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Editor's Choice

Elliptic operators with general Wentzell boundary conditions, analytic semigroups and the angle concavity theorem

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We prove a very general form of the Angle Concavity Theorem, which says that if $\big(T(t)\big)$ defines a one parameter semigroup acting over various L^p spaces (over a fixed measure space), which is analytic in a sector of opening angle θ_p, then the maximal choice for θ_p is a concave function of $1-1/p$. This and related results are applied to give improved estimates on the optimal L^p angle of ellipticity for a parabolic equation of the form $\partial u/\partial t = Au$, where A is a uniformly elliptic second order partial differential operator with Wentzell or dynamic boundary conditions. Similar results are obtained for the higher order equation $\partial u/\partial t = (-1)^{m+1}A^m u$, for all positive integers m.

1 Introduction

Let us consider a uniformly elliptic operator in divergence form of the type

$$Au = \sum_{i,j=1}^{N} \partial_i (a_{ij}(x)\partial_j u) = \nabla \cdot (a(x)\nabla) u,$$

where a_{ij}, $i,j = 1,\dots,N$, are real valued functions in $C^1(\overline{\Omega})$ such that $a_{ij}(x) = a_{ji}(x)$,

$$\alpha_0|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \alpha_1|\xi|^2$$

for some $\alpha_1 \ge \alpha_0 > 0$ and all $x \in \overline{\Omega}$, $\xi = (\xi_1,\dots,\xi_n) \in \mathbf{R^N}$. Here Ω is a bounded domain of $\mathbf{R}^N$ with boundary $\partial\Omega$ in $C^{2+\varepsilon}$.

We are interested in the problem

$$\frac{\partial u}{\partial t} = Au\,, \quad \text{in} \quad \Omega\,, \tag{1.1}$$

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$$Au + \beta \partial_n^a u + \gamma u - q\beta \Delta_{LB} u = 0\,, \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $n := (n_1, \dots, n_N)$ is the unit outer normal,

$$\partial_n^a u := \sum_{i,j=1}^{N} a_{ij}(x)(\partial_j u) n_i = (a(x)\nabla u) \cdot n$$

is the conormal derivative of u with respect to the matrix $a = (a_{ij})$; β, γ belong to $C^1(\partial\Omega)$, with $\beta > 0$, $q \in [0, \infty)$, and Δ_{LB} is the Laplace-Beltrami operator on the boundary.

According to [14] we know that for $q = 0$ the suitable L^p spaces for studying problem (1.1)–(1.2) are the spaces X_p obtained as completions of $C(\overline{\Omega})$ in suitable norms (see Section 3). The boundary condition (1.2) is called the general Wentzell boundary condition $(GWBC)$ and was studied by many authors (see, e.g., [2], [7], [11]– [16]). Note that, whenever (1.1) holds on $\overline{\Omega}$, the boundary condition (1.2) can be regarded as either *dynamic* $\big($thus involving $\frac{\partial u}{\partial t}$ on $\partial\Omega\big)$ or *general Wentzell* (involving Au on the boundary $\partial\Omega$).

Hence the interpretations of the boundary conditions agree for all $t > 0$, whenever we have that the appropriate realization of A generates an analytic semigroup. For a physical interpretation of (1.2) see [18].

Here our main aims will be to show that, under additional conditions, the closure G_p of the realization of A in X_p, $1 \le p \le \infty$, generates an analytic semigroup. In addition, lower bounds for the angles of analyticity can be described and depend only on p and the angles associated with G_∞. Furthermore, lower bounds are given for angles of analyticity of the semigroup generated by $(-1)^{m+1} G_p^m$ for $m = 1, 2, \dots$ and $1 < p < \infty$ (and in some cases for $p = 1, \infty$ also).

The main tools used for the proofs are Neuberger's theorem (see, e.g. [28], [20], Exercise 5.10.5, p. 38), Liskevich-Perelmuter estimates [26], a suitable extension of Stein's interpolation theorem, and results of Goldstein [19] and Huang-Zheng [23].

For some historical remarks and a conjecture, see Section 5.

2 Analytic semigroups and sectors of analyticity

Here we recall some preliminaries concerning analytic semigroups on Banach spaces and we prove the Angle Concavity Theorem. Let X be a complex Banach space with dual space X^*, and denote by $\mathcal{B}(X)$ the Banach space of all bounded linear operators on X. For $0 < \theta \le \frac{\pi}{2}$, let $\Sigma(\theta)$ be the sector

$$\Sigma(\theta) = \{z \in \mathbf{C} \setminus \{0\} : \ |\arg z| < \theta\}.$$

Definition 2.1 Let $0 < \theta \le \frac{\pi}{2}$. We say that $\mathcal{T} = \{T(t) : t \in \Sigma(\theta) \cup \{0\}\} \subset \mathcal{B}(X)$ is an *analytic semigroup in* $\Sigma(\theta)$ if

1. $T(t+s) = T(t)T(s)$, for all $t, s \in \Sigma(\theta)$, $T(0) = I$,
2. $\langle T(\cdot)f, \varphi\rangle$ is analytic in $\Sigma(\theta)$ for all $f \in X$, $\varphi \in X^*$,
3. $\lim_{t\to 0, t\in\Sigma(\alpha)} T(t)f = f$, for all $f \in X$, $\alpha \in (0, \theta)$.

Observe that for an analytic semigroup $\mathcal{T} = \{T(t) : t \in \Sigma(\theta) \cup \{0\}\}$ on X, the restriction $(T(t))_{t\ge 0}$ is a (C_0) semigroup on X. Let $(A, D(A))$ denote the generator of $(T(t))_{t\ge 0}$. We will use the following well-known characterization of analytic semigroups (see [8], [19]).

Theorem 2.2 *Let* $0 < \theta_0 \le \frac{\pi}{2}$. *The following assertions are equivalent.*

1. $(A, D(A))$ *is the generator of an analytic semigroup in* $\Sigma(\theta_0)$ *on* X.
2. *Given* $0 < \theta < \theta_0$, *there exist constants* $M, R > 0$ *such that for any* $\lambda \in \mathbf{C}$, $|\lambda| > R$, $\lambda \in \Sigma\left(\theta + \frac{\pi}{2}\right)$, *we have* $\lambda \in \rho(A)$ *and*
$$\|R(\lambda, A)\| \le \frac{M}{|\lambda|}.$$
3. *For any* $\theta \in (-\theta_0, \theta_0)$, *the operator* $\left(e^{i\theta} A, D(A)\right)$ *generates a* (C_0) *semigroup on* X.

Let us introduce the following

Hypothesis 2.3 *Let* (Ω, Σ, μ) *be a* σ*-finite measure space and let* $L^p = L^p(\Omega, \Sigma, \mu)$. *Let* $\mathcal{M} \left(\subset \bigcap_{p=1}^{\infty} L^p\right)$ *be a subspace of functions on* Ω *such that* $\mathcal{M}$ *is dense in* $L^1 \cap L^p$, $1 \le p < \infty$. *Let*

$$\mathcal{T} = \{T(t) : \, t \in \Sigma(\theta) \cup \{0\}\}$$

be a family of linear operators from $\mathcal{M}$ *to* Σ*-measurable functions on* Ω *such that the families of operators*

$$\mathcal{T}_{p_j} = \{T_{p_j}(t) = T(t) : \, t \in \Sigma(\theta_{p_j}) \cup \{0\}\}, \quad j = 0, 1,$$

extend by continuity to analytic semigroups in $\Sigma(\theta_{p_j})$ *on* L^{p_j} *respectively. Here we assume that*

$$0 < \theta_{p_j} < \theta \le \frac{\pi}{2}, \quad 1 \le p_0 \le p_1 < \infty,$$

so that each $T_{p_j}(t)$ *is the* L^{p_j} *closure of* $T(t)|_{\mathcal{M}}$, $t \in \Sigma(\theta_{p_j}) \cup \{0\}$. *If* $p_j = \infty$, *we assume that* $\left(T_{p_j}(t)\right)$ *is an analytic semigroup on* $\mathcal{M}_\infty$, *the closure of* $\mathcal{M}$ *in the* L^∞ *norm.*

The usual choice of $\mathcal{M}$ is the set of all integrable simple functions on Ω. In the case Ω is a compact Hausdorff space and μ is a Borel measure on Ω, we may take $\mathcal{M} = C(\Omega)$. In the case $p = \infty$, we allow for the possibility that $\left(T_\infty(t)\right)$ is an analytic, but non (C_0) semigroup on $\mathcal{M}_\infty$. For example, consider

$$T(t)f(x) = (4\pi t)^{-N/2} \int_{\mathbf{R}^N} e^{-|x-y|^2/(4t)} f(y)\, dy\,,$$

for $t > 0$ and $x \in \mathbf{R}^n$. The semigroup $\left(T(t)\right)$ on $L^\infty\left(\mathbf{R}^N\right)$ is analytic in the right half plane and contractive for $t > 0$. The largest subspace of $L^\infty\left(\mathbf{R}^N\right)$ on which it is strongly continuous is $BUC\left(\mathbf{R}^N\right)$. For an example of this sort in $C\left([-\infty, \infty]\right)$, see [21].

In order to proceed, let us recall a result by Stein [31]– [32], p. 69, which is a significant extension of the Riesz convexity theorem.

