}(L^q,L^p,\{\lambda\})$.

If, for a given pair $0 < q \le p \le \infty$ and for any $f_k \in F$ and $f_k \in L^p(S)$, there holds

$$E_n(f)_{p,\varphi} \le C(p,q)\left\{\sum_{k=n}^{\infty}\frac{\lambda_k^{dp_1\theta}}{k}E_k(f)_{q,\psi}^{p_1}\right\}^{\frac{1}{p_1}}, \tag{1.3}$$

where $p_1 := \begin{cases} p, & p < \infty \\ 1, & p = \infty \end{cases}$, $\theta := \frac{1}{q} - \frac{1}{p}$ and

$$E_n(f)_{p,w} := \inf_{P_n \in F_n} \|w(f - P_n)\|_p,$$

then we say that F satisfies general multivariate weighted Ul'yanov-type inequality with order $\{\lambda\}$ between $L^q(S)$ and $L^p(S)$, with notation $F \in U_{\varphi,\psi}(L^q, L^p, \{\lambda\})$, and we say that P_n is the best approximation general polynomial of f with weighted function $w(x)$ if $\|w(f - P_n)\|_p = E_n(f)_{p,w}$.

Now, we state two examples of $F \in B_{\varphi,\psi}(L^p, \{\lambda\})$.

Multivariate Bernstein Inequality for Algebraic Polynomial [8] For a bounded convex set $S \subset \mathbb{R}^d$, any direction ξ ($|\xi| = 1$, where $|\xi|$ is the Euclidean norm of ξ), integer r, and $0 < p \le \infty$, there holds

$$\left\|(\varphi^*)^{r/2}\left(\frac{\partial}{\partial \xi}\right)^r P_n\right\|_p \le C(p, r) n^r \|P_n\|_p, \tag{1.4}$$

where the weighted function $\varphi^*(x)$ is defined by

$$\varphi^*(x) := \sup_{0<\mu,\ x+\mu\xi \in S} d(x, x + \mu\xi) \sup_{0>\mu, x+\mu\xi \in S} d(x, x + \mu\xi),$$

and $P_n \in \mathcal{P}_n$.

Multivariate Markov Inequality [8] If S is a bounded convex set in $\mathbb{R}^d$, $0 < p \le \infty$, and $P_n \in \mathcal{P}_n$, then,

$$\|D_k(P_n)\|_p \le C(p, k, S) n^{2k} \|P_n\|_p. \tag{1.5}$$

In this article, we as well prove that (1.2) can be deduced from (1.1) for some weighted functions. In Section 3, we will prove that (1.3) implies (1.4) under some assumptions. As applications of main results of this article, some classical inequalities will be deduced from the obtained results in the last section. As stated in Sections 2 and 3, an important conclusion of this article is that a family of functions equipped with Bernstein-type inequality satisfies Nikol'skiĭ-and-Ul'yanov-type inequalities.

2 Multivariate Weighted Nikol'skiĭ-Type Inequality for F_n

In this section, we will discuss the relation between Bernstein and Nikol'skiĭ-type inequalities.

Theorem 2.1 Let $0 < q < p \le \infty$, $S \subset \mathbb{R}^d$ be a bounded convex body, and $\{\lambda\} = \{\lambda_n \uparrow \infty\}$ be an increasing sequence of positive numbers. Then, the following statements hold:

(i) If $\varphi(x) = \psi(x)$, $D_1(\varphi) \in L^p(S)$, and

$$\|D_1\varphi\|_p \le C_1 \lambda_n \|\varphi\|_p, \tag{2.1}$$

then, $F \in B_{\varphi,\varphi}(L^\infty, \{\lambda\})$ implies $F \in N_{\varphi,\varphi}(L^q, L^p, \{\lambda\})$.

(ii) If $\psi(x) = 1$ and $|\varphi(x)| \le C_2$, then, $F \in B_{\varphi,1}(L^\infty, \{\lambda\})$ implies $F \in N_{\varphi,1}(L^q, L^p, \{\lambda\})$.

Proof We first prove (i). For the sake of brevity, set $M := 4 \max\{C(p)\, \lambda_n, C_1\lambda_n\}$, where constants $C(p)$ and C_1 are the same as those of (1.1) and (2.1), respectively. For $P_n \in F_n$, we can choose a point $t_0 \in S$, such that $|P_n(t_0)\varphi(t_0)| = \max_{t\in S}|P_n(t)\varphi(t)|$. We denote by $\{\xi_1, \xi_2, \cdots, \xi_d\}$ an orthonormal basis of $\mathbb{R}^d$, and write $t_0 := (\mu_1, \mu_2, \cdots, \mu_d)$, $u := (\nu_1, \nu_2, \cdots, \nu_d)$. Then, for every absolute constant $K > d$ and every

