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Complex Zeros of Algebraic Polynomial with Non-Zero Mean Random Coefficients[1]

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In this paper we obtain a formula for the average density of the distribution of complex zeros of an algebraic polynomial with random coefficients. The coefficients are assumed independent identical normally distributed random variables with mean μ and variance σ^2. The value of the average density for the case of $\mu = 0$ and $\sigma^2 = 1$ was obtained previously. Some limits of the distribution of the complex zeros are provided using the presented formula.

KEY WORDS: Number of complex zeros; real roots; complex roots; random algebraic polynomials; Jacobian of transformation.

1. INTRODUCTION

Let

$$P_n(z) = \sum_{j=0}^{n-1} \eta_j z^j \tag{1.1}$$

where $\eta_j = a_j + ib_j$ and a_j and b_j are independent identical normally distributed random variables with mean μ and variance σ^2. For z real and n sufficiently large there are many asymptotic estimates for the expected number of real zeros of $P_n(z)$. These estimates are mainly limited to the case of real coefficients η_j, i.e., $b_j = 0$, $j = 0 \cdots n-1$, and are reviewed in

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Bharucha-Reid and Sambandham.[3] More recent results and geometric study of random polynomials are presented in Edelman and Kostlan.[6] Wilkins[10] gives an important asymptotic value for the expected number of real zeros of such a polynomial which contains the smallest error term.

For complex coefficients, Dunnage[4, 5] gives some estimates for the number of real zeros. However, not until very recently have comprehensive properties of complex zeros of $P_n(z)$, $z \in \mathbb{C}$ been studied. Let $\nu_\mu^n(\Phi)$ be the number of complex points $z \in \Phi$, for which $P_n(z)=0$. Shepp and Vanderbei,[9] assuming real and standard normal distributed coefficients for $P_n(z)$, obtained the value for the density function $h^n(z)$ such that

$$E\nu_0^n(\Phi)=\int_\Phi h^n(z)\,dz$$

Their result has recently been generalized by Ibragimov and Zeitouni[8] to a wider class of distribution of the coefficients. However in their work, just as in Shepp and Vanderbei,[9] the distributions of the coefficients are assumed to be symmetric with respect to zero. A major part of the recent work of Farahmand[7] includes the above results and a comprehensive review of relevant work.

In this paper we consider the effect of nonzero mean μ and a general variance σ^2. Let us assume that Φ is a manifold in the complex plane, $\Phi \subset \mathbb{C}$. Suppose that Φ is a compact and the boundary of Φ does not contain points for which $P_n(z)=0$. Assume the origin, $z=0$, does not belong to Φ.

Denote

$$X_1 \equiv X_1(r,\theta)=\sum_{j=0}^{n-1} r^j(a_j\cos j\theta - b_j \sin j\theta)$$

$$X_2 \equiv X_2(r,\theta)=\sum_{j=0}^{n-1} r^j(a_j\sin j\theta + b_j \cos j\theta)$$

as the real and imaginary parts of $P_n(z)$ and $\mathbf{X}=(X_1, X_2)^\top$. Then for $z=r\exp(\imath\theta)$ the Jacobian of the (random) transformation $(r,\theta)\to(X_1,X_2)$ is

$$\nabla\mathbf{X}=\begin{bmatrix}\partial X_1/\partial r & \partial X_2/\partial r\\ \partial X_1/\partial\theta & \partial X_2/\partial\theta\end{bmatrix}$$

Suppose now that there are no points in Φ for which both equalities $P_n(z)=0$ and $\det \nabla \mathbf{X} = \mathbf{0}$ take place. Then, according to Adler's theorem[1] [Thm. 5.1.1, p. 95] and its corollary [Adler,[1] p. 97], the following equalities hold:

$$Ev_\mu^n(\Phi) = \int_\Phi h^n(r,\theta)\,dr\,d\theta$$

$$h^n(r,\theta) = E\{|\det \nabla \mathbf{X}| \mid \mathbf{X}(z) = \mathbf{0}\}\ p(0,0) \tag{1.2}$$

where $p(x_1, x_2)$ denotes the density of the random vector $\mathbf{X}(z) \equiv \mathbf{X}$. The following theorem is proven in the Section 2:

Theorem 1. Provided all the conditions mentioned earlier are satisfied, then for all n the density function $h^n(z)$ is

$$\begin{aligned} h^n(z) = {} & \frac{1}{2\pi\sigma^2 |z| \sum_{j=0}^{n-1} |z|^{2j}} \exp\left(-\frac{\mu^2}{\sigma^2} \frac{|\sum_{j=0}^{n-1} |z|^j|^2}{\sum_{j=0}^{n-1} |z|^{2j}}\right) \\ & \times \left\{\sigma^2 \left(\sum_{j=1}^{n-1} j^2 |z|^{2j} - \frac{(\sum_{j=0}^{n-1} |z|^{2j})^2}{\sum_{j=0}^{n-1} |z|^{2j}}\right)\right. \\ & \left. + \mu^2 \left|\sum_{j=1}^{n-1} jz^j - \left(\sum_{j=1}^{n-1} j\,|z|^{2j}\right) \frac{\sum_{j=0}^{n-1} z^j}{\sum_{j=0}^{n-1} |z|^{2j}}\right|^2\right\} \end{aligned} \tag{1.3}$$