Theorem 2.4 *Let* $\{U(z) : 0 \le Re\, z \le 1\}$ *be a family of linear operators on* $\mathcal{M}$ *to the* Σ*-measurable functions on* Ω, *where* $\mathcal{M}$ *is as in Hypothesis* 2.3. *Suppose* $0 < \theta_{p_0},\ \theta_{p_1} \le \frac{\pi}{2}$. *Assume* $p_0 < p_1$ *and* $\theta_{p_0} \le \theta_{p_1}$. *(The case of* $\theta_{p_0} \ge \theta_{p_1}$ *is analogous.) Let* S *be the strip*

$$S = \{z \in \mathbf{C} : \ 0 < Re\, z < 1\},$$

with closure

$$\overline{S} = \{z \in \mathbf{C} : \ 0 \le Re\, z \le 1\}.$$

Suppose that for all $f, g \in \mathcal{M}$, *the mapping* $z \to \langle U(z)f, g\rangle = \int_\Omega U(z) f \overline{g}\, d\mu$ *is analytic in* S *and bounded and continuous on* $\overline{S}$. *If*

$$\|U(z)f\|_{p_j} \le M_j \|f\|_{p_j}, \quad \textit{for} \quad Re\, z = j, \quad j = 0, 1\,,$$

then

$$\|U(z)f\|_p \le M\|f\|_p, \quad \textit{for} \quad p_0 < p < p_1, \quad Re\, z = \tau \in (0,1)\,,$$

where

$$\frac{1}{p} = \frac{\tau}{p_1} + \frac{1-\tau}{p_0}\,,$$

for this $\tau \in (0,1)$ *and* $M = M_0^{1-\tau} M_1^{\tau}$.

This is the form of the Stein's interpolation (or convexity) theorem that we need. By using Stein's approach, with suitable modifications, we show how to interpolate on the angles in an analytic semigroup. Note that Stein [32] did this under additional hypotheses, including that $\mathcal{T}_2$ consists of selfadjoint operators.

Theorem 2.5 (Angle Concavity Theorem) *Let Hypothesis* 2.3 *hold. Let* $p_0 < p < p_1$ *and let* $\mathcal{T}_p$ *be the* L^p *closure of* $\mathcal{T}$ *in* L^p*. Then* $\mathcal{T}_p = \{T(t) : t \in \Sigma(\theta_p) \cup \{0\}\}$ *is an analytic semigroup on* L^p *and*

$$\theta_p \geq \tau\theta_{p_1} + (1-\tau)\theta_{p_0},$$

where $\tau \in [0,1]$ *is such that*

$$\frac{1}{p} = \frac{\tau}{p_1} + \frac{1-\tau}{p_0}.$$

Thus, as a function of $1 - \frac{1}{p}$*, the maximal choice of* θ_p *is concave.*

It has been pointed out to us that Theorem 2.5 is 'mathematical folklore' and is known to experts. The first version of it was given by Stein [32]. But we have been unable to find in the literature a statement or proof of the result in the generality stated above. Thus we include a full proof for the sake of completeness.

Proof. Let $(A_p, D(A_p))$ be the generator of $\mathcal{T}_p$, M_j, R_j the constants related to $\mathcal{T}_{p_j}$ as in Theorem 2.2-2. and $\lambda \in \Sigma\left(\theta + \frac{\pi}{2}\right)$, $|\lambda| > R_j$, for $j = 0, 1$.

By the Riesz interpolation theorem,

$$\|R(\lambda, A_p)\| \leq \frac{\max\{M_0, M_1\}}{|\lambda|},$$

for $\lambda \in \Sigma\left(\theta + \frac{\pi}{2}\right)$ and $|\lambda| > \max\{R_0, R_1\}$. Then, by Theorem 2.2, $(A_p, D(A_p))$ generates a semigroup analytic in $\Sigma(\min\{\theta_{p_0}, \theta_{p_1}\})$. The harder part is to get a good estimate for the best possible angle θ_p.

Assume $\theta_{p_0} < \theta_{p_1}$, for there is nothing more to prove in the case of equality. Let $-\theta_{p_0} < \theta_0 < \theta_{p_0}$ (recall $p_0 < p_1$). Let $\eta > 0$, $\theta \in (-\theta_{p_1} - \theta_0, \theta_{p_1} - \theta_0)$, and

$$U(z) = T_{p_1}\left(\eta e^{i\theta_0} e^{i\theta z}\right).$$

According to the notation used in Theorem 2.4, for $z = r + is \in \overline{S}$,

$$\eta e^{i\theta_0} e^{i\theta z} = \eta e^{-\theta s} e^{i(\theta_0 + r\theta)} \in \Sigma\left(\theta_{p_1}\right),$$

since $-\theta_{p_1} < \theta_0 + r\theta < \theta_{p_1}$, for $0 \leq r \leq 1$. Thus, replacing A by $A - \omega I$ for suitable $\omega \geq 0$, we claim that $\{U(z)\}$ is analytic on S and continuous on $\overline{S}$. By semigroup theory, for any $\varepsilon > 0$ we can choose a real constant ω such that the semigroup $\{e^{-\omega t} T_{p_j}(t)\}$, whose generator is $A_{p_j} - \omega I$, is uniformly bounded in the sector $\Sigma(\theta_{p_j} - \varepsilon)$, for $j = 0, 1$. Modify $A_p, \mathcal{T}_p$ in this way and assume $\varepsilon > 0$ is small enough so that $|\theta_0 \pm \theta_{p_0}| > \varepsilon$ and $|\theta_0 + \theta \pm \theta_{p_1}| > \varepsilon$. This modification does not affect the sector of analyticity. Moreover, the mapping $\langle U(z)f, g\rangle$ is bounded on $\overline{S}$ for $f, g \in \mathcal{M}$, thanks to this choice of ω.

For $Re\, z = 0$, i.e. $z = is$, $s \in \mathbf{R}$, and $f \in \mathcal{M}$,

$$\|U(z)f\|_{p_0} = \left\|T_{p_0}\left(\left(\eta e^{-\theta s}\right)e^{i\theta_0}\right)f\right\|_{p_0} \leq M_0\|f\|_{p_0}$$

and for $Re\, z = 1$, i.e. $z = 1 + is$, $s \in \mathbf{R}$, and $f \in \mathcal{M}$,

$$\|U(z)f\|_{p_1} = \left\|T_{p_1}\left(\left(\eta e^{-\theta s}\right)e^{i(\theta_0+\theta)}\right)f\right\|_{p_1} \leq M_1\|f\|_{p_1},$$

because $-\theta_{p_1} < \theta_0 + \theta < \theta_{p_1}$, by hypothesis. By Stein's theorem 2.4, for $p_0 < p < p_1$,

$$\|U(z)f\|_p = \left\|T_p\left(\left(\eta e^{i\theta_0}\right)e^{i\theta(r+is)}\right)f\right\|_p \leq M_0^{1-r} M_1^r \|f\|_p, \tag{2.1}$$

where $z = r + is$, $0 < r < 1$, $\frac{1}{p} = \frac{r}{p_1} + \frac{1-r}{p_0}$.

Solving for r gives

$$r = \frac{p_1}{p}\left(\frac{p - p_0}{p_1 - p_0}\right) \in (0,1). \tag{2.2}$$

In (2.1), T_p is evaluated at $w = \eta e^{-\theta s} e^{i(\theta_0 + r\theta)}$ and

$$\arg(w) = \theta_0 + r\theta = \theta_0 + \frac{p_1}{p}\left(\frac{p - p_0}{p_1 - p_0}\right)\theta.$$

Hence T_p is well defined on the half line

$$\left\{\eta e^{i(\theta_0 + r\theta)} : \eta \geq 0\right\},$$

and $S_{p,\theta}(\eta) = T_p\left(\eta e^{i(\theta_0 + r\theta)}\right)$ defines, for $\eta \geq 0$, a (C_0)-semigroup on L^p. Let us note that

$$(-\theta_{p_1} + \theta_{p_0}, \theta_{p_1} - \theta_{p_0}) \subseteq (-\theta_{p_1} - \theta_{p_0}, \theta_{p_1} - \theta_{p_0});$$

it follows that T_p is well defined in the sector

$$\left\{\eta e^{i(\theta_0 + r\varphi)} : \eta \geq 0,\, |\varphi| < \theta_{p_1} - \theta_{p_0}\right\}.$$

Then T_p is well defined also in the sector

$$\left\{\eta e^{i(\psi + r\varphi)} : \eta \geq 0,\, |\varphi| < \theta_{p_1} - \theta_{p_0},\, |\psi| < \theta_{p_0}\right\}$$

and defines a (C_0) semigroup along each ray in this sector. Thus $\mathcal{T}_p$ is an analytic semigroup in $\Sigma(\theta_p)$, where

$$\theta_p \geq \theta_{p_0} + r(\theta_{p_1} - \theta_{p_0}) = r\theta_{p_1} + (1 - r)\theta_{p_0}.$$

This completes the proof of Theorem 2.5, since $\frac{1}{p} = \frac{r}{p_1} + \frac{1-r}{p_0}$ is equivalent to (2.2). □

Remark 2.6 A modification of the proof of the above theorem shows it includes the case when one of $\theta_{p_0}, \theta_{p_1}$ is zero and the other is positive. In fact, Stein's result [32] used $\theta_2 = \frac{\pi}{2}$ and $\theta_1 = \theta_\infty = 0$.