$$u \in S \cap O(t_0, (KM)^{-1}),$$

where $O(t_0, (KM)^{-1})$ denotes the ball with center t_0 and of radius $(KM)^{-1}$, from Lagrange Mean Value Theorem and (1.1) with $p = \infty$, it follows that there is $\alpha := (x_1, \cdots, x_d)$, satisfying $\min\{\mu_i, \nu_i\} \le x_i \le \max\{\mu_i, \nu_i\}$ $(i = 1, \cdots, d)$, such that

$$\begin{aligned} |P_n(t_0)\varphi(t_0) - P_n(u)\varphi(u)| &= \left|\sum_{k=1}^{d} \frac{\partial}{\partial \xi_k}(P_n\varphi)(\alpha)(\mu_k - \nu_k)\right| \\ &= \left|\sum_{k=1}^{d} \left(\frac{\partial P_n(\alpha)}{\partial \xi_k}\varphi(\alpha) + \frac{\partial \varphi(\alpha)}{\partial \xi_k}P_n(\alpha)\right)(\mu_k - \nu_k)\right| \\ &\le \frac{1}{2}M|P_n(t_0)\varphi(t_0)|\sum_{k=1}^{d}|\mu_k - \nu_k| \\ &\le M|P_n(t_0)\varphi(t_0)|d(KM)^{-1} \\ &\le \frac{d}{K}|P_n(t_0)\varphi(t_0)|, \end{aligned}$$

which implies

$$|P_n(t_0)\varphi(t_0)| \le \frac{K}{K-d}|P_n(u)\varphi(u)|.$$

Therefore, we can deduce

$$\max_{x\in S}|P_n(x)\varphi(x)| \le C(p, q, S)M^{\frac{d}{q}}\|P_n(\cdot)\varphi(\cdot)\|_q. \tag{2.2}$$

Indeed, we denote by $B(S)$ the boundary of S, and divide the proof of (2.2) into two cases.

Case 1 If $t_0 \in S - B(S)$, then, we choose the constant K large enough such that $(KM)^{-1} \le \omega$, where $\omega := \min\limits_{t\in B(S)}\{|t - t_0|\}$, so $O(t_0, (KM)^{-1}) \subseteq S$. Hence,

$$\begin{aligned} \|P_n\varphi\|_q^q &= \int_S |P_n(t)\varphi(t)|^q \mathrm{d}t \ge \int_{O(t_0,(KM)^{-1})} |P_n(t)\varphi(t)|^q \mathrm{d}t \\ &\ge \frac{K-d}{K}|P_n(t_0)\varphi(t_0)|^q \int_{O(t_0,(KM)^{-1})} \mathrm{d}t \ge CM^{-d}|P_n(t_0)\varphi(t_0)|^q. \end{aligned}$$

Case 2 If $t_0 \in B(S)$, then, we choose a point $t_0' \in O(t_0, (KM)^{-1}) \cap S$ and denote by t_0^* the middle point of the segment t_0t_0', and there is a sufficiently large $K' \ge 2$, such that $(K'KM)^{-1} \le \omega'$ where $\omega' := \min\limits_{t\in B(S)}\{|t - t_0^*|\}$, which implies

$$O(t_0^*, (K'KM)^{-1}) \subseteq S \cap O(t_0, (KM)^{-1}).$$

Therefore,

$$\|P_n\varphi\|_q^q = \int_S |P_n(t)\varphi(t)|^q \mathrm{d}t \ge \int_{S\cap O(t_0,(KM)^{-1})} |P_n(t)\varphi(t)|^q \mathrm{d}t$$

$$
\begin{aligned}
&\geq \frac{K-d}{K}|P_n(t_0)\varphi(t_0)|^q \int_{S\cap O(t_0,(KM)^{-1})} \mathrm{d}t \\
&\geq \frac{K-d}{K}|P_n(t_0)\varphi(t_0)|^q \int_{O(t_0^*,(K'KM)^{-1})} \mathrm{d}t \\
&\geq CM^{-d}|P_n(t_0)\varphi(t_0)|^q.
\end{aligned}
$$

Collecting Case 1 and Case 2, we get (2.2). So,

$$
\begin{aligned}
\|P_n\varphi\|_p^p &= \int_S |P_n(t)\varphi(t)|^{p-q}|P_n(t)\varphi(t)|^q \mathrm{d}t \leq |P_n(t_0)\varphi(t_0)|^{p-q} \int_S |P_n(t)\varphi(t)|^q \mathrm{d}t \\
&\leq C(p,q,S)M^{d\frac{p-q}{q}}\|P_n\varphi\|_q^{p-q}\|P_n\varphi\|_q^q \\
&\leq C(p,q,S)M^{d\frac{p-q}{q}}\|P_n\varphi\|_q^p
\end{aligned}
$$

which implies

$$
\|P_n\varphi\|_p \leq C(p,q,S)M^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|P_n\varphi\|_q \leq C(p,q,S)\lambda_n^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|P_n\varphi\|_q.
$$

This completes the proof of (i).

