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Markovian Connection, Curvature and Weitzenböck Formula on Riemannian Path Spaces

Shizan Fang

Laboratoire de Topologie, Département de Mathématiques, Université de Bourgogne, 9, avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France

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We shall consider on a Riemannian path space $\mathbf{P}_{m_o}(M)$ the Cruzeiro–Malliavin's Markovian connection. The Laplace operator will be defined as the divergence of the gradient. We shall compute explicitly the associated curvature tensor. A Weitzenböck formula will be established. To this end, we shall introduce an "inner product" between the tangent processes and simple vector fields.

Let M be a compact Riemannian manifold. Denote by Δ the Laplace–Beltrami operator on M. The classical Bochner–Weitzenböck formula reads

$$[\Delta, \nabla]\, f = \mathrm{Ric}^M(\nabla f). \tag{0.1}$$

In the above formula, the vector Laplace operator Δ is defined with respect to the Levi–Civita connection. If we replace the Levi–Civita connection by another one $\tilde{\nabla}$ compatible with the metric, which satisfies Driver's TSS condition [Dr1], then the formula (0.1) becomes (see [E-LJ-L])

$$\tilde{\Delta}(\nabla f) - \nabla \Delta f = \widehat{\mathrm{Ric}}^M(\nabla f), \tag{0.2}$$

where $\widehat{\mathrm{Ric}}_m^M\, v = \widetilde{\mathrm{Ric}}_m^M\, v + \sum_{\alpha=1}^d (\tilde{\nabla}_{e_\alpha} T)(e_\alpha, v)$ and $\{e_1, ..., e_d\}$ is an orthonormal basis of the tangent space $T_m M$.

The purpose of this work is to establish the above type Bochner–Weitzenböck formula on the Riemannian path space $\mathbf{P}_{m_o}(M)$. In our situation, two structures co-exist: the differential structure via Malliavin calculus and the structure of Itô filtration. The Levi–Civita connection on the path space can be defined (see [CM1]), but it does not preserve the H-vector fields on $\mathbf{P}_{m_o}(M)$, nor the structure of Itô filtration. Morever the Ricci type curvature is a divergent object. In order to overcome these

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difficulties, the notion of the Markovian connection has been introduced by Cruzeiro and Malliavin [CM1]. This new connection has the advantage to preserve the category of adapted vector fields on $\mathbf{P}_{m_o}(M)$, which is vital in stochastic analysis. The price to pay is that the associated torsion tensor is not free, nor TSS.

The paper is organized as follows. In Section 1, we shall recall some elements of differential calculus on $\mathbf{P}_{m_o}(M)$, especially the derivative of Itô map $r_x(\tau)$ seen as a functional on $\mathbf{P}_{m_o}(M)$. It plays a crucial role. In Section 2, we shall emphasize the importance of the role of tangent processes which is the base of the development of the renormalized differential geometry on $\mathbf{P}_{m_o}(M)$: the Lie bracket of two constant H-vector fields is a tangent process (see [CM1]). In Section 3, we shall take account of the Markovian connection. The explicit and simple expression for curvature tensor will be obtained. The new phenomenon here is that the Ricci curvature does exist at a global level (see Theorem 3.5). In this way, we re-find the expression of Ricci curvature which already appeared in [CM1]. In Section 4, we shall consider the Laplace operator ΔF, which is defined by $\Delta F = \sum_{\alpha=1}^{d} \int_0^1 D^2_{\tau,\alpha} F\, d\tau$. This definition is quite natural: the manifold $\mathbf{P}_{m_o}(M)$ is parallelized and $\mathbf{e}_{\tau,\alpha}(s) = \mathbf{1}_{(\tau<s)}\varepsilon_\alpha$ constitutes the basic system of vector fields. We shall prove that the Laplace operator is the divergence with respect to the Markovian connection of the gradient operator. The torsion tensor of two constant vector fields is a tangent process and its role is particular (see for example the formula (0.2)). In Section 5, we shall introduce the "inner product" between a tangent process and a simple H-vector field, which will enable us to develop a tensorial calculus, without using frame bundles over path space as done in [CM2]. In Section 6, we shall define the vector Laplace operator and in Section 7, we shall establish a Weitzenböck formula on $\mathbf{P}_{m_o}(M)$, following the approach of [DL]. Comparing to (0.2), a first order differential operator will appear, which would be vanished if the torsion were TSS. On a finite dimensional manifold, this kind of difficulty has been handled in [E-LJ-L], using the dual connection. In our situation, it seems difficult to handle with it: the dual connection gives arise of tangent processes. Finally, in Section 8, we shall prove that the first order operator will disappear by taking the expectation in the case of $\mathrm{Ric}^M = 0$. This result is in coherence with that in [CM2].

1. GENERAL FRAMEWORK

Let M be a compact Riemannian manifold (of dimension d). We consider the space of paths

$$\mathbf{P}_{m_o}(M) = \{\gamma : [0,1] \to M \text{ continuous};\ \gamma(0) = m_o\}$$

where $m_o \in M$ is a fixed point. Let $A_1, ..., A_d$ be the canonical horizontal vector fields (relative to the Levi–Civita connection) on the orthonormal frame bundle $O(M)$. The horizontal stochastic flow $r_x(\tau)$ over $O(M)$ is defined by the SDE

$$dr_x(\tau) = \sum_{\alpha=1}^{d} A_\alpha(r_x(\tau)) \circ dx(\tau), \qquad r_x(0) = r_o, \tag{1.1}$$

where $r_o \in O(M)$ is a fixed frame at the point m_o, and $x(\tau)$ denotes the canonical Brownian motion on $\mathbb{R}^d$. Denote by X the classical Wiener space of the trajectories of the $\mathbb{R}^d$-valued Brownian motion. Let π: $O(M) \to M$ be the canonical projection. Denote

$$\gamma_x(\tau) = \pi(r_x(\tau)).$$

Then $\tau \to \gamma_x(\tau)$ is a Brownian motion on M, associated to the Laplace–Beltrami operator $\frac{1}{2} \Delta_M$. The map $X \to \mathbf{P}_{m_o}(M)$ defined by $x \to I(x) = \gamma_x$ is a measurable isomorphism.

Denote

$$\mathbb{H} = \left\{ h : [0, 1] \to \mathbb{R}^d; h(0) = 0, |h|_H^2 = \int_0^1 |\dot{h}(\tau)|^2 \, d\tau < +\infty \right\}.$$

A H-vector field Z on $\mathbf{P}_{m_o}(M)$ is the data of $Z(\gamma, \tau) \in T_{\gamma(\tau)} M$ such that $\mathbb{E}(|z|_H^2) < +\infty$ where

$$z(x, \tau) = r_x(\tau)^{-1} \, Z(\gamma_x, \tau). \tag{1.2}$$

In what follows, we shall consider $z(x, \tau)$ as a vector field on $\mathbf{P}_{m_o}(M)$ according to (1.2). Let $h \in \mathbb{H}$. Denote

$$\Gamma_h(\tau) = \int_0^\tau \Omega_s(h(s), \circ dx(s)), \tag{1.3}$$

where Ω_r is the curvature tensor readed at the frame $r \in O(M)$ and

$$\Omega_s = \Omega_{r_x(s)}. \tag{1.4}$$

Consider the semi-martingale $\xi_h(\tau)$ defined by

$$d\xi_h(\tau) = \Gamma_h(\tau) \circ dx(\tau) + \dot{h}(\tau) \, d\tau. \tag{1.5}$$

Let F be a functional on $\mathbf{P}_{m_o}(M)$. Denote $\tilde{f}(x)=F(\gamma_x)$. The derivation D_h along h of F is defined b

$$(D_h F)(\gamma_x)=\left\{\frac{d}{d\varepsilon}\tilde{F}\left(\int_0^{\cdot} e^{\varepsilon \Gamma_h(s)}\circ dx(s)+\varepsilon\, h\right)\right\}_{\varepsilon=0}. \tag{1.6}$$

The following result is well known (see [DR1, FM, CM1]).

THEOREM 1.1. *Fix* $\tau\in[0,1]$. *Considering* $r_x(\tau)$ *as a functional on* $\mathbf{P}_{m_o}(M)$ *throughout the inverse of the map* I, *we have*

$$\langle\theta, D_h r_x(\tau)\rangle=h(\tau), \qquad \langle\omega, D_h r_x(\tau)\rangle=-\Gamma_h(\tau), \tag{1.7}$$

where (θ,ω) *denotes the canonical parallelism on* $O(M)$.

