

## **Complex Zeros of Algebraic Polynomial with Non-Zero Mean Random Coefficients<sup>1</sup>**

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In this paper we obtain a formula for the average density of the distribution of complex zeros of an algebraic polynomial with random coefficients. The coefficients are assumed independent identical normally distributed random variables with mean  $\mu$  and variance  $\sigma^2$ . The value of the average density for the case of  $\mu = 0$  and  $\sigma^2 = 1$  was obtained previously. Some limits of the distribution of the complex zeros are provided using the presented formula.

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**KEY WORDS:** Number of complex zeros; real roots; complex roots; random algebraic polynomials; Jacobian of transformation.

### **1. INTRODUCTION**

Let

$$P_n(z) = \sum_{j=0}^{n-1} \eta_j z^j \quad (1.1)$$

where  $\eta_j = a_j + ib_j$  and  $a_j$  and  $b_j$  are independent identical normally distributed random variables with mean  $\mu$  and variance  $\sigma^2$ . For  $z$  real and  $n$  sufficiently large there are many asymptotic estimates for the expected number of real zeros of  $P_n(z)$ . These estimates are mainly limited to the case of real coefficients  $\eta_j$ , i.e.,  $b_j = 0$ ,  $j = 0 \cdots n-1$ , and are reviewed in

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Bharucha-Reid and Sambandham.<sup>(3)</sup> More recent results and geometric study of random polynomials are presented in Edelman and Kostlan.<sup>(6)</sup> Wilkins<sup>(10)</sup> gives an important asymptotic value for the expected number of real zeros of such a polynomial which contains the smallest error term.

For complex coefficients, Dunnage<sup>(4,5)</sup> gives some estimates for the number of real zeros. However, not until very recently have comprehensive properties of complex zeros of  $P_n(z)$ ,  $z \in \mathbb{C}$  been studied. Let  $\nu_\mu^n(\Phi)$  be the number of complex points  $z \in \Phi$ , for which  $P_n(z) = 0$ . Shepp and Vanderbei,<sup>(9)</sup> assuming real and standard normal distributed coefficients for  $P_n(z)$ , obtained the value for the density function  $h^n(z)$  such that

$$E\nu_0^n(\Phi) = \int_{\Phi} h^n(z) dz$$

Their result has recently been generalized by Ibragimov and Zeitouni<sup>(8)</sup> to a wider class of distribution of the coefficients. However in their work, just as in Shepp and Vanderbei,<sup>(9)</sup> the distributions of the coefficients are assumed to be symmetric with respect to zero. A major part of the recent work of Farahmand<sup>(7)</sup> includes the above results and a comprehensive review of relevant work.

In this paper we consider the effect of nonzero mean  $\mu$  and a general variance  $\sigma^2$ . Let us assume that  $\Phi$  is a manifold in the complex plane,  $\Phi \subset \mathbb{C}$ . Suppose that  $\Phi$  is a compact and the boundary of  $\Phi$  does not contain points for which  $P_n(z) = 0$ . Assume the origin,  $z = 0$ , does not belong to  $\Phi$ .

Denote

$$X_1 \equiv X_1(r, \theta) = \sum_{j=0}^{n-1} r^j (a_j \cos j\theta - b_j \sin j\theta)$$

$$X_2 \equiv X_2(r, \theta) = \sum_{j=0}^{n-1} r^j (a_j \sin j\theta + b_j \cos j\theta)$$

as the real and imaginary parts of  $P_n(z)$  and  $\mathbf{X} = (X_1, X_2)^T$ . Then for  $z = r \exp(i\theta)$  the Jacobian of the (random) transformation  $(r, \theta) \rightarrow (X_1, X_2)$  is

$$\nabla \mathbf{X} = \begin{bmatrix} \partial X_1 / \partial r & \partial X_2 / \partial r \\ \partial X_1 / \partial \theta & \partial X_2 / \partial \theta \end{bmatrix}$$