Now, we turn to prove (ii). Let $M_1 := 2C(p)\lambda_n$, where the constant $C(p)$ is the same as that in (1.1). We take a point $t_1 \in S$, such that

$$
|P_n(t_1)\varphi(t_1)| = \max_{t\in S}|P_n(t)\varphi(t)|.
$$

It is obvious that $\varphi(t_1) \neq 0$. We can choose an absolute constant $K_1 > d$, such that for every $u \in O(t_1,(K_1M)^{-1})$, there holds $|\varphi(t_1)| \leq C_3|\varphi(u)|$. Then, for

$$
u \in S \cap O(t_1,(K_1'M)^{-1}),
$$

where

$$
\max\left\{K_1, \frac{dC_3C_5}{\varphi(t_1)}\right\} \leq K_1' \leq \frac{C_3C_4d}{\varphi(t_1)},
$$

$1 < C_5 < C_4$, and

$$
t_1 := (\mu_1^*,\mu_2^*,\cdots,\mu_d^*), \quad u := (\nu_1^*,\nu_2^*,\cdots,\nu_d^*),
$$

using Lagrange Mean Value Theorem, condition (1.1) when $p=\infty$, and the facts that $|\varphi(x)| \leq C_2$ and $|\varphi(t_1)| \leq C_3|\varphi(u)|$, we verify that there is $\beta := (x_1^*,\cdots,x_d^*)$ satisfying $\min\{\mu_i^*,\nu_i^*\} \leq x_i^* \leq \max\{\mu_i^*,\nu_i^*\}$ $(i=1,\cdots,d)$ such that

$$
\begin{aligned}
|P_n(t_1)\varphi(t_1) - P_n(u)\varphi(t_1)| &\leq |\varphi(t_1)|\left|\sum_{k=1}^d \frac{\partial P_n(\beta)}{\partial \xi_k}(\mu_k^*-\nu_k^*)\right| \\
&\leq C_3\left|\sum_{k=1}^d \frac{\partial P_n(\beta)}{\partial \xi_k}\varphi(\beta)(\mu_k^*-\nu_k^*)\right| \\
&\leq \frac{C_3M}{2}|P_n(t_1)|\sum_{k=1}^d|\mu_k^*-\nu_k^*| \\
&\leq \frac{C_3d}{K_1'}|P_n(t_1)|.
\end{aligned}
$$

Therefore,

$$|P_n(t_1)\varphi(t_1)| \le |P_n(u)\varphi(t_1)| + \frac{C_3 d}{K_1'}|P_n(t_1)|,$$

that is,

$$|P_n(t_1)\varphi(t_1)| \le \frac{C_2C_5}{C_5-1}|P_n(u)|.$$

Using the similar method as proving (2.2), we obtain

$$|P_n(t_1)\varphi(t_1)| \le C(p,q,S)M^{\frac{d}{q}}\|P_n\|_q.$$

Then,

$$\begin{aligned}\|P_n\varphi\|_p^p &= \int_S |P_n\varphi(t)|^{p-q}|P_n\varphi(t)|^q\mathrm{d}t \le \max_{t\in S}|P_n(t)\varphi(t)|^{p-q}M_1\int_S |P_n(t)|^q\mathrm{d}t\\ &\le C(p,q,S)M^{d\frac{p-q}{q}}\|P_n\|_q^{p-q}\|P_n\|_q^q\\ &\le C(p,q,S)M^{d\frac{p-q}{q}}\|P_n\|_q^p,\end{aligned}$$

which implies

$$\|P_n\varphi\|_p \le C(p,q,S)M^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|P_n\|_q \le C(p,q,S)\lambda_n^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|P_n\|_q.$$

The proof of (2) is completed.

3 Multivariate Weighted Ul'yanov-Type Inequality for F_n

In the above section, we have established a relation between multivariate weighted Bernstein and Nikol'skiĭ-type inequalities. Now, let us turn to consider the relation between Nikol'skiĭ and Ul'yanov-type inequalities. In [9], Ditzian and Tikhonov established a relation between Nikol'skiĭ and Ul'yanov-type inequalities for trigonometric polynomials in $L^p(\mathbb{T}^d)$ with a weighted function $w(x)$, where $\mathbb{T}^d$ denotes the d-dimensional cube for the interval $\mathbb{T}$. In this section, we will extend their result in three directions: (1) arbitrary bounded convex body S in $\mathbb{R}^d$; (2) different weighted functions in both sides of inequalities; (3) general polynomial $P_n \in F_n$. In fact, we will prove the following result.