Let F be a cylindrical function on $\mathbf{P}_{m_o}(M)$ at the level of $O(M)$,

$$F(\gamma_x)=\tilde{f}(r_x(\tau_1),\ldots,r_x(\tau_k))$$

with $\tilde{f}\in\mathscr{C}^\infty(O(M)^k)$, $0<\tau_1<\cdots<\tau_k<1$. Using (1.7), we have

$$(D_h F)(\gamma_x)=\sum_{i=1}^k\sum_{\alpha=1}^d(\partial^i_{A_\alpha}\tilde{f})\,h^\alpha(\tau_i)-\sum_{i=1}^k\partial^i_{\Gamma_h(\tau_i)^*}\tilde{f}, \tag{1.8}$$

where $\Gamma_h(\tau_i)^*$ denotes the fundamental vector field on $O(M)$ associated to $\Gamma_h(\tau_i)$ by

$$E^*(r)=\left\{\frac{d}{d\varepsilon}r\cdot e^{\varepsilon E}\right\}_{\varepsilon=0} \qquad \text{for} \quad E\in so(d) \tag{1.9}$$

and ∂^i denotes the derivative with respect to the ith component. When F is a true cylindrical function on $\mathbf{P}_{m_o}(M)$,

$$F(\gamma)=f(\gamma(\tau_1),\ldots,\gamma(\tau_k)), \tag{1.10}$$

then

$$(D_h F)(\gamma_x)=\sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde{f})\,h^\alpha(\tau_i) \tag{1.11}$$

with $\tilde{f}=f\circ\pi$. This identity justifies the definition given in (1.6). Let ∇F be the gradient of F such that $\langle\nabla F, h\rangle_H=D_h F$.

PROPOSITION 1.2. *Let F be a cylindrical function on $\mathbf{P}_{m_o}(M)$ written in the form* (1.10). *Denote $z_{i,\alpha}(\tau)=(\tau\wedge\tau_i)\,\varepsilon_\alpha$ where $\{\varepsilon_1, ..., \varepsilon_d\}$ is an orthonormal basis of $\mathbb{R}^d$. Then*

$$\nabla F=\sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde{f})\,z_{i,\alpha}. \tag{1.12}$$

Proof. It is enough to remark that $h^\alpha(\tau_i)=\langle h, z_{i,\alpha}\rangle_H$. ∎

Remark 1.3. In [CM1] the methodology of the moving frame has been put at the forefront of notations. In this paper for the convenience of computations, we prefer to lift everything to the frame bundle $O(M)$ of M. This idea has also been used in [CM1] in order to obtain the structural equations on $\mathbf{P}_{m_o}(M)$.

2. TANGENT PROCESSES

The major difficulty on the geometry of the Riemannian path space $\mathbf{P}_{m_o}(M)$ comes from the fact that the Lie bracket operation is not stable on the H-vector fields on $\mathbf{P}_{m_o}(M)$. To overcome this difficulty, the notion of tangent processes has been introduced by Cruzeiro and Malliavin [CM1].

A semi-martingale $d\xi_x(t)=a_x(t)\,dx(t)+b_x(t)\,dt$ on $\mathbb{R}^d$ is called a tangent process if $t\to(a_x(t), b_x(t))$ is an adapted process taking values in $so(d)\times\mathbb{R}^d$ such that

$$\mathbb{E}(e^{c\int_0^1 |b_x(t)|^2\,dt})<+\infty \qquad \text{for some} \quad c>0.$$

Consider now $Z_\xi(t)=r_x(t)\,\xi_x(t)$. Then $Z_\xi(t)$ is a vector field on M along the Brownian curves. We shall consider ξ as a vector field on $\mathbf{P}_{m_o}(M)$ throughout Z_ξ. Similarly as in Section 1, define $\Gamma_\xi(\tau)=\int_0^\tau \Omega_s(\xi(s), \circ dx_s)$. Consider $d\xi^*(t)=d\xi(t)+\Gamma_\xi(t)\circ dx(t)$. Then ξ^* is again a tangent process written in the form

$$d\xi^*(t)=q_x(t)\,dx(t)+\dot{h}(t)\,dt \tag{2.1}$$

with

$$q_x(t)=a_x(t)+\Gamma_\xi(t), \qquad \dot{h}(t)=b_x(t)+\tfrac{1}{2}\,\mathrm{Ric}_t\,\xi(t), \tag{2.2}$$

where $\mathrm{Ric}_t=\mathrm{Ric}^M_{r_x(t)}$ and Ric^M denotes the Ricci tensor on M. Let F be a functional on the Wiener space X. We define

$$(D_{q,h}F)(x)=\left\{\frac{d}{d\varepsilon}F\left(\int_0^\cdot e^{\varepsilon\,q(s)}\,dx(s)+\varepsilon\,h\right)\right\}_{\varepsilon=0}.$$

THEOREM 2.1. *Denote*

$$\beta(\tau)=\langle \theta, D_{q,h} r_x(\tau)\rangle, \qquad \omega(\tau)=\langle \omega, D_{q,h} r_x(\tau)\rangle.$$

Then $\beta(\tau)=\xi(\tau)$ *and* $\omega(\tau)=-\Gamma_\xi(\tau)$.

Proof. Applying [FM, p. 257], $(\beta(\tau), \omega(\tau))$ satisfies the linear SDE

$$\begin{cases} d\beta(t)=\dot{h}(t)\,dt-\frac{1}{2}\,dq\cdot dx+(q+\omega)\circ dx(t) \\ d\omega(t)=\Omega_t(\circ dx_t, \beta(t)), \end{cases} \tag{2.3}$$

where $dq \cdot dx$ denotes the stochastic contraction. It is easy to verify that $(\xi(t), -\Gamma_\xi(t))$ is a solution of (2.3). Therefore by unicity, we obtain the result. ∎

DEFINITION 2.2. *Let F be a functional on* $\mathbf{P}_{m_o}(M)$. *Define*

$$(D_\xi F)\circ I=D_{\xi*}(F\circ I). \tag{2.4}$$

Following Proposition 2.1, we have, for any cylindrical function F at the level of $O(M)$,

$$(D_\xi F)(\gamma_x)=\sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde{f})\,\xi^\alpha(\tau_i)-\sum_i \partial^i_{\Gamma_\xi(\tau_i)*}\tilde{f}. \tag{2.5}$$

Remark 2.3. Even in the case where $d\xi_x(t)=a_x(t)\,dx(t)$ with $b_x=0$, in general

$$\mathbb{E}(D_\xi F)=\mathbb{E}(D_{\xi*}(F\circ I))=\mathbb{E}\left(F\int_0^1\langle \tfrac{1}{2}\operatorname{Ric}_s \xi(s), dx(s)\rangle\right) \tag{2.6}$$

which is not necessairely equal to zero.

THEOREM 2.4 (Cruzeiro–Malliavin's Structural Equations). *Let* $z_1, z_2 \in \mathbb{H}$. *Then there exists a tangent process* $[z_1, z_2]$ *such that for all cylindrical functions F*

$$D_{[z_1,z_2]}F=D_{z_1}D_{z_2}F-D_{z_2}D_{z_1}F.$$

Morever, $[z_1, z_2]$ *has the explicit expression*

$$[z_1, z_2](\tau)=\Gamma_{z_2}(\tau)\,z_1(\tau)-\Gamma_{z_1}(\tau)\,z_2(\tau). \tag{2.7}$$

Proof. See [CM1, p. 147–152; Dr2]. ∎

3. MARKOVIAN CONNECTION AND CURVATURE TENSOR

The notion of Markovian connection was introduced by Cruzeiro and Malliavin [CM1] in order to substitute the Levi–Civita connection. Its advantadge is that it preserves the category of adapted vector fields.

Let ξ be a tangent process, z be a H-vector field on $\mathbf{P}_{m_o}(M)$. Following [CM1], we define

$$\overset{\cdot}{\overbrace{\nabla_\xi z}}(\tau) = D_\xi \dot{z}(\tau) - \Gamma_\xi(\tau)\, \dot{z}(\tau). \tag{3.1}$$

Definition 3.1. Let $z_1, z_2, z_3 \in \mathbb{H}$. We define

$$R^P(z_1, z_2)\, z_3 = [\nabla_{z_1}, \nabla_{z_2}]\, z_3 - \nabla_{[z_1, z_2]} z_3.$$

Proposition 3.2 [CM2]. *We have*

$$\overset{\cdot}{\overbrace{R^P(z_1, z_2)\, z_3}} = ([\Gamma_{z_1}, \Gamma_{z_2}] - (D_{z_1} \Gamma_{z_2} - D_{z_2} \Gamma_{z_1}) - \Gamma_{\Gamma_{z_1} z_2 - \Gamma_{z_2} z_1})\, \dot{z}_3.$$

Proof. It follows directly from (2.7) and (3.1). ∎

Using the above expression, we shall obtain the following explicit expression, which was obtained independently by J. R. Norris by another approach.