Suppose now that there are no points in  $\Phi$  for which both equalities  $P_n(z) = 0$  and  $\det \nabla \mathbf{X} = \mathbf{0}$  take place. Then, according to Adler's theorem<sup>(1)</sup> [Thm. 5.1.1, p. 95] and its corollary [Adler,<sup>(1)</sup> p. 97], the following equalities hold:

$$Ev_\mu^n(\Phi) = \int_{\Phi} h^n(r, \theta) dr d\theta$$

$$h^n(r, \theta) = E\{|\det \nabla \mathbf{X}| | \mathbf{X}(z) = \mathbf{0}\} p(0, 0) \tag{1.2}$$

where  $p(x_1, x_2)$  denotes the density of the random vector  $\mathbf{X}(z) \equiv \mathbf{X}$ . The following theorem is proven in the Section 2:

**Theorem 1.** Provided all the conditions mentioned earlier are satisfied, then for all  $n$  the density function  $h^n(z)$  is

$$h^n(z) = \frac{1}{2\pi\sigma^2 |z| \sum_{j=0}^{n-1} |z|^{2j}} \exp\left(-\frac{\mu^2 |\sum_{j=0}^{n-1} |z|^j|^2}{\sigma^2 \sum_{j=0}^{n-1} |z|^{2j}}\right)$$

$$\times \left\{ \sigma^2 \left( \sum_{j=1}^{n-1} j^2 |z|^{2j} - \frac{(\sum_{j=0}^{n-1} |z|^{2j})^2}{\sum_{j=0}^{n-1} |z|^{2j}} \right) \right.$$

$$\left. + \mu^2 \left| \sum_{j=1}^{n-1} jz^j - \left( \sum_{j=1}^{n-1} j |z|^{2j} \right) \frac{\sum_{j=0}^{n-1} z^j}{\sum_{j=0}^{n-1} |z|^{2j}} \right|^2 \right\} \tag{1.3}$$

It is of special interest to study the behavior of  $h^n(z)$  for  $n$  large. Using Theorem 1, in the Section 3 we obtain,

$$\lim_{n \rightarrow \infty} h^n(z) = \frac{1}{\pi\sigma^2 |z|} \times \exp\left(-\frac{\mu^2 ||z|^2 - 1|}{\sigma^2 |z - 1|^2}\right)$$

$$\times \left\{ \frac{\sigma^2 \bar{h}(|z|)}{(|z|^2 - 1)^2} + \frac{\mu^2}{||z|^2 - 1|^3} \left| \frac{|z|^2 - z}{z - 1} \right|^2 \right\} \tag{1.4}$$

where

$$\bar{h}(|z|) = \begin{cases} |z| + |z|^3 - |z|^4 & \text{if } |z| < 1, \\ |z|^2 & \text{if } |z| > 1 \end{cases}$$

## 2. DENSITY OF COMPLEX ZEROS

In order to be able to use (1.2), first note that for values of  $X_1$  and  $X_2$

$$\begin{aligned} \det \nabla \mathbf{X} &= \frac{1}{r} \sum_{j,k=1}^{n-1} jkr^{j+k}(a_j a_k + b_j b_k) \cos(j-k)\theta \\ &\quad + \frac{2}{r} \sum_{j,k=1}^{n-1} jkr^{j+k} a_j b_k \sin(j-k)\theta \\ &= \frac{1}{r} \left( \sum_{j=1}^{n-1} jr^j (a_j \cos j\theta - b_j \sin j\theta) \right)^2 \\ &\quad + \frac{1}{r} \left( \sum_{j=1}^{n-1} jr^j (a_j \sin j\theta + b_j \cos j\theta) \right)^2 \geq 0 \end{aligned} \quad (2.1)$$