Remark 2.7 Consider the special case, when $A_2 = A_2^*$, $\mathcal{T}_2$ is a semigroup analytic in $\Sigma\left(\frac{\pi}{2}\right)$, and the semigroup $\mathcal{T}_p$ satisfies $\|T_p(t)\|_p \leq 1$ for all $1 < p < \infty$ and $t > 0$. (We could just as well suppose $\|T_p(t)\|_p \leq e^{\omega t}$). By the spectral theorem, $(A_2, D(A_2))$ is unitarily equivalent to the operator of multiplication by a nonpositive function on some concrete L^2 space. It follows that

$$\|T_2(t) - I\| \leq 1.$$

But also

$$\|T_p(t) - I\| \leq 2,$$

(or $\|T_p(t) - I\| \leq 1 + e^{\omega t} \leq 2e^{\omega t}$ if $\omega \geq 0$), and so, by Riesz interpolation,

$$\overline{\lim_{t \to 0^+}} \|T_p(t) - I\| < 2$$

for $1 < p < \infty$. By Neuberger's theorem (see [28], or [20], Exercise 5.10.5, p. 38), $\mathcal{T}_p$ is an analytic semigroup. The original suggestion to use Neuberger's Theorem in contexts such as this was made by Wolfgang Arendt in the 1990's; we gratefully acknowledge this good idea.

In the proof of next theorem, we shall need the following result, for which we gratefully acknowledge E. Semenov for pointing this out.

Lemma 2.8 *Let K be a compact subset of $\mathbf{R}^n$. If a linear operator A is bounded on $C(K)$ and on $L^2(K)$, then A is bounded also on $L^\infty(K)$ and $\|A\|_{L^\infty(K)} = \|A\|_{C(K)}$.*

P r o o f. Suppose the contrary; then there exists a function $g \in L^\infty(K)$, such that $\|g\|_{L^\infty(K)} \le 1$ and $\|Ag\|_{L^\infty(K)} > \|A\|_{C(K)}$. Hence, there exists $\varepsilon \in \mathbf{R}^+$ and $E \subset K$, such that measure$(E) \ge \varepsilon$ and, for every $t \in E$, $|Ag(t)| \ge \|A\|_{C(K)} + \varepsilon$. By Lusin's theorem, there is a sequence $\{g_n\}$ in $C(K)$, such that, for every $n \in \mathbf{N} \setminus \{0\}$, $\|g_n\|_{C(K)} \le 1$ and

$$\text{measure}\,(K_n) = \text{measure}\,(\{t \in K | g(t) \ne g_n(t)\}) \le \frac{1}{n}\,.$$

Thus $Ag_n \in C(K)$, $\|Ag_n\| \le \|A\|_{C(K)}$ and

$$\begin{aligned} \|Ag - Ag_n\|_{L^2(K)} &\le \|A\|_{L^2(K)}\, \|g - g_n\|_{L^2(K)} \\ &= \|A\|_{L^2(K)} \left(\int_{K_n} |g(t) - g_n(t)|^2\, dt \right)^{1/2} \\ &\le \frac{2}{\sqrt{n}} \|A\|_{L^2(K)}\,. \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\begin{aligned} \|Ag - Ag_n\|_{L^2(K)} &\ge \left(\int_E \left(|Ag(t)| - |Ag_n(t)|\right)^2 dt \right)^{1/2} \\ &\ge \left(\int_E \left(\|A\|_{C(K)} + \varepsilon - \|A\|_{C(K)}\right)^2 dt \right)^{1/2} \\ &\ge \varepsilon^{3/2}\,. \end{aligned} \tag{2.4}$$

Inequalities (2.3)–(2.4) give a contradiction; so the operator A is bounded in L^∞ and $\|A\|_{L^\infty(K)} \le \|A\|_{C(K)}$. Since, the reverse inequality is obvious, the lemma is proved. □

Theorem 2.9 *Let Hypothesis* 2.3 *hold, and suppose that* $A_2 = A_2^*$ *and that* $\|T_p(t)\| \le e^{\omega t}$ *holds for all* $p \in (1, \infty)$*, all* $t > 0$*, and some* $\omega \ge 0$.

1. *Then* $\mathcal{T}_p$ *is an analytic semigroup in* $\Sigma(\theta_p)$*, where*

$$\theta_p \ge \begin{cases} \dfrac{\pi}{p}\,, & \text{if} \quad 2 \le p < \infty\,, \\ \dfrac{\pi}{p'} = \pi\left(1 - \dfrac{1}{p}\right)\,, & \text{if} \quad 1 < p < 2\,. \end{cases}$$

2. *If* $\mathcal{T}_\infty$ *on* $C(\overline{\Omega})$*,* $\overline{\Omega}$ *being a compact Hausdorff space, is a* (C_0) *and an analytic semigroup in* $\Sigma(\theta^*)$*, where* $0 < \theta^* \le \frac{\pi}{2}$*, then* $\mathcal{T}_p$ *is analytic in* $\Sigma(\theta_p)$*; here*

$$\theta_p \ge \begin{cases} \dfrac{\pi}{p} + \theta^*\left(1 - \dfrac{2}{p}\right)\,, & \text{if} \quad 2 < p < \infty\,, \\ \dfrac{\pi}{p'} + \theta^*\left(1 - \dfrac{2}{p'}\right)\,, & \text{if} \quad 1 \le p \le 2\,, \quad \dfrac{1}{p} + \dfrac{1}{p'} = 1\,. \end{cases}$$

3. *If* $\theta^* = \frac{\pi}{2}$ *in* 2.*, then* $\theta_p = \frac{\pi}{2}$*, for all* $p \in [1, \infty]$.

P r o o f. 1. By Remark 2.7, $\mathcal{T}_p$ is an analytic semigroup for $1 < p < \infty$. Take $p_0 = 2$, $p_1 = q$, and let $q \to 1$ (resp. $q \to \infty$); this gives

$$\theta_p \ge \begin{cases} \dfrac{\pi}{p}\,, & \text{if} \quad p > 2\,, \\ \dfrac{\pi}{p'} = \pi\left(1 - \dfrac{1}{p}\right)\,, & \text{if} \quad p < 2\,, \end{cases}$$

and thus

$$\theta_p \geq \frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right).$$

This is Theorem 1 by Stein in [32], p. 67.

2. For this part, we interpolate between $\mathcal{T}_2$ (analytic in $\Sigma\left(\frac{\pi}{2}\right)$ and $\mathcal{T}_\infty$ (analytic in $\Sigma(\theta^*)$ and acting on $C(\overline{\Omega})$). For $2 < p < \infty$, we have $1 - r = \frac{2}{p}$ (see (2.2)). Thus

$$\theta_p \geq r\theta_{p_1} + (1-r)\theta_{p_0} = r\theta^* + \frac{2}{p}\frac{\pi}{2} = \theta^*\left(1 - \frac{2}{p}\right) + \frac{\pi}{p}. \tag{2.5}$$

The case of θ_p for $p \in (1,2)$ is similar, if $p_0 = 1$ and $p_1 = 2$. The duality argument needed for this makes use of the fact that, if A generates a semigroup analytic in $\Sigma(\theta)$, then so does A^* on X^*, if X is reflexive. But for $A = A_p$ on L^p, this yields $A_p^* = A_{p'}$, for $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$. It remains to show that $\mathcal{T}_1$ generates a semigroup in L^1, analytic in $\Sigma(\theta)$, for $0 < \theta < \theta^*$. Let A_∞ denote the generator of $\mathcal{T}_\infty$ on $C(\overline{\Omega})$, which is analytic in $\Sigma(\theta^*)$. By Theorem 2.2-3, for any $\theta \in (-\theta^*, \theta^*)$, the operator $B_{\theta p} := e^{i\theta}A_p$ generates a (C_0) semigroup on L^p, if $1 < p < \infty$, and on $C(\overline{\Omega})$, if $p = \infty$.

Set $Y_p = L^p(\Omega)$, for $1 < p < \infty$, and $Y_\infty = C(\overline{\Omega})$. Then,

$$\|\exp(tB_{\theta\infty})\|_{L(Y_\infty)} \leq M'_\theta e^{\omega'_\theta t}$$

holds for all $t > 0$ and for some constants $M'_\theta \geq 1$ and $\omega'_\theta \geq 0$. Next, $\|\exp(tB_{\theta 2})\|_{L(Y_2)}$ satisfies the same sort of inequality, but with M'_θ and ω'_θ replaced by M''_θ and ω''_θ, respectively. We thus conclude that

$$\|\exp(tB_{\theta p})\|_{L(Y_p)} \leq M'''_\theta e^{\omega'''_\theta t} \tag{2.6}$$

holds for all $t > 0$, $\theta \in (-\theta^*, \theta^*)$ and $2 \leq p \leq \infty$, where $M'''_\theta = \max\{M'_\theta, M''_\theta\}$ and $\omega'''_\theta = \max\{\omega'_\theta, \omega''_\theta\}$. Note that these constants are independent of p. By duality, the same estimate also holds for $1 < p < 2$. Because $\mathcal{M}$ is dense in Y_1, letting $p \to 1$ in (2.6), we get that this inequality holds for $p = 1$ as well.

Now, Y_1 embeds continuously into Y_∞^*, which by the Riesz representation theorem can be identified with the (finite) complex signed measures on $\overline{\Omega}$. For $f \in Y_\infty \subset Y_1$, $e^{tB_{\theta 1}} f \in Y_\infty \subset Y_1$. Let $\varepsilon > 0$ be given. For a general $f \in Y_1$, consider a suitable sequence $f_n \in Y_\infty$, such that

$$\|f_n - f\|_{Y_1} < \frac{\varepsilon}{3M'''_\theta \exp(\omega'''_\theta)},$$

if $n \geq N_0$, for some N_0. Then, using Lemma 2.8, we get

$$\begin{aligned}
\|\exp(tB_{\theta 1})f - f\|_{Y_1} &\leq \|\exp(tB_{\theta 1})(f - f_n)\|_{Y_1} + \|\exp(tB_{\theta 1})f_n - f_n\|_{Y_1} + \|f_n - f\|_{Y_1} \\
&\leq M'''_\theta \exp(\omega'''_\theta)\|f - f_n\|_{Y_1} + \mu(\overline{\Omega})\|\exp(tB_{\theta\infty})f_n - f_n\|_{Y_\infty} \\
&\quad + \frac{\varepsilon}{3M'''_\theta \exp(\omega'''_\theta)} \\
&< \frac{2}{3}\varepsilon + \mu(\overline{\Omega})\|\exp(tB_{\theta\infty})f_n - f_n\|_{Y_\infty} \\
&< \varepsilon,
\end{aligned}$$

for a fixed n and $0 < t < t_0 = t_0(\varepsilon, n, \mu(\overline{\Omega}))$. This implies $\exp(tB_{\theta 1})(Y_1) \subseteq Y_1$ and $\{\exp(tB_{\theta 1}) : t \geq 0\}$ is a (C_0) semigroup on Y_1. Hence, by Theorem 2.2-3, A_1 generates a semigroup, analytic in $\Sigma(\theta^*)$. Thus the assertion 2 holds.