Theorem 3.3. *We have*

$$\overset{\cdot}{\overbrace{R^P(z_1, z_2)\, z_3}} = \Omega(z_1, z_2)\, \dot{z}_3. \tag{3.2}$$

To prove (3.2), we need the following preparation.

Lemma 3.4.

$$D_{z_1} \Gamma_{z_2}(\tau) = \int_0^\tau [\Gamma_{z_1}(s), \Omega_s(z_2, \circ dx_s)] - \int_0^\tau \Omega_s(\Gamma_{z_1} z_2, \circ dx_s)$$
$$+ \int_0^\tau \Omega_s(z_2, \dot{z}_1)\, ds + \sum_\alpha \int_0^\tau (\mathscr{L}_{A_\alpha} \Omega)(z_2, \circ dx_s)\, z_1^\alpha(s). \tag{3.3}$$

Proof. By definition (1.6), we have

$$D_{z_1} \Gamma_{z_2}(\tau) = \int_0^\tau (D_{z_1} \Omega_{r_x(s)})(z_2(s), \circ dx_s) + \int_0^\tau \Omega_{r_x(s)}(z_2(s), \circ \Gamma_{z_1}(s)\, dx_s)$$
$$+ \int_0^\tau \Omega_{r_x(s)}(z_2(s), \dot{z}_1(s))\, ds. \tag{3.4}$$

Now using (1.7),

$$\text{(i)} \quad D_{z_1}\Omega_{r_x(s)} = \textstyle\sum_\alpha (\mathcal{L}_{A_\alpha}\Omega)_{r_x(s)}\, z_1^\alpha(s) - (\mathcal{L}_{\Gamma_{z_1}(s)*}\Omega)_{r_x(s)}.$$

By the equi-invariance property, for $E \in so(d)$; $a, b \in \mathbb{R}^d$,

$$(\Omega_{re^{\varepsilon E}})(a, b) = e^{-\varepsilon E}\,\Omega_r(e^{\varepsilon E}a, e^{\varepsilon E}b)\, e^{\varepsilon E}.$$

It follows that

$$(\mathcal{L}_{E*}\Omega)(a, b) = -[E, \Omega(a, b)] + \Omega(Ea, b) + \Omega(a, Eb). \tag{3.5}$$

Therefore by (i), we obtain

$$\begin{aligned}(D_{z_1}\Omega_{r_x(s)})(z_2(s), \circ dx_s) = \sum_\alpha (\mathcal{L}_{A_\alpha}\Omega)_{r_x(s)}\,(z_2(s), \circ dx_s)\, z_1^\alpha(s)\\ + [\Gamma_{z_1}(s), \Omega_s(z_2(s), \circ dx_s)]\\ - \Omega_s(\Gamma_{z_1}(s)\, z_2(s), \circ dx_s) - \Omega_s(z_2(s), \circ \Gamma_{z_1}(s)\, dx_s).\end{aligned} \tag{3.6}$$

So combining with (3.4), we obtain (3.3). ∎

Proof of Theorem 3.3. By (3.3), we have

$$\begin{aligned}&D_{z_1}\Gamma_{z_2}(\tau) - D_{z_2}\Gamma_{z_1}(\tau)\\ &\quad = \int_0^\tau [\Gamma_{z_1}(s), \Omega_s(z_2, \circ dx_s)] - \int_0^\tau [\Gamma_{z_2}(s), \Omega_s(z_1, \circ dx_s)]\\ &\quad\quad - \Gamma_{\Gamma_{z_1}z_2 - \Gamma_{z_2}z_1}\\ &\quad\quad + \int_0^\tau (\Omega_s(z_2, \dot z_1) - \Omega_s(z_1, \dot z_2))\, ds\\ &\quad\quad + \sum_\alpha \int_0^\tau (z_1^\alpha(s)(\mathcal{L}_{A_\alpha}\Omega)(z_2, \circ dx_s) - z_2^\alpha(s)(\mathcal{L}_{A_\alpha}\Omega)(z_1, \circ dx_s)).\end{aligned}$$

By the second Bianchi identity,

$$\begin{aligned}&\sum_\alpha z_1^\alpha(s)(\mathcal{L}_{A_\alpha}\Omega)(z_2, \circ dx_s) - z_2^\alpha(s)(\mathcal{L}_{A_\alpha}\Omega)(z_1, \circ dx_s)\\ &\quad = -\sum_\alpha (\mathcal{L}_{A_\alpha}\Omega)(z_1, z_2) \circ dx^\alpha(s).\end{aligned}$$

Now by the Itô formula, and according to SDE (1.1)

$$d\Omega_s(z_1(s), z_2(s)) = \sum_\alpha (\mathscr{L}_{A_\alpha}\Omega)(z_1, z_2) \circ dx^\alpha(s) + (\Omega_s(\dot{z}_1(s), z_2(s)) + \Omega_s(z_1(s), \dot{z}_2(s)))\, ds.$$

Therefore

$$d\,(D_{z_1}\Gamma_{z_2}(\tau) - D_{z_2}\Gamma_{z_1}(\tau)) = d[\Gamma_{z_1}, \Gamma_{z_2}](\tau) - d\Gamma_{\Gamma_{z_1}z_2 - \Gamma_{z_2}z_1}(\tau) - d\,\Omega_\tau(z_1, z_2).$$

Now using the expression in Proposition 3.2, we obtain the result. ∎

In what follows, we shall establish the existence of the Ricci curvature. To this end, we shall compute the two step trace like in [Fr]. Let $\{c_n, n \geqslant 1\}$ be an orthonormal basis of $H([0,1], \mathbb{R})$. Consider

$$h_{n,\alpha}(\tau) = c_n(\tau)\, \varepsilon_\alpha. \tag{3.7}$$

Then $\{h_{n,\alpha}\,;\, n \geqslant 1, \alpha = 1, ..., d\}$ is an orthonormal basis of $\mathbb{H}$.

THEOREM 3.5. *Let $z \in \mathbb{H}$, $k \in \mathscr{C}^2([0,1], \mathbb{R}^d)$. Then the series*

$$\sum_n \sum_\alpha \langle R^P(z, h_{n,\alpha})\, h_{n,\alpha}, k \rangle_H$$

converges in $L^2(X)$ to $\int_0^1 \langle \mathrm{Ric}_t^M z_t, \dot{k}_t \rangle\, dt$.

Proof. Let $m_x(t) = \langle \mathrm{Ric}_t^M z_t, \dot{k}_t \rangle$. Then $m_x(t)$ is a semi-martingale. We have

$$\int_0^1 m_x(t)(c_n(t)^2)'\, dt = m_x(1)\, c_n(1)^2 - \int_0^1 c_n(t)^2\, dm_x(t).$$

Since $\{c_n; n \geqslant 1\}$ is an orthonormal basis of $H([0,1], \mathbb{R})$, then for all $t \in [0,1]$,

$$\sum_n c_n(t)^2 = t.$$

It follows that in $L^2(X)$, $\sum_n \int_0^1 c_n(t)^2\, dm_x(t) = \int_0^1 t\, dm_x(t)$. Therefore in $L^2(X)$,

$$\sum_n \int_0^1 m_x(t)(c_n(t)^2)'\, dt = m_x(1) - \int_0^1 t\, dm_x(t) = \int_0^1 m_x(t)\, dt. \tag{3.8}$$

Now by (3.2),

$$\sum_{\alpha} \langle R^P(z, h_{n,\alpha})\, h_{n,\alpha}, k\rangle_H = \int_0^1 \langle \mathrm{Ric}_t^M\, z_t, \dot{k}_t\rangle\, c_n(t)\, \dot{c}_n(t)\, dt$$

$$= \tfrac{1}{2} \int_0^1 m_x(t)(c_n^2(t))'\, dt.$$

Therefore according to (3.8), we obtain the result. ∎

4. SCALAR LAPLACE OPERATOR

Define the scalar Laplace operator Δ on $\mathbf{P}_{m_o}(M)$ by

$$\Delta F = \sum_{\alpha=1}^{d} \int_0^1 D^2_{\tau,\alpha} F\, d\tau \tag{4.1}$$

where $D_{\tau,\alpha}F$ is the derivative of F localized at (τ, α) such that

$$\sum_{\alpha} \int_0^1 D_{\tau,\alpha} F\, \dot{h}^{\alpha}(\tau)\, d\tau = D_h F.$$