Thus the evaluation of  $h^n$  leads to the computation of the expected value of a quadratic form  $\det \nabla \mathbf{X}$  of i.i.d. random variables, conditioned on two linear combinations. We define

$$\Pi_{\mathbf{ab}} = E\{(\mathbf{a} - E(\mathbf{a}))(\mathbf{b} - E(\mathbf{b}))^\top\}$$

a generalized (unconditional) covariance matrix of 2 vectors  $\mathbf{a}$  and  $\mathbf{b}$ , and

$$\Pi_{\mathbf{ab}, \mathbf{X}} = \Pi_{\mathbf{ab}} - \Pi_{\mathbf{aX}} \Pi_{\mathbf{XX}}^{-1} \Pi_{\mathbf{Xb}}$$

Based on the assumption that all the scalar random variables involved are independent and normally distributed, by standard multivariate analysis [for example, see Anderson<sup>(2)</sup>], we have

$$\text{cov}(\mathbf{a}, \mathbf{b} | \mathbf{X} = \mathbf{0}) = \begin{bmatrix} \Pi_{\mathbf{aa}, \mathbf{X}} & \Pi_{\mathbf{ab}, \mathbf{X}} \\ \Pi_{\mathbf{ba}, \mathbf{X}} & \Pi_{\mathbf{bb}, \mathbf{X}} \end{bmatrix} \quad (2.2)$$

and

$$E(\mathbf{a} | \mathbf{X} = \mathbf{0}) = E(\mathbf{a}) - \Pi_{\mathbf{aX}} \Pi_{\mathbf{XX}}^{-1} E(\mathbf{X})^\top \quad (2.3)$$

The expression for  $E(\mathbf{b} | \mathbf{X} = \mathbf{0})$  is analogous. Now direct computation leads to the equalities

$$\det \Pi_{\mathbf{XX}} = \left( \sigma^2 \sum_{j=0}^{n-1} r^{2j} \right)^2 \quad (2.4)$$

and

$$\Pi_{\mathbf{XX}}^{-1} = \frac{1}{\det \Pi_{\mathbf{XX}}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

According to (2.4),  $\Pi_{\mathbf{XX}}^{-1}$  exists everywhere except the point  $z = 0$ .

Expanding our definitions, we obtain  $\Pi_{\mathbf{aa}, \mathbf{X}}$ ,  $\Pi_{\mathbf{ab}, \mathbf{X}}$ ,  $\Pi_{\mathbf{ba}, \mathbf{X}}$  and  $\Pi_{\mathbf{bb}, \mathbf{X}}$  as follows. Since  $\Pi_{\mathbf{aa}} = \Pi_{\mathbf{bb}} = \sigma^2 \mathbf{I}$  and  $\Pi_{\mathbf{ab}} = \mathbf{0}$ , simple algebra leads to

$$\Pi_{\mathbf{aa}, \mathbf{X}} = \Pi_{\mathbf{bb}, \mathbf{X}} = \sigma^2 \mathbf{I} - \frac{\sigma^4}{\sqrt{\det \Pi_{\mathbf{XX}}}} (r^{j+k} \cos(j-k) \theta)_{n \times n} \quad (2.5)$$

$$\Pi_{\mathbf{ab}, \mathbf{X}} = -\frac{\sigma^4}{\sqrt{\det \Pi_{\mathbf{XX}}}} (r^{j+k} \sin(j-k) \theta)_{n \times n} \quad (2.6)$$

$$E(\mathbf{a} | \mathbf{X} = \mathbf{0}) = \boldsymbol{\mu} - \frac{\mu \sigma^2}{\sqrt{\det \Pi_{\mathbf{XX}}}} \left( r^j \sum_{k=0}^{n-1} r^k (\cos(j-k) \theta + \sin(j-k) \theta) \right)_{j=0}^{n-1} \quad (2.7)$$

and

$$E(\mathbf{b} | \mathbf{X} = \mathbf{0}) = \boldsymbol{\mu} + \frac{\mu \sigma^2}{\sqrt{\det \Pi_{\mathbf{XX}}}} \left( r^j \sum_{k=0}^{n-1} r^k (\sin(j-k) \theta - \cos(j-k) \theta) \right)_{j=0}^{n-1} \quad (2.8)$$