3. The equality $\theta_p = \frac{\pi}{2}$ for $2 \leq p < \infty$ follows by interpolation, for $1 < p < 2$ it follows by duality. For $p = 1$ we use Theorem 2.2-3, as in the proof of part 2 above. This proves 3. □

The theorem also holds if we replace $L^p(\Omega, \Sigma, \mu)$ by $L^p(\Omega, \Sigma, \mu; Y)$, the Y-valued measurable L^p functions on Ω, where Y is any Banach space.

Remark 2.10 The end of the proof of Theorem 2.9-2 above shows that, if $\mathcal{T}_\infty$ generates a semigroup on $C(\overline{\Omega})$ analytic in $\Sigma(\theta^*)$, then so does $\mathcal{T}_1$ on L^1. The converse is not true, but one can nevertheless deduce some consequences. So let $\mathcal{T}_1$ generate a semigroup on L^1 analytic in $\Sigma(\theta^*)$, assume Hypothesis 2.3 holds and $A_2 = A_2^*$. As in the proof of 2. above, by letting $p \to \infty$, for $f \in C(\overline{\Omega})$ we can conclude

$$\|S_{\infty,\theta}(t)f\|_\infty \le M_{\theta_1} e^{\omega_{\theta_1} t} \|f\|_\infty$$

holds for $t \ge 0$, $\theta \in \mathbf{R}$, $|\theta| \le \theta_1 < \theta^*$. The strong continuity of $S_{p,\theta}(t)$ follows for $t > 0$, $|\theta| < \theta^*$. But this does not imply that $\mathcal{S}_{\infty,\theta}$ is of class (C_0). We can conclude that $S_{\infty,\theta}(\cdot)f$ is analytic for $t > 0$ and strongly continuous for $t > 0$, for each $\theta \in (-\theta^*, \theta^*)$ and each $f \in C(\overline{\Omega})$. Then $\mathcal{T}_\infty$ is analytic in $\Sigma(\theta^*)$, but not necessarily strongly continuous at $t = 0$. However, if it consists of positive operators (for $t > 0$), then by Arendt's theorem [3], it is a once integrated semigroup.

We have dealt with the case when the bounded domain is smooth. If $\big(T_1(t)\big)$ is analytic, then, for $t > 0$, $T(t)f \in \bigcap_{n=1}^\infty D\left(A_1^n\right)$, for all $f \in L^1$, and hence for all $f \in C(\overline{\Omega})$. Concluding that $T(t)f \in C(\overline{\Omega})$, for $t > 0$ and $f \in C(\overline{\Omega})$, normally requires the use of Sobolev inequalities and enough regularity for $\partial\Omega$. Without such assumptions it is not clear whether or not $\big(T(t)\big)$ extends to a semigroup on $C(\overline{\Omega})$.

3 Second order uniformly elliptic operators with general Wentzell boundary conditions

In this Section we will assume the following hypothesis.

Hypothesis 3.1 1. *Ω is a bounded domain of $\mathbf{R^N}$ with boundary $\partial\Omega$ in $C^{2+\epsilon}$.*

2. *a_{ij}, $i,j = 1,\dots,N$, are real valued functions in $C^1(\overline{\Omega})$ such that $a_{ij} = a_{ji}$ and there exist $\alpha_1 \ge \alpha_0 > 0$ such that*

$$\alpha_0|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \le \alpha_1|\xi|^2\,, \quad x \in \overline{\Omega}, \quad \xi = (\xi_1,\dots,\xi_N) \in \mathbf{R^N}\,.$$

3. *$\beta, \gamma \in C^1(\partial\Omega)$, $\beta(x) > 0$, $\gamma(x) \in \mathbf{R}$ on $\partial\Omega$, $q \in [0,\infty)$.*

Let us identify every $u \in C(\overline{\Omega})$ with $U = (u|_\Omega, u|_{\partial\Omega})$ and define X_p to be the completion of $C(\overline{\Omega})$ in the norm $|||\cdot|||_p$ given by

$$|||u|||_p := \left(\int_\Omega |u|^p\,dx + \int_{\partial\Omega} |u|^p \,\frac{dS}{\beta}\right)^{\frac{1}{p}}, \quad 1 \le p < \infty\,, \tag{3.1}$$

or, for $p = \infty$, $X_\infty = C(\overline{\Omega})$ and

$$|||u|||_\infty := ||u||_\infty = \sup_{x\in\overline{\Omega}} |u(x)|.$$

In general, a member of X_p is $H = (f,g)$, where $f \in L^p(\Omega)$, $g \in L^p\left(\partial\Omega, dS/\beta\right)$. Note that f may not have a trace on $\partial\Omega$, and even if f does, this trace needs not equal g. For $p = 2$, X_2 is a Hilbert space with respect to the inner product

$$\langle H_1, H_2\rangle_{X_2} := \langle f_1, f_2\rangle_{L^2(\Omega)} + \langle g_1, g_2\rangle_{L^2(\partial\Omega, dS/\beta)}\,, \quad H_i = (f_i, g_i) \in X_2, \quad i = 1,2\,;$$

$X_\infty = C(\overline{\Omega})$ is equipped with the sup-norm. Define the formal differential operator A such that

$$Au := \sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j u) = \nabla\cdot(a(x)\nabla u)\,;$$

here $n = (n_1,\dots,n_N)$ is the unit outer normal and $\partial_n^a u$ the conormal derivative of u with respect to (a_{ij}), i.e.

$$\partial_n^a u := \sum_{i,j=1}^N a_{ij}(\partial_j u) n_i = (a(x)\nabla u)\cdot n\,. \tag{3.2}$$

We say that the general Wentzell boundary condition holds for a (sufficiently smooth) function u if

$$Au + \beta\partial_n^a u + \gamma u - q\beta\Delta_{LB} u = 0\,, \quad \text{on} \quad \partial\Omega\,. \tag{3.3}$$

The next result extends [14], Theorem 3.1 in the context of the space X_2.

Theorem 3.2 *Under the Hypothesis* 3.1*, the realization A_2 of A in X_2, with domain*

$$D(A_2) := \left\{U = (u|_\Omega, u|_{\partial\Omega}) \in C^2(\overline{\Omega}) :\ u|_{\partial\Omega} \textit{ satisfies } (3.3)\right\}\,, \tag{3.4}$$

has the following properties:

1. *A_2 is symmetric on X_2;*
2. *A_2 is bounded above on X_2;*
3. *The closure G_2 of A_2 is selfadjoint and thus generates a cosine function and a quasicontraction (contraction, if $\gamma \geq 0$) semigroup on X_2, which is analytic in the right half plane.*

Proof. Observe that if $U = (u|_\Omega, u|_{\partial\Omega}) \in D(A_2)$ then $Au \in C(\overline{\Omega}) \subset X_2$ and A_2U can be identified by the pair $(Au|_\Omega, Au|_{\partial\Omega})$. Now let $V = (v|_\Omega, v|_{\partial\Omega}) \in D(A_2)$ and use the divergence theorem, the boundary condition, and Stokes' theorem on the boundary to evaluate

$$\begin{aligned}
\langle A_2U, V\rangle_{X_2} &= \sum_{i,j=1}^N \int_\Omega \partial_i\big(a_{ij}(x)\partial_j u(x)\big)\overline{v}(x)\,dx + \int_{\partial\Omega} Au(x)\overline{v}(x)\,\frac{dS}{\beta(x)}\\
&= -\sum_{i,j=1}^N \int_\Omega a_{ij}(x)\big(\partial_j u(x)\big)\partial_i\overline{v}(x)\,dx\\
&\quad + \int_{\partial\Omega}\left[\sum_{i,j=1}^N a_{ij}(x)\big(\partial_j u(x)\big)n_i + \frac{Au(x)}{\beta(x)}\right]\overline{v}(x)\,dS\\
&= -\sum_{i,j=1}^N \int_\Omega a_{ij}(x)\big(\partial_j u(x)\big)\partial_i\overline{v}(x)\,dx\\
&\quad - \int_{\partial\Omega}\gamma(x)u(x)\overline{v}(x)\frac{dS}{\beta(x)} - q\int_{\partial\Omega}\nabla_\tau u(x)\nabla_\tau\overline{v}(x)\,dS\\
&= \langle U, A_2V\rangle_{X_2}\,,
\end{aligned}$$

where ∇_τ is the tangential gradient on $\partial\Omega$ and $\nabla_\tau\cdot\nabla_\tau = \Delta_{LB}$. Then the assertion 1 follows.