Let F be a cylindrical function on $\mathbf{P}_{m_o}(M)$ in the form (1.10). Then

$$D_{\tau,\alpha} F = \sum_i (\partial^i_{A_\alpha} \tilde{f})\, \mathbf{1}_{(\tau < \tau_i)},$$

$$D_{\tau,\alpha} D_{\tau,\alpha} F = \sum_i D_{\tau,\alpha}(\partial^i_{A_\alpha} \tilde{f})\, \mathbf{1}_{(\tau<\tau_i)} = \sum_{i,j} \langle \partial^j(\partial^i_{A_\alpha} \tilde{f}), D_{\tau,\alpha} r_x(\tau_j)\rangle\, \mathbf{1}_{(\tau<\tau_i)}.$$

By (1.7), we see that $D_{\tau,\alpha} r_x(\tau_j) = \mathbf{1}_{(\tau<\tau_j)} A_\alpha - \Gamma_{\tau,\alpha}(\tau_j)^*$ where

$$\Gamma_{\tau,\alpha}(\tau_j) = \mathbf{1}_{(\tau<\tau_j)} \int_\tau^{\tau_j} \Omega_s(\varepsilon_\alpha, \circ dx_s). \tag{4.2}$$

It follows that

$$D_{\tau,\alpha} D_{\tau,\alpha} F = \sum_{i,j} (\partial^j_{A_\alpha} \partial^i_{A_\alpha} \tilde{f})\, \mathbf{1}_{(\tau<\tau_i \wedge \tau_j)} - \sum_i \sum_\beta (\partial^i_{A_\alpha} \tilde{f}) \langle \Gamma_{\tau,\alpha}(\tau_i)\, \varepsilon_\alpha, \varepsilon_\beta\rangle\, \mathbf{1}_{(\tau<\tau_i)}.$$

Since

$$\sum_\alpha \Gamma_{\tau,\alpha}(\tau_i)\, \varepsilon_\alpha = \mathbf{1}_{(\tau<\tau_i)} \int_\tau^{\tau_i} \sum_\alpha \Omega_s(\varepsilon_\alpha, \circ dx_s)\, \varepsilon_\alpha = -\mathbf{1}_{(\tau<\tau_i)} \int_\tau^{\tau_i} \mathrm{Ric}_s \circ dx_s,$$

and

$$\int_0^{\tau_i} \left(\int_{\tau}^{\tau_i} \langle \mathrm{Ric}_s\, \varepsilon_\beta, \circ dx(s) \rangle \right) d\tau = \int_0^{\tau_i} \tau \langle \mathrm{Ric}_\tau\, \varepsilon_\beta, \circ dx(\tau) \rangle,$$

then applying the definition (4.1), we obtain

$$\Delta F = \sum_{i,j} \sum_{\alpha} (\partial^j_{A_\alpha} \partial^i_{A_\alpha} \tilde{f})(\tau_i \wedge \tau_j) + \sum_i \sum_\alpha (\partial^i_{A_\alpha} \tilde{f}) \int_0^{\tau_i} \tau \langle \mathrm{Ric}_\tau\, \varepsilon_\alpha, \circ dx(\tau) \rangle. \tag{4.3}$$

Remark. A similar expression has been computed by Kazumi [Ka].

In what follows, we shall prove that the above Laplace operator can be written as the divergence with respect to the Markovian connection of the gradient.

DEFINITION 4.1. Let Z be a H-vector field on $\mathbf{P}_{m_o}(M)$. We define

$$\mathrm{div}(Z) = \sum_n \sum_\alpha \langle \nabla_{h_{n,\alpha}} Z, h_{n,\alpha} \rangle_H,$$

if the series conveges in $L^2(X)$.

Remark that this notion is algebraic, diffrent from one defined as the adjoint operator with respect to some measure.

THEOREM 4.2. *Consider the Markovian connection* ∇ *on* $\mathbf{P}_{m_o}(M)$. *Then*

$$\Delta F = \mathrm{div}(\nabla F) \qquad \text{for all cylindrical function } F. \tag{4.4}$$

We shall firstly establish the following result.

LEMMA 4.3. *Let* $k \in \mathbb{H}$. *Then* $\mathrm{div}(k) = 0$. *More precisely,*

$$\sum_n \sum_\alpha \langle \nabla_{h_{n,\alpha}} k, h_{n,\alpha} \rangle_H = 0 \qquad \text{in } L^2(X). \tag{4.5}$$

Proof. First consider $k(t) = a(t)\, e$, with $a \in H([0,1], \mathbb{R})$ and $e \in \mathbb{R}^d$. Then

$$\overset{\cdot}{\overbrace{\nabla_{h_{n,\alpha}} k}}(\tau) = -\Gamma_{h_{n,\alpha}}(\tau)\, e\dot{a}(\tau),$$

$$\langle \nabla_{h_{n,\alpha}} k, h_{n,\alpha} \rangle_H = -\int_0^1 \langle \Gamma_{h_{n,\alpha}}(\tau)\, e, \varepsilon_\alpha \rangle\, \dot{c}_n(\tau)\, \dot{a}(\tau)\, d\tau.$$

Define $u_n(\tau)=\sum_{\alpha=1}^{d} \Gamma_{h_{n,\alpha}}(\tau)\,\varepsilon_\alpha$. Then $u_n(\tau)$ has the expression

$$u_n(\tau)=-\int_0^\tau c_n(s)\,\mathrm{Ric}_s^M \circ dx_s. \tag{4.6}$$

We have

$$\sum_\alpha \langle \nabla_{h_{n,\alpha}} k, h_{n,\alpha}\rangle_H=\int_0^1 \langle u_n(\tau), e\rangle\, \dot c_n(\tau)\,\dot a(\tau)\,d\tau.$$

Define $d_n(t)=\int_0^t \dot c_n(\tau)\,\dot a(\tau)\,d\tau$. Then

$$\int_0^1 \langle u_n(\tau), e\rangle\, \dot c_n(\tau)\,\dot a(\tau)\,d\tau$$

$$=\langle u_n(1), e\rangle\, d_n(1)-\int_0^1 d_n(\tau)\,d\langle u_n(\tau), e\rangle=I_{n,1}+I_{n,2}.$$

By (4.6), we obtain

$$I_{n,1}=-\left(\int_0^1 \dot c_n(\tau)\left(\int_\tau^1 \langle \mathrm{Ric}_s^M e, \circ dx_s\rangle\right) d\tau\right)\left(\int_0^1 \dot c_n(\tau)\,\dot a(\tau)\,d\tau\right).$$

It follows that

$$\sum_n I_{n,1}=-\int_0^1 a(\tau)\langle \mathrm{Ric}_\tau^M e, \circ dx_\tau\rangle. \tag{4.7}$$

To see the convergence of the above series holds also in $L^2(X)$ and for later use, put:

$$A_{N,1}=\sum_{n=1}^N I_{n,1}.$$

We have:

$$A_{N,1}^2\leqslant\left(\int_0^1\left(\int_\tau^1 \langle \mathrm{Ric}_s^M e, \circ dx_s\rangle\right)^2 d\tau\right)\left(\int_0^1 \dot a(\tau)^2\,d\tau\right).$$

Therefore there exists a constant $c>0$ such that

$$\mathbb{E}(A_{N,1}^2)\leqslant c\int_0^1 \dot a(\tau)^2\,d\tau. \tag{4.8}$$

On the other hand,

$$I_{n,2}=\int_0^1\left(\int_0^\tau \dot{c}_n(s)\,\dot{a}(s)\,ds\right)c_n(\tau)\langle \mathrm{Ric}_\tau^M e,\ \circ dx_\tau\rangle.$$

We have

$$\sum_n\left(\int_0^\tau \dot{c}_n(s)\,\dot{a}(s)\,ds\right)c_n(\tau)=\int_0^\tau \dot{a}(s)\,ds=a(\tau).$$

Put $A_{N,2}=\sum_{n=1}^N I_{n,2}$. Then

$$\mathbb{E}(A_{N,2}^2)\leqslant c\int_0^1 \dot{a}(\tau)^2\,d\tau, \tag{4.9}$$

and

$$\begin{aligned}\mathbb{E}&\left(\left|A_{N,2}-\int_0^1 a(\tau)\langle \mathrm{Ric}_\tau^M e,\ \circ dx_\tau\rangle\right|^2\right)\\ &\leqslant c\mathbb{E}\int_0^1\left(\sum_{n>N}\left(\int_0^\tau \dot{c}_n(s)\,\dot{a}(s)\,ds\right)c_n(\tau)\right)^2 d\tau\\ &\leqslant c\left(\int_0^1 \dot{a}(s)^2\,ds\right)\int_0^1\sum_{n>N}c_n^2(\tau)\,d\tau\to 0.\end{aligned}$$