where  $\boldsymbol{\mu} = (\mu, \dots, \mu)^\top$ . Having obtained formulae (2.5)–(2.8), we can now derive  $E(a_j b_k | \mathbf{X} = \mathbf{0})$  and  $E(a_j a_k | \mathbf{X} = \mathbf{0})$ . Further we use these expressions in computing the value of  $E\{\det \nabla \mathbf{X} | \mathbf{X} = \mathbf{0}\}$ . After all the necessary simplifications, as used (2.1),  $\det \nabla \mathbf{X}$  is always nonnegative;  $E\{\det \nabla \mathbf{X} | \mathbf{X} = \mathbf{0}\}$  looks as follows:

$$E\{\det \nabla \mathbf{X} | \mathbf{X} = \mathbf{0}\} = \frac{2\sigma^2}{r} \sum_{j=1}^{n-1} j^2 r^{2j} - \frac{2\sigma^2}{r \sum_{j=0}^{n-1} r^{2j}} \sum_{j,k=1}^{n-1} jkr^{2(j+k)} + I_1 + I_2 + I_3 \quad (2.9)$$

where

$$I_1 = \frac{2\mu^2}{r} \left\{ \left( \sum_{j=1}^{n-1} jr^j \cos j\theta \right)^2 + \left( \sum_{j=1}^{n-1} jr^j \sin j\theta \right)^2 \right\} \quad (2.10)$$

$$I_2 = \frac{2\mu^2}{r \sum_{j=0}^{n-1} r^{2j}} \left\{ -2 \sum_{j=0}^{n-1} r^j \cos j\theta \sum_{j=1}^{n-1} jr^{2j} \sum_{j=1}^{n-1} jr^j \cos j\theta \right. \\ \left. - 2 \sum_{j=0}^{n-1} r^j \sin j\theta \sum_{j=1}^{n-1} jr^{2j} \sum_{j=1}^{n-1} jr^j \sin j\theta \right\} \quad (2.11)$$

and

$$I_3 = \frac{2\mu^2}{r(\sum_{j=0}^{n-1} r^{2j})^2} \left( \sum_{j=1}^{n-1} jr^{2j} \right)^2 \left\{ \left( \sum_{j=0}^{n-1} r^j \cos j\theta \right)^2 + \left( \sum_{j=0}^{n-1} r^j \sin j\theta \right)^2 \right\} \quad (2.12)$$

After regrouping the terms in  $I_1$ ,  $I_2$ ,  $I_3$ , with a little algebra we can write

$$\begin{aligned} I_1 + I_2 + I_3 &= \frac{2\mu^2}{r} \left\{ \left( \sum_{j=1}^{n-1} jr^j \cos j\theta - \frac{\sum_{j=1}^{n-1} jr^{2j}}{\sum_{j=0}^{n-1} r^{2j}} \sum_{j=0}^{n-1} r^j \cos j\theta \right)^2 \right. \\ &\quad \left. + \left( \sum_{j=1}^{n-1} jr^j \sin j\theta - \frac{\sum_{j=1}^{n-1} jr^{2j}}{\sum_{j=0}^{n-1} r^{2j}} \sum_{j=0}^{n-1} r^j \sin j\theta \right)^2 \right\} \\ &= \frac{2\mu^2}{|z|} \left| \sum_{j=1}^{n-1} jz^j - \frac{\sum_{j=1}^{n-1} j|z|^{2j}}{\sum_{j=0}^{n-1} |z|^{2j}} \sum_{j=0}^{n-1} z^j \right|^2 \end{aligned} \quad (2.13)$$

Now since the joint density of two random normal variables  $X_1$  and  $X_2$  is

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2 \sum_{j=0}^{n-1} r^{2j}} \exp\left(-\frac{(x_1 - E(x_1))^2 + (x_2 - E(x_2))^2}{2\sigma^2 \sum_{j=0}^{n-1} r^{2j}}\right)$$

we can show that for our case

$$p(0, 0) = \frac{1}{2\pi\sigma^2 |z| \sum_{j=0}^{n-1} |z|^{2j}} \exp\left(-\frac{\mu^2 \sum_{j=0}^{n-1} |z|^j}{\sigma^2 \sum_{j=0}^{n-1} |z|^{2j}}\right) \quad (2.14)$$

Therefore, from (2.9), (2.13), and (2.14), using the  $z$  variable instead of  $r$  and  $\theta$ , we obtain the final formula (1.3).