It is of special interest to study the behavior of $h^n(z)$ for n large. Using Theorem 1, in the Section 3 we obtain,

$$\begin{aligned} \lim_{n\to\infty} h^n(z) = {} & \frac{1}{\pi\sigma^2 |z|} \times \exp\left(-\frac{\mu^2}{\sigma^2} \frac{||z|^2-1|}{|z-1|^2}\right) \\ & \times \left\{\frac{\sigma^2 \bar{h}(|z|)}{(|z|^2-1)^2} + \frac{\mu^2}{||z|^2-1|^3} \left|\frac{|z|^2-z}{(z-1)^2}\right|^2\right\} \end{aligned} \tag{1.4}$$

where

$$\bar{h}(|z|) = \begin{cases} |z| + |z|^3 - |z|^4 & \text{if } \ |z|<1, \\ |z|^2 & \text{if } \ |z|>1 \end{cases}$$

2. DENSITY OF COMPLEX ZEROS

In order to be able to use (1.2), first note that for values of X_1 and X_2

$$\begin{aligned}\det \nabla \mathbf{X} &= \frac{1}{r} \sum_{j,k=1}^{n-1} jkr^{j+k}(a_j a_k + b_j b_k) \cos(j-k)\,\theta \\ &\quad + \frac{2}{r} \sum_{j,k=1}^{n-1} jkr^{j+k} a_j b_k \sin(j-k)\,\theta \\ &= \frac{1}{r} \left(\sum_{j=1}^{n-1} jr^j (a_j \cos j\theta - b_j \sin j\theta) \right)^2 \\ &\quad + \frac{1}{r} \left(\sum_{j=1}^{n-1} jr^j (a_j \sin j\theta + b_j \cos j\theta) \right)^2 \geqslant 0 \end{aligned} \tag{2.1}$$

Thus the evaluation of h^n leads to the computation of the expected value of a quadratic form $\det \nabla \mathbf{X}$ of i.i.d. random variables, conditioned on two linear combinations. We define

$$\Pi_{\mathbf{ab}} = E\{(\mathbf{a} - E(\mathbf{a}))(\mathbf{b} - E(\mathbf{b}))^{\top}\}$$

a generalized (unconditional) covariance matrix of 2 vectors **a** and **b**, and

$$\Pi_{\mathbf{ab},\mathbf{X}} = \Pi_{\mathbf{ab}} - \Pi_{\mathbf{aX}} \Pi_{\mathbf{XX}}^{-1} \Pi_{\mathbf{Xb}}$$

Based on the assumption that all the scalar random variables involved are independent and normally distributed, by standard multivariate analysis [for example, see Anderson[2]], we have

$$\operatorname{cov}(\mathbf{a}, \mathbf{b} \mid \mathbf{X} = \mathbf{0}) = \begin{bmatrix} \Pi_{\mathbf{aa},\mathbf{X}} & \Pi_{\mathbf{ab},\mathbf{X}} \\ \Pi_{\mathbf{ba},\mathbf{X}} & \Pi_{\mathbf{bb},\mathbf{X}} \end{bmatrix} \tag{2.2}$$

and

$$E(\mathbf{a} \mid \mathbf{X} = \mathbf{0}) = E(\mathbf{a}) - \Pi_{\mathbf{aX}} \Pi_{\mathbf{XX}}^{-1} E(\mathbf{X})^{\top} \tag{2.3}$$

The expression for $E(\mathbf{b} \mid \mathbf{X} = 0)$ is analogous. Now direct computation leads to the equalities

$$\det \Pi_{\mathbf{XX}} = \left(\sigma^2 \sum_{j=0}^{n-1} r^{2j} \right)^2 \tag{2.4}$$

and

$$\Pi_{\mathbf{XX}}^{-1} = \frac{1}{\det \Pi_{\mathbf{XX}}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

According to (2.4), $\Pi_{\mathbf{XX}}^{-1}$ exists everywhere except the point $z = 0$.