It follows that in $L^2(X)$,

$$\sum_n I_{n,2}=\int_0^1 a(\tau)\,\langle \mathrm{Ric}_\tau^M e,\ \circ dx_\tau\rangle.$$

Therefore combining with (4.7), we obtain $\sum_n(I_{n,1}+I_{n,2})=0$ in $L^2(X)$. For general case, it is enough to write $k(t)=\sum_\alpha\langle k(t),\varepsilon_\alpha\rangle\,\varepsilon_\alpha$. ∎

LEMMA 4.4. *Let ξ be a tangent process. Then*

$$D_\xi(\partial^i_{A_\alpha}\tilde{f})=\sum_j\sum_\beta(\partial^j_{A_\beta}\partial^i_{A_\alpha}\tilde{f})\,\xi^\beta(\tau_j)-\sum_\beta(\partial^i_{A_\beta}\tilde{f})\langle\Gamma_\xi(\tau_i)\,\varepsilon_\alpha,\varepsilon_\beta\rangle. \tag{4.10}$$

Proof. We have $\partial^j_{\Gamma_\xi(\tau_j)^*}\partial^i_{A_\alpha}\tilde{f}=\partial^i_{A_\alpha}\partial^j_{\Gamma_\xi(\tau_j)^*}\tilde{f}=0$ if $i\neq j$, and

$$\partial^i_{\Gamma_\xi(\tau_i)^*}\partial^i_{A_\alpha}\tilde{f}=\partial^i_{[\Gamma_\xi(\tau_i)^*,\,A_\alpha]}\tilde{f}=\sum_\beta(\partial^i_{A_\beta}\tilde{f})\langle\Gamma_\xi(\tau_i)\,\varepsilon_\alpha,\varepsilon_\beta\rangle.$$

Now according to (2.5), we obtain the result. ∎

Proof of Theorem 4.1. Denote $z_{i,\alpha}(\tau)=(\tau\wedge\tau_i)\,\varepsilon_\alpha$. Then by (1.12), we have

$$\nabla_{h_{n,\alpha}}(\nabla F)=\sum_{i,\beta} D_{h_{n,\alpha}}(\partial^i_{A_\beta}\tilde{f})\,z_{i,\beta}+\sum_{i,\beta}(\partial^i_{A_\beta}\tilde{f})\,\nabla_{h_{n,\alpha}}z_{i,\beta},$$

$$\langle\nabla_{h_{n,\alpha}}(\nabla F),h_{n,\alpha}\rangle=\sum_{i} D_{h_{n,\alpha}}(\partial^i_{A_\alpha}\tilde{f})\,c_n(\tau_i)+\sum_{i,\beta}(\partial^i_{A_\beta}\tilde{f})\langle\nabla_{h_{n,\alpha}}z_{i,\beta},h_{n,\alpha}\rangle.$$

By Lemma 4.4, $D_{h_{n,\alpha}}(\partial^i_{A_\alpha}\tilde{f})=\sum_j(\partial^j_{A_\alpha}\partial^i_{A_\alpha}\tilde{f})\,c_n(\tau_j)-\sum_\beta(\partial^i_{A_\beta}\tilde{f})\langle\Gamma_{h_{n,\alpha}}(\tau_i)\,\varepsilon_\alpha,\varepsilon_\beta\rangle$. Since

$$\sum_\alpha\langle\Gamma_{h_{n,\alpha}}(\tau_i)\,\varepsilon_\alpha,\varepsilon_\beta\rangle=-\int_0^{\tau_i}\langle\mathrm{Ric}^M_s\,\varepsilon_\beta,\circ dx_s\rangle\,c_n(s),$$

then in $L^2(X)$ holds

$$\sum_n\sum_\alpha\langle\Gamma_{h_{n,\alpha}}(\tau_i)\,\varepsilon_\alpha,\varepsilon_\beta\rangle\,c_n(\tau_i)=-\int_0^{\tau_i}s\langle\mathrm{Ric}^M_s\,\varepsilon_\beta,\circ dx_s\rangle.$$

Now using Lemma 4.2, the series $\sum_n\sum_\alpha\langle\nabla_{h_{n,\alpha}}(\nabla F),h_{n,\alpha}\rangle$ converges in $L^2(X)$ to the quantity

$$\sum_{i,j}\sum_\alpha(\partial^j_{A_\alpha}\partial^i_{A_\alpha}\tilde{f})(\tau_i\wedge\tau_j)+\sum_{i,\beta}(\partial^i_{A_\beta}\tilde{f})\int_0^{\tau_i}s\langle\mathrm{Ric}^M_s\,\varepsilon_\beta,\circ dx_s\rangle=\Delta F.\quad\blacksquare$$

Remark. By Theorem 4.2 and (4.5), we see that the Laplace operator defined in (4.1) is the same as that introduced in [AM, p. 481].

5. TORSION TENSOR

Let $h,k\in\mathbb{H}$. The torsion $T(h,k)$ between h and k with respect to the Markovian connection ∇ is defined by

$$T(h,k)=\nabla_h k-\nabla_k h-[h,k].\tag{5.1}$$

By (2.7), (3.1), and the first Bianchi identity, $T(h,k)$ has the following expression (see [CM1]):

$$T(h,k)(\tau)=\int_0^\tau\Omega_s(h(s),k(s))\circ dx(s).\tag{5.2}$$

We see that $T(h,k)$ is a tangent process. In order to develop a tensorial calculus in our context, we shall define

Definition 5.1. Let $\xi(t)$ be a tangent process. We define

(i) $\langle z, \xi\rangle = \int_0^1 \langle \dot z(t), \circ d\xi(t)\rangle$ if z is an adapted vector field;

(ii) $\langle Z, \xi\rangle = \sum_i f_i \langle z_i, \xi\rangle$ if $Z = \sum_{\text{finite}} f_i z_i$ with z_i adapted.

Remark 5.2. (i) If $Z = \nabla F = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde f)\, z_{i,\alpha}$, then by (2.5),

$$\langle \nabla F, \xi\rangle = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde f) \langle z_{i,\alpha}, \xi\rangle = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde f)\, \xi^\alpha(\tau_i) = D_\xi F.$$

(ii) The definition of $\langle Z, \xi\rangle$ is in fact the anticipative Stratanovich integral; see [CM1, p. 158].

Proposition 5.3. *Let F be a cylindrical function. We have*

$$\langle \nabla^2 F, h \wedge k\rangle = -\langle \nabla F, T(h,k)\rangle. \tag{5.3}$$

Proof. We have

$$\langle \nabla^2 F, h \otimes k\rangle = \langle \nabla_h \nabla F, k\rangle = D_h D_k F - \langle \nabla F, \nabla_h k\rangle.$$

In the same way, $\langle \nabla^2 F, k \otimes h\rangle = D_k D_h F - \langle \nabla F, \nabla_k h\rangle$. Then

$$\langle \nabla^2 F, h \wedge k\rangle = (D_h D_k - D_k D_h)\, F - \langle \nabla F, \nabla_h k - \nabla_k h\rangle.$$

Since $\nabla_h k - \nabla_k h = [h,k] + T(h,k)$, so we obtain (5.3). ∎

Theorem 5.4. *There exists a tangent process denoted by $\nabla_h T(h,k)$ such that*

$$D_h D_{T(h,k)} F - \langle \nabla_h \nabla F, T(h,k)\rangle = \langle \nabla F, \nabla_h T(h,k)\rangle \tag{5.4}$$

for any cylindrical function F.