Theorem 3.1 Let $\{\lambda\} = \{\lambda_n\}_{n=0}^{\infty}$ be an increasing sequence of non-negative real numbers. Suppose S is a bounded convex body in $\mathbb{R}^d$, $0<q<p\le\infty$, F_n is dense in $L^p(S)$, and there is a sequence of natural numbers $\{u_n\}$ satisfying

$$u_0 = n, \quad \frac{u_{n+1}}{u_n} \ge b_1 > 1. \tag{3.1}$$

Also, $\{\lambda_n\}_{n=0}^{\infty}$ and $\{u_n\}$ are assumed to satisfy

$$1 > b_3 \ge \frac{\lambda_{u_n}}{\lambda_{u_{n+1}}} \ge b_2 > 0. \tag{3.2}$$

Then, if (1.2) is valid, then there is a constant $C(S,p,q)$, such that

$$E_n(f)_{p,\varphi} \le C(S,p,q)\left\{\lambda_n^{dp_1\theta}E_n(f)_{q,\psi}^{p_1} + \sum_{k=n+1}^{\infty}\frac{\lambda_k^{dp_1\theta}}{k}E_k(f)_{q,\psi}^{p_1}\right\}^{\frac{1}{p_1}}. \tag{3.3}$$

Remark The sequences $\{\lambda_k\}$ and $\{u_k\}$ satisfying the conditions of Theorem 3.1 exist. For example, $u_k = n2^k,\ \lambda_k = k^a\ (a > 0)$.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2 Let a be a non-negative integer, and $\{\lambda\} = \{\lambda_n \uparrow \infty\}$ be an increasing sequence of non-negative real numbers, and for which there exists a sequence $\{u_i\}$ of natural numbers satisfying (3.1) such that, for any $n \geq 1$, (3.2) holds. Then,

$$\lambda^r_{u_{n+1}} \leq C(r,a) \sum_{k=u_{n-1}+1}^{u_n} \frac{\lambda^r_{k+a}}{k+a} \quad (n \geq 1).$$

Proof By (3.2), we have

$$0 < b_2^{2r} \leq \frac{\lambda^r_{u_{n-1}}}{\lambda^r_{u_{n+1}}}, \quad n \geq 1.$$

Then,

$$\begin{aligned}\sum_{k=u_{n-1}+1}^{u_n} \frac{\lambda^r_{k+a}}{k+a} &\geq (u_n - u_{n-1}) \frac{\lambda^r_{u_{n-1}+1+a}}{u_n + a} \geq \frac{1}{a+1} \frac{u_n - u_{n-1}}{u_n} \lambda^r_{u_{n-1}} \\ &\geq \frac{1}{a+1} \frac{u_n - u_{n-1}}{u_n} b_2^{2r} \lambda^r_{u_{n+1}} \geq \frac{1}{a+1} (1 - \frac{1}{b_1}) \lambda^r_{u_{n+1}} \\ &\geq C(a,r) \lambda^r_{u_{n+1}}.\end{aligned}$$

This finishes the proof of Lemma 3.2.

Lemma 3.3 (Extended Hölder inequality, see page 18 of [19])

(1) If $g_1, g_2, \cdots, g_n \in L^p$, then, there are $\alpha_k > 0,\ k = 1, \cdots, n$, satisfying $\sum\limits_{k=1}^{n} \alpha_k = 1$ such that

$$\int g_1 \cdots g_n \leq \|g_1\|_{\frac{1}{\alpha_1}} \cdots \|g_n\|_{\frac{1}{\alpha_n}}. \tag{3.4}$$

(2) If $g_1, g_2, \cdots, g_n \in L^p$, then, there are $\alpha_k > 0,\ k = 1, \cdots, n$, satisfying $\sum\limits_{k=1}^{n} \alpha_k = 1$ such that

$$\sum_v g_v(1) \cdots g_v(n) \leq \left(\sum_v |g_v(1)|^{\frac{1}{\alpha_1}} \right)^{\alpha_1} \cdots \left(\sum_v |g_v(n)|^{\frac{1}{\alpha_n}} \right)^{\alpha_n}. \tag{3.5}$$

Lemma 3.4 Let $P_n \in F_n$ be the general polynomial of the best approximation of f with weighted function $\psi(t)$. Then, under the conditions of Theorem 3.1, there is a constant $C(p,q,S)$, such that

$$\left\| \varphi \sum_{k=0}^{m} (P_{u_{k+1}} - P_{u_k}) \right\|_p \leq C(p,q,S) \left(\sum_{k=0}^{m} \left(\lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)} E_{u_k}(f)_{q,\psi} \right)^{p_1} \right)^{\frac{1}{p_1}}. \tag{3.6}$$