Expanding our definitions, we obtain $\Pi_{\mathbf{aa},\mathbf{X}}$, $\Pi_{\mathbf{ab},\mathbf{X}}$, $\Pi_{\mathbf{ba},\mathbf{X}}$ and $\Pi_{\mathbf{bb},\mathbf{X}}$ as follows. Since $\Pi_{\mathbf{aa}} = \Pi_{\mathbf{bb}} = \sigma^2 \mathbf{I}$ and $\Pi_{\mathbf{ab}} = \mathbf{0}$, simple algebra leads to

$$\Pi_{\mathbf{aa},\mathbf{X}} = \Pi_{\mathbf{bb},\mathbf{X}} = \sigma^2 \mathbf{I} - \frac{\sigma^4}{\sqrt{\det \Pi_{\mathbf{XX}}}} (r^{j+k} \cos(j-k)\, \theta)_{n \times n} \tag{2.5}$$

$$\Pi_{\mathbf{ab},\mathbf{X}} = -\frac{\sigma^4}{\sqrt{\det \Pi_{\mathbf{XX}}}} (r^{j+k} \sin(j-k)\, \theta)_{n \times n} \tag{2.6}$$

$$E(\mathbf{a} \mid \mathbf{X} = \mathbf{0}) = \boldsymbol{\mu} - \frac{\mu \sigma^2}{\sqrt{\det \Pi_{\mathbf{XX}}}} \left(r^j \sum_{k=0}^{n-1} r^k (\cos(j-k)\, \theta + \sin(j-k)\, \theta) \right)_{j=0}^{n-1} \tag{2.7}$$

and

$$E(\mathbf{b} \mid \mathbf{X} = \mathbf{0}) = \boldsymbol{\mu} + \frac{\mu \sigma^2}{\sqrt{\det \Pi_{\mathbf{XX}}}} \left(r^j \sum_{k=0}^{n-1} r^k (\sin(j-k)\, \theta - \cos(j-k)\, \theta) \right)_{j=0}^{n-1} \tag{2.8}$$

where $\boldsymbol{\mu} = (\mu, ..., \mu)^{\mathrm{T}}$. Having obtained formulae (2.5)–(2.8), we can now derive $E(a_j b_k \mid \mathbf{X} = \mathbf{0})$ and $E(a_j a_k \mid \mathbf{X} = \mathbf{0})$. Further we use these expressions in computing the value of $E\{\det \nabla \mathbf{X} \mid \mathbf{X} = \mathbf{0}\}$. After all the necessary simplifications, as used (2.1), $\det \nabla \mathbf{X}$ is always nonnegative; $E\{\det \nabla \mathbf{X} \mid \mathbf{X} = 0\}$ looks as follows:

$$E\{\det \nabla \mathbf{X} \mid \mathbf{X} = \mathbf{0}\} = \frac{2\sigma^2}{r} \sum_{j=1}^{n-1} j^2 r^{2j} - \frac{2\sigma^2}{r \sum_{j=0}^{n-1} r^{2j}} \sum_{j,k=1}^{n-1} jk r^{2(j+k)} + I_1 + I_2 + I_3 \tag{2.9}$$

where

$$I_1 = \frac{2\mu^2}{r} \left\{ \left(\sum_{j=1}^{n-1} j r^j \cos j\theta \right)^2 + \left(\sum_{j=1}^{n-1} j r^j \sin j\theta \right)^2 \right\} \tag{2.10}$$

$$I_2 = \frac{2\mu^2}{r \sum_{j=0}^{n-1} r^{2j}} \left\{ -2 \sum_{j=0}^{n-1} r^j \cos j\theta \sum_{j=1}^{n-1} j r^{2j} \sum_{j=1}^{n-1} j r^j \cos j\theta \right.$$
$$\left. -2 \sum_{j=0}^{n-1} r^j \sin j\theta \sum_{j=1}^{n-1} j r^{2j} \sum_{j=1}^{n-1} j r^j \sin j\theta \right\} \tag{2.11}$$

and

$$I_3 = \frac{2\mu^2}{r(\sum_{j=0}^{n-1} r^{2j})^2} \left(\sum_{j=1}^{n-1} jr^{2j}\right)^2 \left\{\left(\sum_{j=0}^{n-1} r^j \cos j\theta\right)^2 + \left(\sum_{j=0}^{n-1} r^j \sin j\theta\right)^2\right\} \tag{2.12}$$

After regrouping the terms in I_1, I_2, I_3, with a little algebra we can write

$$\begin{aligned} I_1 + I_2 + I_3 &= \frac{2\mu^2}{r} \left\{\left(\sum_{j=1}^{n-1} jr^j \cos j\theta - \frac{\sum_{j=1}^{n-1} jr^{2j}}{\sum_{j=0}^{n-1} r^{2j}} \sum_{j=0}^{n-1} r^j \cos j\theta\right)^2\right. \\ &\quad \left. + \left(\sum_{j=1}^{n-1} jr^j \sin j\theta - \frac{\sum_{j=1}^{n-1} jr^{2j}}{\sum_{j=0}^{n-1} r^{2j}} \sum_{j=0}^{n-1} r^j \sin j\theta\right)^2\right\} \\ &= \frac{2\mu^2}{|z|} \left|\sum_{j=1}^{n-1} jz^j - \frac{\sum_{j=1}^{n-1} j\,|z|^{2j}}{\sum_{j=0}^{n-1} |z|^{2j}} \sum_{j=0}^{n-1} z^j\right|^2 \end{aligned} \tag{2.13}$$