Proof. Let $F = f(\gamma(\tau_1), ..., \gamma(\tau_k))$, $f \in \mathscr{C}^\infty(M^k)$. Denote $z_{i,\alpha} = (\tau \wedge \tau_i)\, \varepsilon_\alpha$. Then $\nabla F = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde f)\, z_{i,\alpha}$ and

$$\nabla_h(\nabla F) = \sum_{i,\alpha} D_h(\partial^i_{A_\alpha} \tilde f)\, z_{i,\alpha} + \sum_{i,\alpha} (\partial^i_{A_\alpha}) \tilde f)\, \nabla_h z_{i,\alpha}. \tag{5.5}$$

Remark that $z_{i,\alpha}$ and $\nabla_h z_{i,\alpha}$ are adapted vector fields, so by Definition 5.1(ii),

$$\text{(i)} \quad \langle \nabla_h(\nabla F), T(h,k)\rangle = \sum_{i,\alpha} D_h(\partial^i_{A_\alpha} \tilde f) \langle z_{i,\alpha}, T(h,k)\rangle + \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde f) \langle \nabla_h z_{i,\alpha}, T(h,k)\rangle.$$

On the other hand, $D_{T(h,k)}F=\sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde{f})\langle z_{i,\alpha}, T(h,k)\rangle$ and

$$\text{(ii)}\quad D_h D_{T(h,k)}F=\sum_{i,\alpha} D_h(\partial^i_{A_\alpha}\tilde{f})\langle z_{i,\alpha}, T(h,k)\rangle + \textstyle\sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde{f})\, D_h\langle z_{i,\alpha}, T(h,k)\rangle.$$

According to Definition 1.6, we have

$$\begin{aligned} D_h\langle z_{i,\alpha}, T(h,k)\rangle &= D_h\int_0^1 \langle \dot{z}_{i,\alpha}, \Omega_\tau(h,k)\circ dx_\tau\rangle\rangle \\ &= \int_0^1 \langle \dot{z}_{i,\alpha}, (D_h\Omega_{r_x(\tau)})(h,k)\circ dx_\tau)\rangle \\ &\quad + \int_0^1 \langle \dot{z}_{i,\alpha}, \Omega_\tau(h,k)\,\Gamma_h(\tau)\circ dx_\tau)\rangle \\ &\quad + \int_0^1 \langle \dot{z}_{i,\alpha}, \Omega_\tau(h,k)\,\dot{h}(\tau)\rangle\, d\tau. \end{aligned}$$

Define the tangent process $\nabla_h T(h,k)$ by

$$\begin{aligned} d\nabla_h T(h,k)(s) &= ((D_h\Omega_{r_x(s)})(h,k)+[\Omega_s(h,k), \Gamma_h(s)])\circ dx(s) \\ &\quad + \Omega_s(h,k)\,\dot{h}(s)\, ds. \end{aligned}\tag{5.6}$$

Then $D_h\langle z_{i,\alpha}, T(h,k)\rangle - \langle \nabla_h z_{i,\alpha}, T(h,k)\rangle = \langle z_{i,\alpha}, \nabla_h T(h,k)\rangle$. Therefore by (i) and (ii), we obtain

$$D_h D_{T(h,k)}F - \langle \nabla_h(\nabla F), T(h,k)\rangle = \langle \nabla F, \nabla_h T(h,k)\rangle. \quad \blacksquare$$

PROPOSITION 5.5. *Define* $(\nabla_h T)(h,k)=\nabla_h T(h,k)-T(\nabla_h h,k)-T(h,\nabla_h k)$. *Then* $(\nabla_h T)(h,k)$ *is a tangent process which has the expression*

$$\begin{aligned} d(\nabla_h T)(h,k) = \Big[&\sum_\alpha (\mathscr{L}_{A_\alpha}\Omega)(h,k)\,h^\alpha - \Omega(\Gamma_h h+\nabla_h h, k) \\ &- \Omega(h, \Gamma_h k+\nabla_h k)\Big]\circ dx + \Omega(h,k)\,\dot{h}\, d\tau. \end{aligned}\tag{5.7}$$

Proof. Using (3.6),

$$\begin{aligned} (D_h\Omega_{r_x(s)})(h,k) &= \sum_\alpha (\mathscr{L}_{A_\alpha}\Omega)_{r_x(s)}(h(s),k(s))\,h^\alpha(s) + [\Gamma_h(s), \Omega_s(h,k)] \\ &\quad - \Omega_s(\Gamma_h(s)\,h(s), k(s)) - \Omega_s(h(s), \Gamma_h(s)\,k(s)). \end{aligned}$$

By (5.6), we have

$$\text{(i)}\quad d(\nabla_h T(h,k)) = \left[\sum_\alpha (\mathcal{L}_{A_\alpha}\Omega)(h,k)\,h^\alpha - \Omega(\Gamma_h hh, k) - \Omega(h, \Gamma_h k)\right]\circ dx + \Omega(h,k)\,\dot h\,d\tau.$$

On the other hand, by (5.2),

$$\text{(ii)}\quad dT(\nabla_h h, k)(s) + dT(h, \nabla_h k)(s) = (\Omega_s(\nabla_h h, k) + \Omega_s(h, \nabla_h k))\circ dx(s).$$

Now combining (i) and (ii), we obtain (5.7). ∎

6. VECTOR LAPLACE OPERATOR

LEMMA 6.1. *Let $z, k \in \mathbb{H}$. Then*

$$\sum_n \sum_\alpha \langle \nabla_{h_{n,\alpha}} h_{n,\alpha} \otimes k, \nabla z\rangle_{H\otimes H} = 0 \qquad \textit{in } L^2(X). \tag{6.1}$$

Proof. We have

$$\text{(i)}\quad \langle \nabla_{h_{n,\alpha}} h_{n,\alpha}\otimes k, \nabla z\rangle_{H\otimes H} = -\int_0^1 \langle \Gamma_{\nabla_{h_{n,\alpha}} h_{n,\alpha}}(\tau)\,\dot z(\tau), \dot k(\tau)\rangle\,d\tau,$$

and $\langle \nabla_{h_{n,\alpha}} h_{n,\alpha}(s), \varepsilon_\beta\rangle = \langle \nabla_{h_{n,\alpha}} h_{n,\alpha}, z_{s,\beta}\rangle_H$ where $z_{s,\beta}(\tau) = (s\wedge\tau)\,\varepsilon_\beta$ and

$$\Gamma_{\nabla_{h_{n,\alpha}} h_{n,\alpha}}(\tau) = \sum_\beta \int_0^\tau \Omega_s(\varepsilon_\beta, \circ dx_s)\langle \nabla_{h_{n,\alpha}} h_{n,\alpha}, z_{s,\beta}\rangle_H.$$

By (4.8) and (4.9), we see easily

$$\sum_n\sum_\alpha \langle \Gamma_{\nabla_{h_{n,\alpha}} h_{n,\alpha}}(\tau)\,\varepsilon_\gamma, \varepsilon_{\gamma'}\rangle = 0 \qquad \text{in } L^2(X), \quad \text{uniformly in} \quad \tau\in[0,1]. \tag{6.2}$$

Now by (i), we obtain (6.1). ∎

PROPOSITION 6.2.

$$\sum_n\sum_\alpha \langle \nabla\nabla F, \nabla_{h_{n,\alpha}} h_{n,\alpha}\otimes k\rangle_{H\otimes H} = 0 \qquad \textit{in } L^2(X). \tag{6.3}$$

Proof. We have by (1.8), $\nabla F = \sum_{i,\beta} (\partial^i_{A_\beta} \tilde{f})\, z_{i,\beta}$. Let z be an adapted vector field, then by Lemma 4.3,

$$\nabla_z(\nabla F) = \sum_{i,j} \sum_{\beta,\gamma} (\partial^j_{A_\gamma} \partial^i_{A_\beta} \tilde{f})\, z^\gamma(\tau_j)\, z_{i,\beta}$$

$$- \sum_{i} \sum_{\beta,\gamma} (\partial^i_{A_\gamma} \tilde{f}) \langle \Gamma_z(\tau_i)\, \varepsilon_\beta, \varepsilon_\gamma \rangle\, z_{i,\beta} + \sum_{i,\beta} (\partial^i_{A_\beta} \tilde{f})\, \nabla_z z_{i,\beta}. \tag{6.4}$$

Let $z = \nabla_{h_{n,\alpha}} h_{n,\alpha}$. By Lemma 4.2,

$$\sum_n \sum_\alpha (\nabla_{h_{n,\alpha}} h_{n,\alpha})^\gamma\, (\tau_j) = \sum_n \sum_\alpha \langle \nabla_{h_{n,\alpha}} h_{n,\alpha}, z_{j,\gamma} \rangle_H = 0 \qquad \text{in } L^2(X).$$

By (6.2), $\sum_n \sum_\alpha \langle \Gamma_{\nabla_{hn,\alpha} h_{n,\alpha}}(\tau_i)\, \varepsilon_\beta, \varepsilon_\gamma \rangle = 0$ in $L^2(X)$. Now according to (6.1), we obtain the result. ∎

LEMMA 6.3.

$$\sum_n \sum_\alpha D_{h_{n,\alpha}} \Gamma_{h_{n,\alpha}}(\tau) \qquad \textit{converges in } L^2(X), \quad \textit{uniformly in } \tau \in [0,1]. \tag{6.5}$$

Proof. Using (3.3), we have

$$D_{h_{n,\alpha}} \Gamma_{h_{n,\alpha}}(\tau) = \int_0^\tau \left[\int_0^s c_n(\xi)\, \Omega_\xi(\varepsilon_\alpha, \circ dx_\xi), c_n(s)\, \Omega_s(\varepsilon_\alpha, \circ dx_s) \right]$$

$$- \int_0^\tau \Omega_s \left(\int_0^s c_n(s)\, c_n(\xi)\, \Omega_\xi(\varepsilon_\alpha, \circ dx_\xi)\, \varepsilon_\alpha\ dx_s \right)$$

$$+ \int_0^\tau c_n^2(s) (\mathscr{L}_{A_\alpha} \Omega)_{r_x(s)}\, (\varepsilon_\alpha, \circ dx_s)\, \varepsilon_\alpha.$$