Proof By (1.2), we have, for $p \leq 1$ and $p_1 = p$,

$$\begin{aligned}\left\| \varphi \sum_{k=0}^{m} (P_{u_{k+1}} - P_{u_k}) \right\|_p^p &\leq \sum_{k=0}^{m} \|\varphi(P_{u_{k+1}} - P_{u_k})\|_p^p \\ &\leq C(p,q,S) \sum_{k=0}^{m} \lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|\psi(P_{u_{k+1}} - P_{u_k})\|_q^p \\ &\leq C(p,q,S) \sum_{k=0}^{m} \lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)} E_{u_k}(f)^p_{q,\psi}.\end{aligned}$$

For $p \ge 1$ and $p_1 = 1$, by (1.2), we may write

$$\begin{aligned}\left\|\varphi\sum_{k=0}^{m}(P_{u_{k+1}}-P_{u_k})\right\|_p &\le \sum_{k=0}^{m}\|\varphi(P_{u_{k+1}}-P_{u_k})\|_p\\ &\le C(p,q,S)\sum_{k=0}^{m}\lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\varphi(P_{u_{k+1}}-P_{u_k})\|_q\\ &\le 2C(p,q,S)\sum_{k=0}^{m}\lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}E_{u_k}(f)_{q,\psi}.\end{aligned}$$

In fact, we only need $p_1 = 1$ for $p = \infty$. To complete the proof, we need to settle the case $1 < p < \infty$ and $p_1 = p$, which is the hard part of the proof. For the sake of brevity, set $\phi_k = \phi_k(t) := |(P_{u_{k+1}}(t) - P_{u_k}(t))\varphi(t)|$ and $\phi_k^* = \phi_k^*(t) := |(P_{u_{k+1}}(t) - P_{u_k}(t))\psi(t)|$, and choose $r = [p] + 1$, where $[x]$ denotes the largest integer not larger than x. Recalling $1 < p < \infty$, we have

$$\begin{aligned}&\left\|\varphi\sum_{k=0}^{m}(P_{u_{k+1}}-P_{u_k})\right\|_p\\ &\le \left[\int\left(\sum_{k=0}^{m}\phi_k(t)\right)^p \mathrm{d}t\right]^{\frac{1}{p}} \le \left[\int\left(\sum_{k=0}^{m}\phi_k^{\frac{p}{r}}(t)\right)^r \mathrm{d}t\right]^{\frac{1}{p}}\\ &= \left[\sum_{k_1=0}^{m}\cdots\sum_{k_r=0}^{m}\int\phi_{k_1}^{\frac{p}{r}}(t)\cdots\phi_{k_r}^{\frac{p}{r}}(t)\mathrm{d}t\right]^{\frac{1}{p}}\\ &\le \left[\sum_{k_1=0}^{m}\cdots\sum_{k_r=0}^{m}\int\left(\prod_{1\le i<j\le r}\phi_{k_i}^{\frac{p}{r}}(t)\phi_{k_j}^{\frac{p}{r}}(t)\right)^{\frac{1}{r-1}}\mathrm{d}t\right]^{\frac{1}{p}}\\ &\le \left[\sum_{k_1=0}^{m}\cdots\sum_{k_r=1}^{m}\prod_{1\le i<j\le r}\left(\int\phi_{k_i}^{\frac{p}{2}}(t)\phi_{k_j}^{\frac{p}{2}}(t)\mathrm{d}t\right)^{\frac{2}{r(r-1)}}\right]^{\frac{1}{p}}\\ &\le C(p,q,S)\left[\sum_{k_1=0}^{m}\cdots\sum_{k_r=0}^{m}\left\{\prod_{1\le i<j\le r}\left(\lambda_{u_{k_i+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_{k_i}^*\|_q\lambda_{u_{k_j+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_{k_j}^*\|_q\right)^p\right.\right.\\ &\qquad\left.\left.\times\left(\frac{\lambda_{u_{k_i+1}}}{\lambda_{u_{k_j+1}}}\right)^{d\frac{p-q}{p+q}}\right\}^{\frac{1}{r(r-1)}}\right]^{\frac{1}{p}}\\ &\le C(p,q,S)\left[\sum_{k_1=0}^{m}\cdots\sum_{k_r=0}^{m}\prod_{s=1}^{r}\left(\lambda_{u_{k_s+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_{k_s}^*\|_q\right)^{\frac{p}{r}}\prod_{t=1}^{r}\left(\frac{\lambda_{u_{\min\{k_s,k_t\}+1}}}{\lambda_{u_{\max\{k_s,k_t\}+1}}}\right)^{\frac{d(p-q)}{2(p+q)r(r-1)}}\right]^{\frac{1}{p}}\\ &\le C(p,q,S)\left(\prod_{s=1}^{r}\left[\sum_{k_1=0}^{m}\cdots\sum_{k_r=0}^{m}\left(\|\phi_{k_s}^*\|_q\lambda_{u_{k_s+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\right)^p\prod_{t=1}^{r}\left(\frac{\lambda_{u_{\min\{k_s,k_t\}+1}}}{\lambda_{u_{\max\{k_s,k_t\}+1}}}\right)^{\frac{d(p-q)}{2(p+q)(r-1)}}\right]^{\frac{1}{r}}\right)^{\frac{1}{p}}\end{aligned}$$