Now let us consider $U = (u|_\Omega, u|_{\partial\Omega}) \in D(A_2)$. From the previous calculation we deduce that

$$\begin{aligned}
\langle A_2U, U\rangle_{X_2} &= -\sum_{i,j=1}^N \int_\Omega a_{ij}(x)\big(\partial_j u(x)\big)\partial_i\overline{u}(x)\,dx\\
&\quad - \int_{\partial\Omega}\gamma(x)|u(x)|^2\frac{dS}{\beta(x)} - q\int_{\partial\Omega}|\nabla_\tau u(x)|^2\,dS\,.
\end{aligned}$$

If $\gamma \geq 0$, then A_2 is nonpositive. Otherwise from $\gamma = \gamma_+ - \gamma_-$ it follows that

$$\langle A_2U, U\rangle_{X_2} \leq \|\gamma_-\|_\infty\,\|U\|_{X_2}^2\,,$$

hence $A_2 - \|\gamma_-\|_\infty I$ is nonpositive on X_2. Thus A_2 is bounded above and quasidissipative on X_2 and assertion 2 holds. This implies that A_2 is closable and its closure G_2 is still quasidissipative and bounded above. In order to complete the proof of the assertion 3 (with the aid of the spectral theorem), we will show that the range condition is satisfied for G_2, i.e. if $\lambda \in \mathbf{C}$, $Re\,\lambda > \|\gamma_-\|_\infty$ and $h : \overline{\Omega} \to \mathbf{C}$ is given, h sufficiently smooth on Ω, then the elliptic equation

$$\lambda U - A_2U = H = (h|_\Omega, h|_{\partial\Omega}) \tag{3.5}$$

is satisfied by some $U = (u|_\Omega, u|_{\partial\Omega}) \in D(A_2)$. Indeed, in (3.5), consider $\langle \lambda U - A_2 U, V\rangle_{X_2}$, where $V = (v|_\Omega, v|_{\partial\Omega}) \in X_2$, $v \in C(\overline{\Omega}) \cap H^1(\Omega)$. The result is

$$\lambda \int_\Omega u\overline{v}\, dx - \int_\Omega (A_2 u)\, \overline{v}\, dx + \int_{\partial\Omega} (\lambda u - A_2 u)\, \overline{v}\, dS = \int_\Omega h\overline{v}\, dx + \int_{\partial\Omega} h\overline{v}\, \frac{dS}{\beta}\,. \tag{3.6}$$

By the divergence theorem and (3.3), we obtain

$$\begin{aligned} &\lambda \int_\Omega u\overline{v}\, dx + \sum_{i,j=1}^N \int_\Omega a_{ij}\big(\partial_j u\big)\partial_i \overline{v}\, dx + q \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau \overline{v}\; dS + \int_{\partial\Omega} (\lambda + \gamma)\, u\overline{v}\, \frac{dS}{\beta} \\ &= \int_\Omega h\overline{v}\, dx + \int_{\partial\Omega} h\overline{v}\, \frac{dS}{\beta}\,. \end{aligned} \tag{3.7}$$

Let $L(U, V)$ be the left-hand side of (3.7) and let $F(V)$ be the corresponding right-hand side. Let $\mathcal{H}$ be the Hilbert space obtained as completion of $C^1(\overline{\Omega})$ in the norm

$$\|U\|_{\mathcal{H}} := \left(\int_\Omega |u|^2\, dx + \int_\Omega \sum_{i,j=1}^N a_{ij}(x)\big(\partial_i u\big)\partial_j \overline{u}\, dx + q \int_{\partial\Omega} |\nabla_\tau u|^2\, dS + \int_{\partial\Omega} |u|^2\, \frac{dS}{\beta} \right)^{\frac{1}{2}},$$

where $U = (u|_\Omega, u|_{\partial\Omega}) \in \mathcal{H}$; $\mathcal{H} = H^1(\Omega) \times L^2(\partial\Omega)$ (algebraically) if $q = 0$, while $\mathcal{H} = H^1(\Omega) \times H^1(\partial\Omega)$ if $q > 0$. L is a bounded sesquilinear form on $\mathcal{H}$ and F is a bounded conjugate linear functional on $\mathcal{H}$:

$$\begin{aligned} |L(U, V)| &\le \max\{|\lambda| + \|\gamma\|_\infty + q, 1\}\, \|U\|_{\mathcal{H}}\, \|V\|_{\mathcal{H}} = c_1(\lambda)\|U\|_{\mathcal{H}}\, \|V\|_{\mathcal{H}}, \\ |F(V)| &\le \|H\|_{X_2}\, \|V\|_{\mathcal{H}}\,. \end{aligned}$$

Also

$$\begin{aligned} Re\, L(U, U) &\ge (Re\,\lambda)\, \|u\|^2_{L^2(\Omega)} + Re \int_\Omega \sum_{i,j=1}^N a_{ij}(x)\big(\partial_i u(x)\big)\partial_j \overline{u}(x)\, dx \\ &\quad + Re\,(\lambda - \|\gamma_-\|_\infty)\, \|u\|^2_{L^2(\partial\Omega, dS/\beta)} + q\, \|\nabla_\tau u\|^2_{L^2(\partial\Omega, dS)} \\ &\ge c_2(\lambda)\, \|U\|^2_{\mathcal{H}} \end{aligned}$$

and $c_2(\lambda) > 0$. By the Lax-Milgram lemma, there exists a unique $U \in \mathcal{H}$ such that $L(U, V) = F(V)$ holds for all $V \in \mathcal{H}$. That is, (3.7) holds for $U = (u|_\Omega, u|_{\partial\Omega})$ and this U is our weak solution of (3.5) with (3.3). Since $a_{ij}, \gamma, \beta, h, \partial\Omega$ are smooth enough, then $U \in C^2(\overline{\Omega})$. Indeed, for $H \in C^4(\overline{\Omega})$, the solution U of $\lambda U - A_2 U = H$ is in $D(A_2)$. If H is simply in X_2, choose $H_n \in X_2 \cap C^4(\overline{\Omega})$ such that $H_n \to H$ in X_2. Then, by elliptic regularity and the compactness of the Sobolev imbedding, $U \in D(G_2)$; thus G_2, the closure of A_2, satisfies the assertion 3. □

Here is our main result for second order operators.

Theorem 3.3 *Assume Hypothesis* 3.1 *and assume further that* $q = 0$, $\partial\Omega$ *and all the coefficients* a_{ij}, β, γ *are of class* C^∞. *Then the closure* G_p *of the realization* A_p *of* A *in* X_p, $1 \le p \le \infty$, *with domain*

$$D(A_p) := \{U \in D(A_2) \cap X_p : \; AU \in X_p\}\,,$$

is quasi m*-dissipative. For* $p \in [1, \infty]$, *the semigroup* T_p *generated by* G_p *is analytic in the sector*

$$\Sigma(\theta_p) := \left\{ z \in \mathbf{C} : \; Re\, z > 0, \; \left| \frac{Im\, z}{Re\, z} \right| \; < \; \theta_p \right\},$$

and

$$\theta_p \ge \max\{\theta_p^*, \theta_p^{**}\}\,,$$

where

$$\theta_p^* = \begin{cases} \pi\left(1-\frac{1}{p}\right) + \left(\frac{2}{p}-1\right)\theta_\infty^*, & \textit{if} \quad 1 \le p \le 2, \\ \frac{\pi}{p} + \left(1-\frac{2}{p}\right)\theta_\infty^*, & \textit{if} \quad 2 \le p \le \infty, \end{cases}$$

$$\theta_p^{**} = \frac{\pi}{2} - \arctan\frac{|p-2|}{2\sqrt{p-1}}, \quad \textit{if} \quad 1 \le p \le \infty,$$

and

$$\theta_\infty^* = \theta_1^* = \arctan\left(\frac{2\sqrt{\alpha_0\alpha_1}}{\alpha_1-\alpha_0}\right),$$

where α_0, α_1 *are the moduli of ellipticity of* $a(x)$, *according to Hypothesis* 3.1-2. *When* A *is a multiple of the Laplacian, then* $\alpha_0 = \alpha_1$, *and* $\theta_p = \frac{\pi}{2}$, *for* $1 \le p \le \infty$.

Remark 3.4 In case $\theta_\infty^* < \frac{\pi}{2} - 1$, we can improve the lower bound for θ_p even more $\big($for $\frac{1}{p}$ close to 1 and 0$\big)$; see Remark 3.5 below. If $q > 0$, $\theta_p^* = \theta_{p'}^*, \ge \frac{\pi}{p}$ for $2 \le p < \infty$ holds.

If we replace θ_∞^* by 0, we do not need the extra C^∞ regularity assumption. Our proof makes use of regularity results concerning the Dirichlet-to-Neumann operator. These results are proved using pseudodifferential operator techniques, and in such proofs, C^∞ regularity is normally assumed. These results are undoubtedly true under much less regularity, but it is technically quite difficult to chase through the proofs to determine what minimal regularity is sufficient.

Relevant historical remarks are given in Section 5.