Now using the Doob's maximal martingale inequality, a straightforward calculs gives the result. ∎

LEMMA 6.4. *Let* $z \in \mathbb{H}$. *Then*

$$\Delta z = \sum_n \sum_\alpha \nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}} z \qquad \textit{converges in } L^1(X, \mathbb{H}). \tag{6.6}$$

Proof. By Definition (3.1), we have

(i) $\overbrace{\nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}} z}^{\cdot}(\tau) = -D_{h_{n,\alpha}} \Gamma_{h_{n,\alpha}}(\tau)\, \dot{z}(\tau) + \Gamma_{h_{n,\alpha}}(\tau)\, \Gamma_{h_{n,\alpha}}(\tau)\, \dot{z}(\tau).$

Using the explicit expression of $\Gamma_{h_{n,\alpha}}(\tau)$ and the maximal martingale inequality, it is easy to see that

$$\sum_n \sum_\alpha \Gamma_{h_{n,\alpha}}(\tau)\, \Gamma_{h_{n,\alpha}}(\tau) \tag{6.7}$$

converges in $L^1(X)$ uniformly in $\tau \in [0, 1]$. Now by (i) and (6.5), we obtain (6.6). Morever,

$$\mathbb{E}(|\Delta z|_H) \leqslant c\, |z|_H. \quad ▮$$

THEOREM 6.5. *Let F be a cylindrical function on $\mathbf{P}_{m_o}(M)$. Then*

$$\sum_n \sum_\alpha \nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}}(\nabla F) \qquad \textit{converges in } L^1(X, H).$$

Proof. By (6.4),

$$\nabla_h(\nabla F) = \sum_{i,j} \sum_{\beta,\gamma} (\partial^j_{A_\gamma} \partial^i_{A_\beta} \tilde{f})\, h^\gamma(\tau_j)\, z_{i,\beta} - \sum_i \sum_{\beta,\gamma} (\partial^i_{A_\gamma} \tilde{f}) \langle \Gamma_h(\tau_i)\, \varepsilon_\beta, \varepsilon_\gamma \rangle\, z_{i,\beta} + \sum_{i,\beta} (\partial^i_{A_\beta} \tilde{f})\, \nabla_h z_{i,\beta}.$$

Then

$$\begin{aligned}\nabla_h \nabla_h(\nabla F) = &\sum_{i,j} \sum_{\beta,\gamma} \{D_h(\partial^j_{A_\gamma} \partial^i_{A_\beta} \tilde{f})\, h^\gamma(\tau_j)\, z_{i,\beta} \\ &+ (\partial^j_{A_\gamma} \partial^i_{A_\beta} \tilde{f})\, h^\gamma(\tau_j)\, \nabla_h z_{i,\beta}\} \\ &- \sum_i \sum_{\beta,\gamma} \{D_h(\partial^i_{A_\gamma} \tilde{f}) \langle \Gamma_h(\tau_i)\, \varepsilon_\beta, \varepsilon_\gamma \rangle\, z_{i,\beta} \\ &+ (\partial^i_{A_\gamma} \tilde{f})(\langle D_h \Gamma_h(\tau_i)\, \varepsilon_\beta, \varepsilon_\gamma \rangle\, z_{i,\beta} \\ &+ \langle \Gamma_h(\tau_i)\, \varepsilon_\beta, \varepsilon_\gamma \rangle\, \nabla_h z_{i,\beta})\} \\ &+ \sum_{i,\beta} (\partial^i_{A_\beta} \tilde{f})\, \nabla_h \nabla_h z_{i,\beta} + D_h(\partial^i_{A_\beta} \tilde{f})\, \nabla_h z_{i,\beta}.\end{aligned}$$

Now by (1.6), (3.1), (6.5), (6.6), and (6.7), we see that

$$\sum_n \sum_\alpha \nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}}(\nabla F) \qquad \text{converges in } L^1(X, H). \quad ▮$$

DEFINITION 6.6.

$$\Delta(\nabla F) = \sum_n \sum_\alpha \nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}}(\nabla F) \qquad \textit{in } L^1(X, H). \tag{6.8}$$

PROPOSITION 6.7. *Let* $k \in \mathbb{H}$. *Then*

$$\langle \Delta(\nabla F), k \rangle = \sum_n \sum_\alpha \langle \nabla^3 F, h_{n,\alpha} \otimes h_{n,\alpha} \otimes k \rangle_{\otimes^3 H}. \tag{6.9}$$

Proof. We have

$$\langle \nabla^3 F, h_{n,\alpha} \otimes h_{n,\alpha} \otimes k \rangle_{\otimes^3 H}$$
$$= \langle \nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}} (\nabla F), k \rangle_H - \langle \nabla^2 F, \nabla_{h_{n,\alpha}} \otimes k \rangle_{H \otimes H}.$$

Therefore (6.9) follows from (6.3) and (6.8). ∎

Remark 6.8. In [CM2], the vector Laplace operator is defined by using the frame bundle on the path space.

7. WEITZENBOCK FORMULA

In what follows, we shall establish the commutation formula for $[\nabla, \Delta]$. In the context of Ornstein–Uhlenbeck operator on $\mathbf{P}_{m_o}(M)$, several works have been done (see [CFM], [CM1, 2]. For a general discussion on a finite dimensional manifold, we refer to [E-LJ-L]. Our approach here follows that of [DL]. However, in our situation, the torsion is not free, nor antisymmetric.

PROPOSITION 7.1. *Let F be a cylindrical function*, $h, k \in \mathbb{H}$. *Then*

$$\langle \nabla^3 F, h \otimes h \otimes k \rangle - \langle \nabla^3 F, h \otimes k \otimes h \rangle$$
$$= -\langle \nabla^2 F, h \otimes T(h, k) \rangle - \langle \nabla F, (\nabla_h T)(h, k) \rangle, \tag{7.1}$$

where $\langle \nabla^2 F, h \otimes T(h, k) \rangle = \langle \nabla_h (\nabla F), T(h, k) \rangle$.

Proof.

$$\langle \nabla^3 F, h \otimes (h \wedge k) \rangle = \langle \nabla_h (\nabla^2 F), h \wedge k \rangle$$
$$= D_h \langle \nabla^2 F, h \wedge k \rangle - \langle \nabla^2 F, \nabla_h (h \wedge k) \rangle$$
$$= D_h \langle \nabla^2 F, h \wedge k \rangle - \langle \nabla^2 F, \nabla_h h \wedge k) \rangle$$
$$- \langle \nabla^2 F, h \wedge \nabla_h k) \rangle.$$

Now by (5.3) and (5.4), the above expression is equal to

$$-D_h\langle \nabla F, T(h,k)\rangle + \langle \nabla F, T(\nabla_h h, k)\rangle + \langle \nabla F, T(h, \nabla_h k)\rangle$$
$$= -\langle \nabla_h \nabla F, T(h,k)\rangle - \langle \nabla F, (\nabla_h T)(h,k)\rangle.$$

So we obtain the result. ∎

PROPOSITION 7.2.

$$\langle \nabla^3 F, (h\wedge k)\otimes h\rangle_{\otimes^3 H} = \langle R^P(h,k)(\nabla F), h\rangle_H - \langle \nabla^2 F, T(h,k)\otimes h\rangle,$$

where $\langle \nabla^2 F, T(h,k)\otimes h\rangle = \langle \nabla_{T(h,k)}\nabla F, h\rangle_H$.