$$\le C(p,q,S)\left(\left[\sum_{k_1=0}^{m}\left(\|\phi_{k_1}^*\|_q\lambda_{u_{k_1+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\right)^p\sum_{k_2=0}^{m}\cdots\sum_{k_r=0}^{m}\prod_{t=1}^{r}\left(\frac{\lambda_{u_{\min\{k_1,k_t\}+1}}}{\lambda_{u_{\max\{k_1,k_t\}+1}}}\right)^{\frac{d(p-q)}{(p+q)(r-1)}}\right]^{\frac{1}{r}}\right)^{\frac{r}{p}}$$

$$\le C(p,q,S)\left(\sum_{k=0}^{m}\lambda_{u_k}^{dp\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_k^*\|_q^p\right)^{\frac{1}{p}}$$

$$\le C(p,q,S)\left(\sum_{k=0}^{m}\left(\lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}E_{u_k}(f)_{q,\psi}\right)^p\right)^{\frac{1}{p}},$$

where the equality

$$\left(\prod_{n=1}^{r}a_n\right)^{r-1}=\prod_{1\le i<j\le r}a_ia_j \quad \text{for } r>1$$

is used in the third inequality, and (3.4) is used in the fourth inequality with $\alpha_k=\frac{2}{r(r-1)},\ n=\frac{r(r-1)}{2}$, and $g_v=\phi_{k_i}^{\frac{p}{r(r-1)}}(t)\phi_{k_j}^{\frac{p}{r(r-1)}}(t),\ i<j, v=1,\cdots,n$. Considering that $p_2=\frac{(p+q)p}{2q}>q$ and $p_3=\frac{p+q}{2}>q$, by (1.2) and Hölder inequality with powers $\alpha=\frac{p+q}{q}$ and $\alpha'=\frac{p+q}{p}$, we obtain

$$\begin{aligned}\int\phi_{k_i}^{\frac{p}{2}}(t)\phi_{k_j}^{\frac{p}{2}}(t)\mathrm{d}t&\le\left(\int\phi_{k_i}^{\frac{(p+q)p}{2q}}(t)\mathrm{d}t\right)^{\frac{q}{p+q}}\left(\int\phi_{k_j}^{\frac{p+q}{2}}(t)\mathrm{d}t\right)^{\frac{p}{p+q}}\\&\le\|\phi_{k_i}\|_{p_2}^{\frac{p}{2}}\|\phi_{k_j}\|_{p_3}^{\frac{p}{2}}\\&\le C(p,q,S)\left[\lambda_{u_{k_i+1}}^{d\left(\frac{1}{q}-\frac{1}{p_2}\right)}\|\phi_{k_i}^*\|_q\lambda_{u_{k_j+1}}^{d\left(\frac{1}{q}-\frac{1}{p_3}\right)}\|\phi_{k_j}^*\|_q\right]^{\frac{p}{2}}\\&=C(p,q,S)\left[\lambda_{u_{k_i+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_{k_i}^*\|_q\lambda_{u_{k_j+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_{k_j}^*\|_q\right]^{\frac{p}{2}}\left(\frac{\lambda_{u_{k_i+1}}}{\lambda_{u_{k_j+1}}}\right)^{\frac{p-q}{2(p+q)}}.\end{aligned}$$

So, the fifth inequality holds. Symmetry between i and j in the last equality allows us to exchange i and j if $u_{k_i}>u_{k_j}$ and replace $\left(\frac{\lambda_{u_{k_i+1}}}{\lambda_{u_{k_j+1}}}\right)^{\frac{p-q}{2(p+q)}}$ by $\left(\frac{\lambda_{u_{\min\{k_i,k_j\}+1}}}{\lambda_{u_{\max\{k_i,k_j\}+1}}}\right)^{\frac{p-q}{2(p+q)}}$, then the sixth inequality follows from the inequality

$$\prod_{1\le i<j\le r}a_{k_i}a_{k_j}\left(\frac{\lambda_{u_{\min\{k_i,k_j\}+1}}}{\lambda_{u_{\max\{k_i,k_j\}+1}}}\right)^{\iota}=\prod_{s=1}^{r}a_{k_s}^{r-1}\prod_{t=1}^{r}\left(\frac{\lambda_{u_{\min\{k_s,k_t\}+1}}}{\lambda_{u_{\max\{k_s,k_t\}+1}}}\right)^{\frac{\iota}{2}}$$

with

$$a_{k_s}=\left(\lambda_{u_{k_s+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi_{k_s}\|_q\right)^{\frac{p}{r(r-1)}}$$