Proof. Let us show that A_p is quasidissipative on X_p, for all $p \in (1,\infty)$. First, suppose $p \in [2,\infty)$. Let $JU := |U|^{p-2}\overline{U}\chi_{\{U\neq 0\}}$ be the duality map of X_p (modulo a positive constant multiple which depends on $|||U|||_p$, for $U \neq 0$). Let $U = (u|_\Omega, u|_{\partial\Omega}) \in D(A_p)$, $0 \neq U \in D(A_p)$. We have

$$\begin{aligned}\langle A_pU, JU\rangle &= \int_\Omega Au\left(|u|^{p-2}\,\overline{u}\right)\chi_{\{u\neq 0\}}\,dx + \int_{\partial\Omega} Au\left(|u|^{p-2}\overline{u}\right)\chi_{\{u\neq 0\}}\,\frac{dS}{\beta} \\ &= \int_\Omega Au\left(|u|^{p-2}\overline{u}\right)\chi_{\{u\neq 0\}}\,dx - \int_{\partial\Omega}\beta\left(\partial_n^a u\right)|u|^{p-2}\overline{u}\,\chi_{\{u\neq 0\}}\,\frac{dS}{\beta} - \int_{\partial\Omega}\gamma u\,|u|^{p-2}\,\overline{u}\,\chi_{\{u\neq 0\}}\,\frac{dS}{\beta} \\ &= -\sum_{i,j=1}^N \int_\Omega a_{ij}\left(\partial_j u\right)\partial_i\left(|u|^{p-2}\overline{u}\right)\chi_{\{u\neq 0\}}\,dx - \int_{\partial\Omega}\gamma|u|^p\,\frac{dS}{\beta},\end{aligned}$$

by the divergence theorem. Observe that, if we call

$$Z := \sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u\partial_i\left(|u|^{p-2}\overline{u}\right)\chi_{\{u\neq 0\}}\,dx,$$

then $Re\,Z \ge 0$. Indeed,

$$\begin{aligned}Z &= \sum_{i,j=1}^N \int_\Omega a_{ij}(\partial_j u)\left(\partial_i\left(|u|^{p-2}\overline{u}\right)\right)\chi_{\{u\neq 0\}}\,dx \\ &= \int_\Omega \sum_{i,j=1}^N a_{ij}(\partial_j u)\left(\partial_i\overline{u}\right)\left(|u|^{p-2}\right)\chi_{\{u\neq 0\}}\,dx \\ &\quad + (p-2)\int_\Omega \sum_{i,j=1}^N a_{ij}(\partial_j u)\,\overline{u}\,|u|^{p-4} Re\left(u\partial_i\overline{u}\right)\chi_{\{u\neq 0\}}\,dx,\end{aligned}$$

since

$$\partial_j |u|^q = \partial_j \left[\left(|u|^2 \right)^{\frac{q}{2}} \right] = \frac{q}{2} |u|^{2(q/2-1)} \left(u \partial_j \overline{u} + (\partial_j u) \overline{u} \right) = q\, |u|^{q-2}\, Re\, (u \partial_j \overline{u})\,,$$

where $|u|^2 = u\overline{u} \neq 0$. From

$$Re\, (\overline{u} \partial_i u)\, Re\, (u \partial_i \overline{u}) = [Re\, (u \partial_i \overline{u})]^2 \geq 0$$

and the positive definiteness of (a_{ij}), it follows that $Re\, Z \geq 0$.

Note that the above proof becomes very simple in the real case, since, if u is real, then $|u|^{p-2}\overline{u}$ becomes the odd version of u^{p-1}. Then we get

$$Z = (p-1) \int_\Omega \sum_{i,j=1}^N a_{ij} (\partial_j u)(\partial_i u) \chi_{\{u \neq 0\}} |u|^{p-2}\, dx \geq 0\,.$$

Thus we obtain that

$$Re\, \langle A_p U, JU \rangle \leq 0, \quad \text{if} \quad \gamma \geq 0.$$

In the general case for γ, the operator A_p is quasidissipative, since

$$Re\, \langle A_p U - \|\gamma_-\|_\infty U, JU \rangle \leq 0.$$

By symmetry and duality or by arguing, for example, as in the proof of Lemma 4.4.3 in [10], it can be seen that the same inequality holds for $p \in (1,2)$ too.

Now, A_p quasidissipative implies

$$\|U\| \leq \frac{\|(\lambda - A_p)U\|}{Re \lambda - \|\gamma_-\|_\infty}\,, \tag{3.8}$$

which proves that the range is closed. Let us show that is dense, too.

Let $H \in \bigcap_{q=1}^\infty X_q$; this latter space is dense in X_p, for all $p \in [1, \infty)$. There is a unique solution $U \in X_2$ of

$$\lambda U - A_2 U = H\,,$$

for all λ, such that $Re\, \lambda > \|\gamma_-\|_\infty$. Let $1 \leq p \leq 2$. Then $U \in X_2$ implies $U \in X_p$, and, since $A_q = \overline{A_{|X_q}}$ for all q, $\lambda U - A_p U = H$ and

$$|||U|||_p \leq \frac{1}{Re\, \lambda - \|\gamma_-\|_\infty}\, |||H|||_p\,. \tag{3.9}$$

Since A_p is quasi m-dissipative, the range $R\,(\lambda I - A_p)$ is both dense and closed, hence it is in X_p. By duality, $\lambda I - A_{p'} = \lambda I - (A_p)^*$ maps $D\,(A_{p'})$ onto $X_{p'}$ for $1 < p \leq 2$, that is for $2 \leq p' < \infty$.

Using Neuberger's theorem [28], it follows that $\mathcal{T}_p$ generates an analytic semigroup on X_p, $1 < p < \infty$.

Now let us consider the case $p = \infty$, i.e. $X_\infty = C(\overline{\Omega})$. If (3.8) holds, let $p \to \infty$ in (3.9). We deduce that

$$\|U\|_\infty \leq \frac{\|H\|_\infty}{Re\, \lambda - \|\gamma_-\|_\infty},$$

for $Re\, \lambda > \|\gamma_-\|_\infty$. Moreover, U belongs to $X_\infty = C(\overline{\Omega})$, by elliptic regularity. Thus A_∞ is quasidissipative in $C(\overline{\Omega})$. Moreover, if $H \in C(\overline{\Omega})$ there exists a solution U to the equation $\lambda U - G_p U = H$, for any $p > 1$ and $Re\, \lambda > \|\gamma_-\|_\infty$. This implies that A_∞ is closable and $u \in \bigcap_{p>1} X_p \cap D(G_p)$. Let $G_\infty = \overline{A}_\infty$. Since the range condition is essentially p-independent, G_∞ generates a (C_0) semigroup on $C(\overline{\Omega})$.

Analogous arguments as in [14], Section 4, lead to the conclusion that A_1 is quasidissipative and closable in X_1. In addition, for $Re\, \lambda$ sufficiently large, $G_1 = \overline{A}_1$ is m-dissipative and thus G_1 generates a (C_0) semigroup on X_1.

Concerning the analyticity, observe that, according to [7], 4.3, whenever $a_{ij}, \beta, \gamma, \partial\Omega$ belong to $C^\infty(\overline{\Omega})$, we have that the realization B_∞ of A, with domain

$$D(B_\infty) = \left\{ f \in \bigcap_{p\geq 1} W^{2,p}_{loc}(\Omega) : \partial^a_n f \in C(\overline{\Omega}),\ Af \in C(\overline{\Omega}),\ f \text{ satisfies } (3.3) \right\},$$

generates an analytic semigroup on $C(\overline{\Omega})$. Since $(G_\infty, D(G_\infty)) = (B_\infty, D(B_\infty))$, we get that $(G_\infty, D(G_\infty))$ does too.

On the other hand, as a consequence of Theorem 2.9-2., we can deduce that, if θ^* is the angle of analyticity of the semigroup generated by G_∞, then G_1 generates an analytic semigroup having the same angle of analyticity. Moreover, if $\theta^* > 0$, then, for any $1 \leq p \leq \infty$, the angle of analyticity θ_p of the semigroup generated by G_p is always greater or equal to θ^*.

Escher [9] showed that the operator A_∞ generates an analytic semigroup on $C(\overline{\Omega})$, with some positive angle θ_∞. Actually Escher assumed $\beta = 1$, but it is easy to remove that restriction, as Engel and Fragnelli [7] showed. Escher's argument was quite ingenious and made use of Dirichlet-to-Neumann map. Recently, this result was extended by deLaubenfels and Emamirad [4], using ideas of [25]; see also [5]. DeLaubenfels and Emamirad obtained (in a different and also very clever way) the lower bound

$$\theta^*_\infty \geq \arctan\left(\frac{2\sqrt{\alpha_0\alpha_1}}{\alpha_1 - \alpha_0}\right). \tag{3.10}$$

The angle of analyticity of T_p on X_p is θ_p where $\theta_2 = \frac{\pi}{2}$ and $\theta_\infty \geq \theta^*_\infty$, since $\theta^{**}_\infty = 0$. Using the Angle Concavity Theorem 2.5, it follows that T_p generates a semigroup analytic in $\Sigma(\theta_p)$ on X_p where

$$\theta_p \geq \frac{\pi}{p} + \left(1 - \frac{2}{p}\right)\theta^*_\infty,$$

for $2 \leq p \leq \infty$. By duality,

$$\theta_p = \theta_{p'} \geq \pi\left(1 - \frac{1}{p}\right) + \left(\frac{2}{p} - 1\right)\theta^*_\infty,$$

for $1 < p \leq 2$, and this also holds for $p = 1$, by Theorem 2.9-2.