### 3. ASYMPTOTICS

In this section we study the asymptotic value of  $h^n(z)$  when  $n$  is large. Converting the geometric progressions in the final formula (1.3), we can rewrite this formula as follows:

$$h^n(z) = \frac{|z|^2 - 1}{\pi\sigma^2 |z|} \times I_1 \times I_2$$

where

$$I_1 = \exp\left(-\frac{\mu^2}{\sigma^2} \left| \frac{z^n - 1}{z - 1} \right|^2 \frac{|z|^2 - 1}{|z|^{2n} - 1}\right),$$

$$I_2 = \frac{\sigma^2}{|z|^{2n} - 1} \left\{ \sum_{j=1}^{n-1} j^2 |z|^{2j} - \frac{(\sum_{j=1}^{n-1} j^2 |z|^{2j})^2}{\sum_{j=0}^{n-1} |z|^{2j}} \right\}$$

$$+ \frac{\mu^2}{|z|^{2n} - 1} \left| \sum_{j=1}^{n-1} j z^j - \sum_{j=1}^{n-1} j |z|^{2j} \frac{\sum_{j=0}^{n-1} z^j}{\sum_{j=0}^{n-1} |z|^{2j}} \right|^2$$

Let us recall the expressions for two internal sums in  $I_2$

$$\sum_{j=1}^{n-1} j |z|^{2j} = \frac{|z|^2}{(1 - |z|^2)^2} - \frac{n |z|^{2n}}{1 - |z|^2} - \frac{|z|^{2n+2}}{(1 - |z|^2)^2},$$

$$\sum_{j=1}^{n-1} j^2 |z|^{2j} = \frac{|z|(1 + |z|^2)}{(1 - |z|^2)^3} - \left\{ \frac{n^2 |z|^{2n}}{1 - |z|^2} + \frac{2n |z|^{2n+2}}{(1 - |z|^2)^2} + \frac{|z|^{2n+2} (|z|^2 + 1)}{(1 - |z|^2)^3} \right\}$$

It is clearly seen that when  $n$  tends to infinity we obtain two generally different asymptotic expressions for  $h^n(z)$  depending on whether  $|z| < 1$  or  $|z| > 1$ . Indeed,

$$\lim_{n \rightarrow \infty} I_1 = \begin{cases} \exp\left(-\frac{\mu^2}{\sigma^2} \frac{1 - |z|^2}{|z - 1|^2}\right) & \text{if } |z| < 1, \\ \exp\left(-\frac{\mu^2}{\sigma^2} \frac{|z|^2 - 1}{|z - 1|^2}\right) & \text{if } |z| > 1; \end{cases}$$

$$\lim_{n \rightarrow \infty} I_2 = \begin{cases} -\sigma^2 \left( \frac{|z| + |z|^3 - |z|^4}{(1 - |z|^2)^3} \right) - \mu^2 \left| \frac{|z|^2 - z}{(z - 1)^2 (|z|^2 - 1)} \right|^2 & \text{if } |z| < 1, \\ \frac{\sigma^2 |z|^2}{(|z|^2 - 1)^3} + \mu^2 \left| \frac{|z|^2 - z}{(z - 1)^2 (|z|^2 - 1)} \right|^2 & \text{if } |z| > 1 \end{cases}$$

Introducing a new notation

$$\bar{h}(|z|) = \begin{cases} |z| + |z|^3 - |z|^4 & \text{if } |z| < 1, \\ |z|^2 & \text{if } |z| > 1 \end{cases}$$

we can easily obtain our asymptotic expression as a single equality given in (1.4).

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