and

$$\iota=\frac{d(p-q)}{(p+q)r(r-1)}.$$

And (3.5) with $\alpha_k=\frac{1}{r},\ n=r$ and $v=(k_1,\cdots,k_r)$ yields the seventh inequality. Observing that all r factors of the product in the seventh inequality are equal and their common value is

$$\left[\sum_{k_1=0}^{m}\left(\|\phi_{k_1}^*\|_q\lambda_{u_{k_1+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\right)^p\sum_{k_2=0}^{m}\cdots\sum_{k_r=0}^{m}\prod_{t=1}^{r}\left(\frac{\lambda_{u_{\min\{k_1,k_t\}+1}}}{\lambda_{u_{\max\{k_1,k_t\}+1}}}\right)^{\frac{p-q}{2(p+q)(r-1)}}\right]^{\frac{1}{r}},$$

we have by (3.2)

$$\begin{aligned}\sum_{k=0}^{m}\left(\frac{\lambda_{u_{\min\{k_1,k\}+1}}}{\lambda_{u_{\max\{k_1,k\}+1}}}\right)^{\iota} &\le C(p,q,S)\sum_{k=0}^{m} b_3^{\iota(\max\{k_1,k\}-\min\{k_1,k\})}\\ &= C(p,q,S)\sum_{k=0}^{m} b_3^{\iota(|k_1-k|)} \le C(p,q,S)\sum_{k=0}^{\infty} b_3^{k\iota}\\ &\le C(p,q,S).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k_2=0}^{m}\cdots\sum_{k_r=0}^{m}\prod_{t=1}^{r}\left(\frac{\lambda_{u_{\min\{k_1,k_t\}+1}}}{\lambda_{u_{\max\{k_1,k_t\}+1}}}\right)^{\iota} &= \sum_{k_2=0}^{m}\cdots\sum_{k_r=0}^{m}\prod_{t=2}^{r}\left(\frac{\lambda_{u_{\min\{k_1,k_t\}+1}}}{\lambda_{u_{\max\{k_1,k_t\}+1}}}\right)^{\iota}\\ &\le \prod_{t=2}^{r}\left(\sum_{k_t=0}^{m}\left(\frac{\lambda_{u_{\min\{k_1,k_t\}+1}}}{\lambda_{u_{\max\{k_1,k_t\}+1}}}\right)^{\iota}\right)\\ &\le C(p,q,S).\end{aligned}$$

Thus, the eighth and ninth inequalities are obvious. This completes the proof of Lemma 3.4.

Now, we proceed the proof of our main result in this section.

Proof of Theorem 3.1 Let $P_n \in F_n$ be the general polynomial of the best approximation of $f \in L^q$ with the weighted function ψ. By the density of F_n, we have

$$(f\varphi - P_{u_m}\varphi)(t) = \sum_{k=m}^{\infty}\{P_{u_{j+1}}(t)\varphi(x) - P_{u_j}(t)\varphi(t)\}.$$

Using Lemmas 3.5 and 3.2 with $a = 0$, we have

$$\begin{aligned}E_n(f)_{p,\varphi} &\le \|f\varphi - P_n\varphi\|_p = \lim_{m\to\infty}\left\|\sum_{k=0}^{m}(P_{u_{k+1}}\varphi - P_{u_k}\varphi)\right\|_p\\ &\le C(p,q,S)\lim_{m\to\infty}\left(\sum_{k=0}^{m}\left(\lambda_{u_{k+1}}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}E_{u_k}(f)_{q,\psi}\right)^{p_1}\right)^{\frac{1}{p_1}}\\ &\le C(p,q,S)\lim_{m\to\infty}\left(\sum_{k=0}^{m}E_{u_k}(f)_{q,\psi}^{p_1}\lambda_{u_{k+1}}^{dp_1\left(\frac{1}{q}-\frac{1}{p}\right)}\right)^{\frac{1}{p_1}}\\ &\le C(p,q,S)\lim_{m\to\infty}\left(\left(\lambda_{u_1}^{d\left(\frac{1}{q}-\frac{1}{p}\right)}E_{u_0}(f)_{q,\psi}\right)^{p_1}\right.\\ &\qquad\left.+\sum_{k=1}^{m}\sum_{j=u_{k-1}+1}^{u_m}\frac{\lambda_j^{dp_1\left(\frac{1}{q}-\frac{1}{p}\right)}}{j}E_j(f)_{q,\psi}^{p_1}\right)^{\frac{1}{p_1}}\\ &\le C(S,p,q)\left\{\lambda_n^{dp_1\theta}E_n(f)_{q,\psi}^{p_1} + \sum_{k=n+1}^{\infty}\frac{\lambda_k^{dp_1\theta}}{k}E_k(f)_{q,\psi}^{p_1}\right\}^{\frac{1}{p_1}}.\end{aligned}$$

This completes the proof of Theorem 3.1.