Note that, since the semigroup generated by $G_p - \|\gamma_-\|_\infty I$ is submarkovian, i.e. positive and contractive, then, according to the result by Liskevich and Perelmuter [26] (see also [29], Theorem 3.13), we can deduce that, for $1 < p < \infty$, the sector of analyticity has angle θ_p such that

$$\theta_p \geq \frac{\pi}{2} - \arctan\frac{|p-2|}{2\sqrt{p-1}},$$

while

$$\frac{\pi}{2} - \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right) > \begin{cases} \dfrac{\pi}{p}, & \text{if } 2 < p < \infty, \\ \pi\left(1 - \dfrac{1}{p}\right), & \text{if } 1 < p < 2. \end{cases}$$

Under the C^∞ hypothesis of [7], we also have that

$$\theta_p \geq \begin{cases} \dfrac{\pi}{p} + \theta^*_\infty\left(1 - \dfrac{2}{p}\right), & \text{if } 2 < p < \infty, \\ \dfrac{\pi}{p'} + \theta^*_\infty\left(1 - \dfrac{2}{p'}\right), & \text{if } 1 < p \leq 2. \end{cases}$$

Thus the proof is complete. □

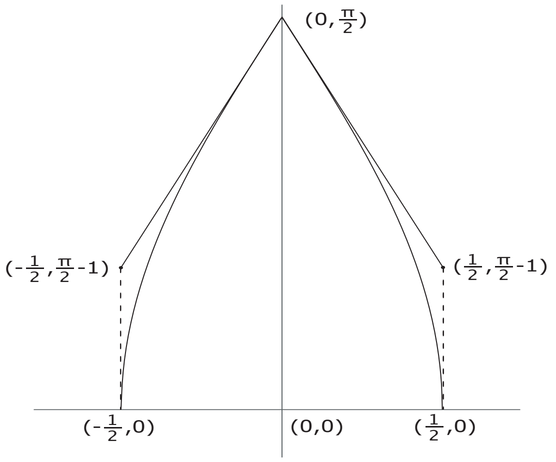

Fig. 1

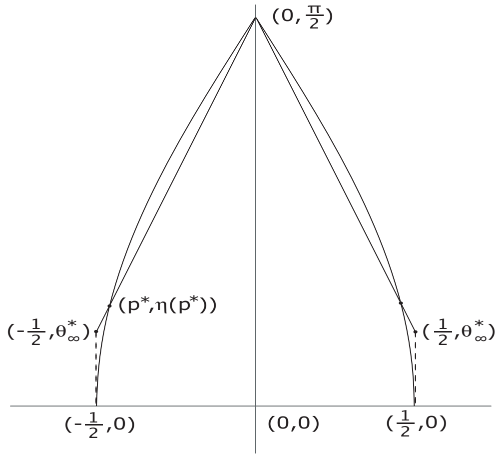

Fig. 2

Remark 3.5 As before, let

$$\theta_p^* = \pi\left(1-\frac{1}{p}\right) + \left(\frac{2}{p}-1\right)\theta_\infty^*, \quad \theta_p^{**} = \frac{\pi}{2} - \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

for $1 \le p \le 2$ and $\theta_p^* = \theta_{p'}^*$, $\theta_p^{**} = \theta_{p'}^{**}$ for $2 \le p \le \infty$. Here, of course, $p^{-1} + (p')^{-1} = 1$. Then $\theta_p \ge \max\{\theta_p^*, \theta_p^{**}\}$. Consider the function

$$\theta = G(p) = \frac{\pi}{2} - \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

for $1 \le p \le \infty$. Write this as $\theta = F(x)$ for $x = 1 - \frac{1}{p} \in [0,1]$. Call the graph of F the *Liskevich-Perelmuter* (or *L-P*) *curve*. Since F is an even function of $x - \frac{1}{2}$, a straightforward calculation shows that its derivative from the left at $x = \frac{1}{2}$, $F'\left(\frac{1}{2}^-\right)$ is 2. The line L_0 of slope 2 in the x-θ plane passing through $Q_1 = \left(\frac{1}{2}, \frac{\pi}{2}\right)$ hits the θ-axis at $Q_0 = \left(0, \frac{\pi}{2} - 1\right)$ and $\frac{\pi}{2} - 1$ is approximately 0.57, about 36% of $\frac{\pi}{2}$ which is approximately 1.57 (see Figure 1). For the sake of simplicity, we used $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for $x - \frac{1}{2}$ in order to make the graph symmetrical with respect to the θ axis.

Consider $\theta_1^* = \theta_\infty^*$, the deLaubenfels-Emamirad angle. If $\theta_1^* \ge \frac{\pi}{2} - 1$, then the straight line interpolation in the x-θ plane determined by the Angle Concavity Theorem 2.5 is the best we can do, i.e. our best estimate for θ_p is θ_p^*. But when $0 < \theta_1^* < \frac{\pi}{2} - 1$ we can do better.

The line segment L_1 from $(0, \theta_1^*) = Q_2$ to $\left(\frac{1}{2}, \frac{\pi}{2}\right) = Q_1$ intersects the L-P curve at exactly one point, $\left(x_3, \widetilde{\theta}\right) = Q_3$, satisfying $0 < x_3 < \frac{1}{2}$. Let m_1 be the slope of this line L_1 (see Figure 2).

Rotate this line L_1 counterclockwise, that is, increase the slope (starting with m_1) until the rotating line is tangent to the *L-P* curve at some point, which we denote by $Q_4 = \left(x_4, \widehat{\theta}\right)$. This point is uniquely determined and $0 < x_4 < \frac{1}{2}$, $\theta_1^* < \widehat{\theta} < \frac{\pi}{2}$. Call this line L_2 (see Figure 3). The curve $\theta(p) = \max\left\{\theta_p^*, \theta_p^{**}\right\}$ is not concave if $0 < \theta_1^* < \pi/2 - 1$. Its concave envelope is precisely the curve we just constructed.

Apply the Angle Concavity Theorem 2.5 to our semigroup, taking $p_0 = 1$ and $p_1 = \frac{1}{1-x_4}$, which means that p_1 is the value of p that x_4 represents. The result is a lower bound for θ_p which strictly exceeds $\max\left\{\theta_p^*, \theta_p^{**}\right\}$ for $1 < p < p_1$. Of course, by symmetry, this strictly increases the lower bound for $\theta_{p'} = \theta_p$ for $p_1' < p < \infty$.

To find good numerical approximations of x_4 and $\widehat{\theta}$ one can use Matlab or a similar tool.

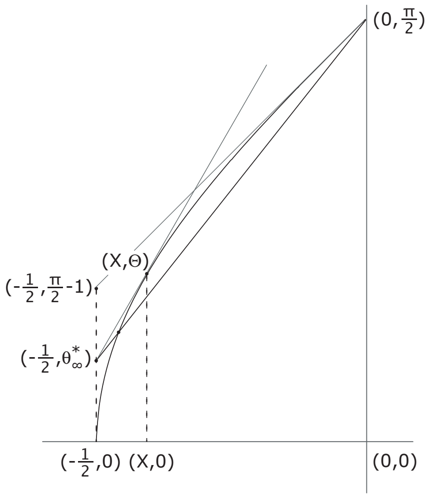

Fig. 3

4 Higher order uniformly elliptic operators

For the operator A under consideration, i.e. $A = \sum_{i,j=1}^{N} \partial_i \left(a_{ij}(x)\partial_j\right)$, with general Wentzell boundary condition 3.3, we want to consider the Cauchy problem for operators of higher order

$$\frac{du}{dt} = (-1)^{m+1} A^m u, \quad u(0) = f\,, \tag{4.1}$$

where m is a positive integer. (Recall that A is *negative* in some sense and in solving $\frac{du}{dt} = Bu$, B must be *negative* in some sense. This is the heuristic explanation for the $(-1)^{m+1}$ factor.) We begin with $p = 2$. Since, according to the results of the previous section, the operator G_2 on X_2 is selfadjoint and bounded above, problem (4.1) is governed by a semigroup analytic in $\Sigma\left(\frac{\pi}{2}\right)$ on X_2, for all m. But in X_p, for $p \neq 2$, $(-1)^{m+1} G_p^m$ is not quasidissipative for any $m \geq 2$, i.e. $(-1)^{m+1} G_p^m$ does not generate a (C_0) semigroup $(T_p(t))_{t\geq 0}$ on X_p satisfying $\|T_p(t)\| \leq e^{\omega t}$ for all $t > 0$ and some real number ω. Thus the study of (4.1) for $m \geq 2$ must proceed differently from the study of (4.1) for $m = 1$.

Lemma 4.1 *Let A generate a semigroup in X, analytic in $\Sigma(\theta)$, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. Then $-A^2$ generates a semigroup, analytic in $\Sigma\left(2\theta - \frac{\pi}{2}\right)$. In particular, if A generates a semigroup analytic in $\Sigma\left(\frac{\pi}{2}\right)$, then so does $-A^{2^n}$ for $n = 1, 2, \ldots$*

This was proved by J. A. Goldstein in [19]. We shall use it to solve (4.1) for $m = 2^n$. Later we shall use an extension of Lemma 4.1 to handle the case of general m. Now let G_p generate a semigroup in L^p, analytic in $\Sigma\left(\theta_{p,1}\right)$. We work in the context of Hypothesis 2.3 (see Section 2). Note that $X_p = L^p\left(\overline{\Omega}, dx|_\Omega \oplus \frac{dS}{\beta}|_{\partial\Omega}\right)$ is contained in L^p. We treat the case of $G_2^* = G_2$. Let Θ_p be the lower bound for θ_p constructed in Remark 3.5. Then for $x = 1 - \frac{1}{p} \in \left[0, \frac{1}{2}\right]$, the graph of $F(x) = \Theta_p$ is the upper envelope of the graphs in Figure 2. Note that $\Theta_{p'} = \Theta_p$ for all $p \in [1, \infty]$ where $p^{-1} + (p')^{-1} = 1$. For convenience, in the discussion we restrict ourselves to the case $p > 2$. We use Lemma 4.1. Then $-G_p^2$ generates a semigroup on X_p, analytic in $\Sigma\left(\theta_{p,2}\right)$, provided $\theta_{p,1} > \frac{\pi}{4}$. In this case the angle $\theta_{p,2}$ is $2\left(\frac{\pi}{p}\right) - \frac{\pi}{2} = \pi\left(\frac{4-p}{2p}\right)$. Next $-G_p^4$ generates a semigroup on X_p, analytic in $\Sigma\left(\theta_{p,3}\right)$, whenever $\theta_{p,2} > \pi\left(\frac{2}{p} - \frac{1}{2}\right) > \frac{\pi}{4}$, or $p < \frac{8}{3}$, in which case the angle $\theta_{p,3}$ is at least

$$2\pi\left(\frac{2}{p} - \frac{1}{2}\right) - \frac{\pi}{2} = \pi\left(\frac{8-3p}{2p}\right).$$

By induction and duality, we conclude that $-G_p^{2^n}$ generates a semigroup on X_p, analytic in $\Sigma(\theta_{p,n+1})$, whenever $\underline{\beta}_n(p) < p < \overline{\beta}_n(p)$ with $\underline{\beta}_n(p) < \frac{2^{n+1}}{2^n+1}$ and $\overline{\beta}_n(p) > \frac{2^{n+1}}{2^n-1}$ (this implies $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2^{n+1}}$), in which case the (lower bound for the best) angle is

$$\theta_{p,n+1} = \theta_{p',n+1} = \pi \left(\frac{2^{n+1} - (2^n - 1)p}{2p} \right).$$

Again $2 < p < \frac{2^{n+1}}{2^n - 1}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. By knowing Θ_p^* and getting a good approximation for Θ_p, we can improve this result a lot when Θ_p^* is close to zero. Now we consider $-(-G_p)^m$ for general $m > 0$. (Here m needs not be an integer.) For this we need fractional powers. We want to take positive powers of *positive* operators, and generators are *negative*, so to regard the m-th power of a generator A as a generator itself, we use $-(-A)^m$.