Now since the joint density of two random normal variables X_1 and X_2 is

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2 \sum_{j=0}^{n-1} r^{2j}} \exp\left(-\frac{(x_1 - E(x_1))^2 + (x_2 - E(x_2))^2}{2\sigma^2 \sum_{j=0}^{n-1} r^{2j}}\right)$$

we can show that for our case

$$p(0, 0) = \frac{1}{2\pi\sigma^2 \, |z| \sum_{j=0}^{n-1} |z|^{2j}} \exp\left(-\frac{\mu^2}{\sigma^2} \frac{|\sum_{j=0}^{n-1} |z|^j|^2}{\sum_{j=0}^{n-1} |z|^{2j}}\right) \tag{2.14}$$

Therefore, from (2.9), (2.13), and (2.14), using the z variable instead of r and θ, we obtain the final formula (1.3).

3. ASYMPTOTICS

In this section we study the asymptotic value of $h^n(z)$ when n is large. Converting the geometric progressions in the final formula (1.3), we can rewrite this formula as follows:

$$h^n(z) = \frac{|z|^2 - 1}{\pi\sigma^2 \, |z|} \times I_1 \times I_2$$

where

$$I_1 = \exp\left(-\frac{\mu^2}{\sigma^2} \left| \frac{z^n - 1}{z - 1} \right|^2 \frac{|z|^2 - 1}{|z|^{2n} - 1} \right),$$

$$I_2 = \frac{\sigma^2}{|z|^{2n} - 1} \left\{ \sum_{j=1}^{n-1} j^2 |z|^{2j} - \frac{(\sum_{j=1}^{n-1} j^2 |z|^{2j})^2}{\sum_{j=0}^{n-1} |z|^{2j}} \right\}$$

$$+ \frac{\mu^2}{|z|^{2n} - 1} \left| \sum_{j=1}^{n-1} jz^j - \sum_{j=1}^{n-1} j |z|^{2j} \frac{\sum_{j=0}^{n-1} z^j}{\sum_{j=0}^{n-1} |z|^{2j}} \right|^2$$

Let us recall the expressions for two internal sums in I_2

$$\sum_{j=1}^{n-1} j |z|^{2j} = \frac{|z|^2}{(1 - |z|^2)^2} - \frac{n |z|^{2n}}{1 - |z|^2} - \frac{|z|^{2n+2}}{(1 - |z|^2)^2},$$

$$\sum_{j=1}^{n-1} j^2 |z|^{2j} = \frac{|z|(1 + |z|^2)}{(1 - |z|^2)^3} - \left\{ \frac{n^2 |z|^{2n}}{1 - |z|^2} + \frac{2n |z|^{2n+2}}{(1 - |z|^2)^2} + \frac{|z|^{2n+2} (|z|^2 + 1)}{(1 - |z|^2)^3} \right\}$$

It is clearly seen that when n tends to infinity we obtain two generally different asymptotic expressions for $h^n(z)$ depending on whether $|z| < 1$ or $|z| > 1$. Indeed,

$$\lim_{n \to \infty} I_1 = \begin{cases} \exp\left(-\dfrac{\mu^2}{\sigma^2} \dfrac{1 - |z|^2}{|z - 1|^2} \right) & \text{if} \quad |z| < 1, \\ \exp\left(-\dfrac{\mu^2}{\sigma^2} \dfrac{|z|^2 - 1}{|z - 1|^2} \right) & \text{if} \quad |z| > 1; \end{cases}$$

$$\lim_{n \to \infty} I_2 = \begin{cases} -\sigma^2 \left(\dfrac{|z| + |z|^3 - |z|^4}{(1 - |z|^2)^3} \right) - \mu^2 \left| \dfrac{|z|^2 - z}{(z - 1)^2 (|z|^2 - 1)} \right|^2 & \text{if} \quad |z| < 1, \\ \dfrac{\sigma^2 |z|^2}{(|z|^2 - 1)^3} + \mu^2 \left| \dfrac{|z|^2 - z}{(z - 1)^2 (|z|^2 - 1)} \right|^2 & \text{if} \quad |z| > 1 \end{cases}$$

Introducing a new notation

$$\bar{h}(|z|) = \begin{cases} |z| + |z|^3 - |z|^4 & \text{if} \quad |z| < 1, \\ |z|^2 & \text{if} \quad |z| > 1 \end{cases}$$

we can easily obtain our asymptotic expression as a single equality given in (1.4).

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