4 Applications

Theorem 2.1 shows that the general Bernstein-type inequality (1.1) implies the general Nikol'skiĭ-type inequality (1.2), and Theorem 3.1 shows that the general Ul'yanov-type inequality (1.3) can be deduced from the general Nikol'skiĭ-type inequality (1.2) under some conditions. In contrast, for algebraic polynomials, there are some such weighted Bernstein-type inequalities, which were shown in [5, 8, 12, 13], etc. Also, there are many weighted functions, such as x^γ, e^x, etc., satisfying Theorems 2.1 and 3.1.

The following Corollary 4.1 is a combination of Theorem 2.1 and (1.4).

Corollary 4.1 For a bounded convex set $S \subset \mathbb{R}^d$, $P_n \in \mathcal{P}_n$, and $0 < p < q \le \infty$, we have

$$\left\|(\varphi^*)^{1/2} P_n\right\|_p \le C(p,q,r) n^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|P_n\|_q,$$

where $\varphi^*(x)$ is defined in Section 1.

From Theorem 3.1, it follows

Corollary 4.2 For a bounded convex set $S \subset \mathbb{R}^d$ and $0 < q < p \le \infty$, there holds

$$E_n(f)_{p,(\varphi^*)^{1/2}} \le C(S,p,q) \left\{ n^{dp_1\theta} E_n(f)_{q,1}^{p_1} + \sum_{k=n+1}^{\infty} k^{dp_1\theta-1} E_k(f)_{q,1}^{p_1} \right\}^{\frac{1}{p_1}}.$$

By combination of (1.5) and Theorem 2.1 with $\varphi(x) = \psi(x) = 1$, we have

Corollary 4.3 If S is a bounded convex set in $\mathbb{R}^d$, $0 < q < p \le \infty$, and $P_n \in \mathcal{P}_n$ is a polynomial, then there is a constant $C(p,k,S)$, such that

$$\|P_n\|_p \le C(p,k,S) n^{2d\left(\frac{1}{q}-\frac{1}{p}\right)} \|P_n\|_q.$$

Combining Corollary 4.3 with Theorem 3.1 gives

Corollary 4.4 For a bounded convex set $S \subset \mathbb{R}^d$, $0 < q < p \le \infty$, and a polynomial $P_n \in \mathcal{P}_n$, there is a constant $C(p,k,S)$, such that

$$E_n(f)_{p,1} \le C(S,p,q) \left\{ n^{2dp_1\theta} E_n(f)_{q,1}^{p_1} + \sum_{k=n+1}^{\infty} k^{2dp_1\theta-1} E_k(f)_{q,1}^{p_1} \right\}^{\frac{1}{p_1}}.$$

Now, we turn to the case of one dimension. The following lemma 4.1 is Theorem 1.6 of [13].

Lemma 4.1 Let $W(x) = e^{-Q(x)}$ and $0 < p \le \infty$, and let $\beta > \frac{-1}{p}$ if $p < \infty$ and $\beta \ge 0$ if $p = \infty$. Then, for $n \ge 1$, $P_n \in \mathcal{P}_n$, and a bounded interval I,

$$\|P_n'(x) W(x) x^\beta\|_{L^p(\mathrm{I})} \le C n^2 \|P(x) W(x) x^\beta\|_{L^p(\mathrm{I})},$$

where $Q(x)$ was defined in Definition 1.1 of [13].

It is proved that the weighted function $W(x)x^\beta$ satisfies the restriction of Theorem 2.1. Therefore, we have

Corollary 4.5 Let $W(x) = e^{-Q(x)}$, and let $\beta > \frac{-1}{p}$ if $p < \infty$ and $\beta \ge 0$ if $p = \infty$. Then, for $0 < q < p \le \infty$, $n \ge 1$, $P_n \in \mathcal{P}_n$, and a bounded interval I,

$$\|P_n(x) W(x) x^\beta\|_{L^p(\mathrm{I})} \le C n^{2\left(\frac{1}{q}-\frac{1}{p}\right)} \|P(x) W(x) x^\beta\|_{L^q(\mathrm{I})}.$$

From Theorem 3.1, it also follows that

Corollary 4.6 Let $W(x)=e^{-Q(x)}$, and let $\beta>\frac{-1}{p}$ if $p<\infty$ and $\beta\geq 0$ if $p=\infty$. Then, for $0<q<p\leq\infty$, $n\geq 1$, $P_n\in\mathcal{P}_n$, and a bounded interval I,

$$E_n(f)_{p,W(x)x^\beta}\leq C\left\{n^{2p_1\theta}E_n(f)^{p_1}_{q,W(x)x^\beta}+\sum_{k=n+1}^{\infty}k^{2p_1\theta-1}E_k(f)^{p_1}_{q,W(x)x^\beta}\right\}^{\frac{1}{p_1}}.$$

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