Let $\mathcal{T} = \big(T(t)\big)$ be a semigroup analytic in $\Sigma\,(\theta_1)$ on a Banach space X, where $0 < \theta_1 \le \frac{\pi}{2}$. Let $0 < \theta_0 < \theta_1$. Then there is an $\omega \ge 0$ such that $\{e^{-\omega t} T(t) : t \in \Sigma\,(\theta_0)\}$ is a semigroup analytic in $\Sigma\,(\theta_0)$ and uniformly bounded; its generator is $A - \omega I$, if A generates $\mathcal{T}$.

Thus we may assume, with no real loss in generality, that $\mathcal{T}$ is uniformly bounded and A boundedly invertible, i.e., $0 \in \rho(A)$. Let $\alpha > 0$, not necessarily an integer. For A as above, we define

$$(-A)^{-\alpha} f = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha} (\lambda I + A)^{-1} f \, d\lambda$$

for $f \in X$, where $\lambda^{-\alpha}$ refers to the principal branch of the power and $\Gamma \subset \rho(-A)$ connects, with a smooth curve in $\rho(-A)$ avoiding $\{z \in \mathbf{R} : z \le 0\}$, the rays $\big\{z = re^{\pm i\varphi} : r \ge M\big\}$, for suitable $M > 0$ and $\varphi \in \big(\frac{\pi}{2}, \frac{\pi}{2} + \theta_1\big)$. Furthermore, define $(-A)^{\alpha}$ as the inverse of $(-A)^{-\alpha}$.

Lemma 4.2 (Huang-Zheng [23]) *Let X be a Banach space and A the generator of a bounded analytic semigroup in $\Sigma(\theta)$, where $0 < \theta \le \frac{\pi}{2}$; furthermore, let $0 \in \rho(A)$. Then $-(-A)^{\alpha}$ generates a bounded analytic semigroup in $\Sigma\big(\frac{\pi}{2} - \big(\frac{\pi}{2} - \theta\big)\,\alpha\big)$, provided that $1 < \alpha < \frac{\pi}{\pi - 2\theta}$.*

Replacing A by $A - \omega I$, for some $\omega \ge 0$, ensures that the semigroup generated by A is bounded in $\Sigma(\theta - \varepsilon)$ (for $\varepsilon > 0$) and $0 \in \rho(A)$. But the powers $(A - \omega I)^{\alpha}$ of $A - \omega I$, are not easily related to powers A^{α} of A. However, if $M = \alpha$ is a positive integer, this is not a problem, for the following reasons.

We say that $(B, D(B))$ is a *Kato perturbation* of $\big(A, D(A)\big)$ (cf. [20]) if $D(A) \subset D(B)$ and for every $\varepsilon > 0$ there is a nonnegative constant c_2 such that $\|Bf\| \le \varepsilon \|Af\| + c_2 \|f\|$, for all f in $D(A)$. Hille proved (see e.g. [20]) that if $(A, D(A))$ generates a semigroup, analytic in $\Sigma(\theta)$, so does $A + B$, if $(B, D(B))$ is a Kato perturbation of $(A, D(A))$.

When m is an integer, then by the binomial theorem,

$$(A - \omega I)^m = A^m + \sum_{j=0}^{m-1} d_j A^j = A^m + B\,;$$

here $(B, D(B))$ is a Kato perturbation of A^m and of $(A - \omega I)^m$, thus A^m and $(A - \omega I)^m$ both generate semigroups, analytic in $\Sigma(\theta)$.

When $\theta > \frac{\pi}{4}$ and $m = 2$, Lemma 4.2 reduces to Lemma 4.1 above.

Theorem 4.3 *Suppose $\alpha > 1$ and not a power of 2. Choose the unique n such that $2^n < \alpha < 2^{n+1}$. Let A satisfy the hypotheses of Lemma 4.2 and suppose, in addition, that $-A^{2^n}$ generates a semigroup, analytic in $\Sigma\,(\theta_n)$. Take $\nu = \frac{\alpha}{2^n} \in (1, 2)$. Then $-\big((-A)^{2^n}\big)^{\nu} = -(-A)^{\alpha}$ generates a semigroup, analytic in $\Sigma(\varphi)$, where*

$$\varphi \ge \varphi_0 = \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta_n\right) \frac{\alpha}{2^n}\,. \tag{4.2}$$

Notice that $\varphi = \frac{\pi}{2}$ when $\theta_n = \frac{\pi}{2}$ and $\varphi > 0$ provided $\alpha < 2^n \pi / (\pi - 2\theta_n)$, which can be rephrased equivalently as $\theta_n > \frac{\pi}{2}\left(1 - \frac{2^n}{\alpha}\right)$. Using the lower bound of Remark 3.5, we can improve the lower bound.

Observe that the angle φ_0 in (4.2) is a linear function of α in $\big[2^n, 2^{n+1}\big]$. By *linear function* here we use the usual one variable calculus definition: $y = ax + b$ is a linear function of x, whose graph is a line.

Notice that we only applied Lemma 4.2 with index ν satisfying $1 < \nu < 2$.

Thus all these arguments work well, in particular, for the operators considered in Section 3.

5 Historical remarks about analyticity for second order Wentzell operators

For the uniformly elliptic operator A defined by

$$Au = \nabla \cdot (a(x)\nabla u)$$

where $a(x)$ is a sufficiently smooth, uniformly elliptic, real hermitian matrix on $\overline{\Omega} \subset\subset \mathbf{R}^{\mathbf{N}}$, the classical boundary conditions are of Robin type:

$$\alpha \partial_n^a u + \beta u = 0 \quad \text{on} \quad \partial\Omega \, ,$$

where $\alpha, \beta \in C^1(\partial\Omega)$ are real valued, with $\alpha(x)^2 + \beta(x)^2 > 0$ for all $x \in \partial\Omega$. It is a classical result, going back to Agmon, Douglis, and Nirenberg [1], that the realization A_p of A on $L^p(\Omega)$ generates a semigroup, analytic in $\Sigma\left(\frac{\pi}{2}\right)$, for $1 < p < \infty$. This was eventually extended to $L^1(\Omega)$ and $C(\overline{\Omega})$ with the major contribution being due to Stewart [33]. Maximal regularity results in $L^p(\Omega)$ for operators with dynamic Robin boundary condition were established by Prüss (see, e.g., [30]).

For the Wentzell boundary condition (1.2) (with $q = 0$), we showed in [14] that A_p generates a semigroup for $1 < p < \infty$, analytic in $\Sigma(\theta_p)$. Analyticity in $C(\overline{\Omega})$ was obtained by Warma [34] in dimension $N = 1$, so that $\Omega = (0, 1)$. Xiao and Liang [35] improved Warma's result by showing that A_∞ generates a cosine function (see also [23]). Other interesting results concerning the generation of cosine functions by means of uniformly elliptic operators having boundary conditions of type (1.2), with $q = 0$, can be found in [24] and [27]. Amann and Escher [2] considered the special case of $\beta = 1$ and got analyticity of the semigroup on $L^p(\Omega, dx) \oplus L^p(\partial\Omega, dS)$, $1 < p < \infty$, but no quasidissipativity results. In a major work, Escher [9] showed that the Dirichlet-to-Neumann map generates an analytic semigroup on $C(\overline{\Omega})$ and that this implies the $p = \infty$ result for the Wentzell boundary condition problem with $\beta \equiv 1$ and $q = 0$. Engel and Fragnelli [7] extended Escher's result with a simpler proof, and Engel [6] showed that, for the Laplacian, the angle θ_∞^* is $\frac{\pi}{2}$. This was the first lower bound for θ_∞^* in dimension $N \geq 2$. Using different techniques, deLaubenfels and Emamirad [4] made a breakthrough by getting a good lower bound based on the moduli of ellipticity α_0, α_1 of A. This was the first explicit positive lower bound for θ_∞^* for general operators. All these results assumed $q = 0$. Some time ago we conjectured that in the Wentzell case, $\theta_p = \frac{\pi}{2}$ for all $p \in [1, \infty]$, just like in the classical Robin case. We delayed writing this paper until we became convinced that this conjecture seems very difficult to prove. Neverthless, we still suspect that the conjecture is valid:

$$\theta_p = \frac{\pi}{2}$$

for all $p \in [1, \infty]$.

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