Proof. We have

$$\langle \nabla^3 F, h\otimes k\otimes h\rangle = \langle \nabla_h \nabla_k(\nabla F), h\rangle - \langle \nabla^2 F, \nabla_h k\otimes h\rangle$$

and

$$\langle \nabla^3 F, k\otimes h\otimes h\rangle = \langle \nabla_k \nabla_h(\nabla F), h\rangle - \langle \nabla^2 F, \nabla_k h\otimes h\rangle.$$

Therefore

$$\begin{aligned}\langle \nabla^3 F, (h\wedge k)\otimes h\rangle_{\otimes^3 H} &= \langle [\nabla_h, \nabla_k](\nabla F), h\rangle - \langle \nabla^2 F, (\nabla_h k - \nabla_k h)\otimes h\rangle\\ &= \langle [\nabla_h, \nabla_k](\nabla F), h\rangle - \langle \nabla_{[h,k]}(\nabla F), h\rangle\\ &\quad - \langle \nabla_{T(h,k)}(\nabla F), h\rangle\\ &= \langle R^P(h,k)(\nabla F), h\rangle - \langle \nabla^2 F, T(h,k)\otimes h\rangle. \quad ∎\end{aligned} \tag{7.2}$$

LEMMA 7.3. *Let* $k\in\mathbb{H}$. *Then*

$$\sum_n \sum_\alpha \langle \nabla^2 F, \nabla_k(h\otimes h)\rangle = 0 \qquad \textit{in } L^1(X). \tag{7.3}$$

Proof. Let $h = h_{n,\alpha}$, $z = \nabla_k h$, $z_{i,\alpha}(\tau) = (\tau\wedge\tau_i)\,\varepsilon_\alpha$. By (6.4),

$$\begin{aligned}\langle \nabla_z(\nabla F), h\rangle_H &= \sum_{i,j}\sum_{\alpha,\beta} (\partial^j_{A_\beta}\partial^i_{A_\alpha}\tilde f)(\nabla_k h)^\beta(\tau_j)\langle z_{i,\alpha}, h\rangle_H\\ &\quad + \sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde f)\langle \Gamma_z(\tau_i)\, z_{i,\alpha}, h\rangle_H + \sum_{i,\alpha}(\partial^i_{A_\alpha}\tilde f)\langle \nabla_z z_{i,\alpha}, h\rangle_H,\end{aligned}$$

$$\langle \nabla_h(\nabla F), z\rangle_H = \sum_{i,j}\sum_{\alpha,\beta} (\partial^j_{A_\beta}\partial^i_{A_\alpha}\tilde{f})\, h^\beta(\tau_j)\langle z_{i,\alpha}, \nabla_k h\rangle_H$$
$$+\sum_{i,\alpha} (\partial^i_{A_\alpha}\tilde{f})\langle \Gamma_h(\tau_i)\, z_{i,\alpha}, \nabla_k h\rangle_H$$
$$+\sum_{i,\alpha} (\partial^i_{A_\alpha}\tilde{f})\langle \nabla_h z_{i,\alpha}, \nabla_k h\rangle_H.$$

To prove (7.3), it is sufficient to establish in $L^1(X)$:

$$\text{(i)}\qquad \sum_h (\nabla_k h)^\beta(\tau_j)\langle z_{i,\alpha}, h\rangle_H + \sum_h h^\beta(\tau_j)\langle z_{i,\alpha}, \nabla_k h\rangle_H = 0;$$

$$\text{(ii)}\qquad \sum_h \langle \Gamma_{\nabla_k h}(\tau_i)\, z_{i,\alpha}, h\rangle_H + \sum_h \langle \Gamma_h(\tau_i)\, z_{i,\alpha}, \nabla_k h\rangle_H = 0;$$

$$\text{(iii)}\qquad \sum_n\sum_\beta \langle \nabla_{\nabla_k h_{n,\beta}} z_{i,\alpha}, h\rangle_H + \sum_n\sum_\beta \langle \nabla_h z_{i,\alpha}, \nabla_k h_{n,\beta}\rangle_H = 0.$$

We have

$$(\nabla_k h)^\beta(\tau_j)\langle z_{i,\alpha}, h\rangle_H = \langle \nabla_k h, z_{j,\beta}\rangle_H\,\langle z_{i,\alpha}, h\rangle_H$$
$$= -\langle h, \nabla_k z_{j,\beta}\rangle_H\,\langle z_{i,\alpha}, h\rangle_H.$$

It follows that $\sum_h (\nabla_k h)^\beta(\tau_j)\langle z_{i,\alpha}, h\rangle_H = -\langle \nabla_k z_{j,\beta}, z_{i,\alpha}\rangle_H$. In the same way, we see that $\sum_h h^\beta(\tau_j)\langle z_{i,\alpha}, \nabla_k h\rangle_H = -\langle z_{j,\beta}, \nabla_k z_{i,\alpha}\rangle_H$. So we obtain (i). For (ii), remark first

$$\Gamma_{\nabla_k h}(\tau_i) = -\int_0^{\tau_i} \Omega_\tau\left(\int_0^\tau \Gamma_k(s)\,\dot{h}(s)\,ds,\ \circ dx(\tau)\right)$$

$$\text{(iv)}\qquad = -\sum_\gamma \int_0^{\tau_i}\left(\int_0^\tau \langle \Gamma_k(s)\,\dot{h}(s), \varepsilon_\gamma\rangle\,ds\right)\Omega_\tau(\varepsilon_\gamma,\ \circ dx(\tau))$$

$$= \sum_\gamma \int_0^{\tau_i} \langle \Gamma_k(s)\,\varepsilon_\gamma, \dot{h}(s)\rangle\left(\int_s^{\tau_i} \Omega_\tau(\varepsilon_\gamma,\ \circ dx(\tau))\right)ds.$$

Then

$$\langle \Gamma_{\nabla_k h}(\tau_i)\, z_{i,\alpha}, h\rangle_H = \langle \Gamma_{\nabla_k h}(\tau_i)\,\varepsilon_\alpha, h(\tau_i)\rangle$$
$$= \sum_{\gamma'} \langle \Gamma_{\nabla_k h}(\tau_i)\,\varepsilon_\alpha, \varepsilon_{\gamma'}\rangle\langle h, z_{i,\gamma'}\rangle_H$$
$$= \sum_{\gamma,\gamma'} \langle h, z_{i,\gamma'}\rangle_H \int_0^{\tau_i} \langle \Gamma_k(s)\,\varepsilon_\gamma, \dot{h}(s)\rangle$$
$$\times\left(\int_s^{\tau_i} \langle \Omega_\tau(\varepsilon_\gamma,\ \circ dx_\tau)\,\varepsilon_\alpha, \varepsilon_{\gamma'}\rangle\right)ds.$$

Therefore,

$$\sum_h \langle \Gamma_{\nabla_k h}(\tau_i)\, z_{i,\alpha}, h\rangle_H$$

$$\text{(v)} \qquad = \sum_{\gamma,\gamma'} \int_0^{\tau_i} \langle \Gamma_k(s)\, \varepsilon_\gamma, \varepsilon_{\gamma'}\rangle \left(\int_s^{\tau_i} \langle \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\, \varepsilon_\alpha, \varepsilon_{\gamma'}\rangle \right) ds$$

$$= \sum_\gamma \int_0^{\tau_i} \left\langle \Gamma_k(s)\, \varepsilon_\gamma, \int_s^{\tau_i} \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\, \varepsilon_\alpha \right\rangle ds.$$

On the other hand, in the same way, we find

$$\langle \Gamma_h(\tau_i)\, z_{i,\alpha}, \nabla_k h\rangle_H = \sum_{\gamma,\gamma'} \int_0^{\tau_i} \dot h^{\gamma'}(\tau) \int_\tau^{\tau_i} \langle \Omega_s(\varepsilon_{\gamma'}, \circ dx_s)\, \varepsilon_\alpha, \varepsilon_\gamma\rangle\, d\tau$$

$$\cdot \int_0^{\tau_i} \langle \Gamma_k(\tau)\, \varepsilon_\gamma, \dot h(\tau)\rangle\, d\tau.$$

It follows that

$$\sum_h \langle \Gamma_h(\tau_i)\, z_{i,\alpha}, \nabla_k h\rangle_H$$

$$\text{(vi)} \qquad = \sum_{\gamma,\gamma'} \int_0^{\tau_i} \langle \Gamma_k(\tau)\, \varepsilon_\gamma, \varepsilon_{\gamma'}\rangle \int_\tau^{\tau_i} \langle \Omega_s(\varepsilon_{\gamma'}, \circ dx_s)\, \varepsilon_\alpha, \varepsilon_\gamma\rangle\, d\tau$$

$$= -\sum_{\gamma'} \int_0^{\tau_i} \langle \Gamma_k(\tau)\, \varepsilon_{\gamma'}\rangle \int_\tau^{\tau_i} \langle \Omega_s(\varepsilon_{\gamma'}, \circ dx_s)\, \varepsilon_\alpha\rangle\, d\tau.$$

So by (v) and (vi), we obtain (ii). To prove (iii), we use the integration by parts

$$\langle \nabla_{\nabla_k h} z_{i,\alpha}, h\rangle_H = -\int_0^{\tau_i} \langle \Gamma_{\nabla_k h}(\tau)\, \varepsilon_\alpha, \dot h(\tau)\rangle\, d\tau$$

$$= -\langle \Gamma_{\nabla_k h}(\tau_i)\, \varepsilon_\alpha, h(\tau_i)\rangle + \int_0^{\tau_i} \langle h(\tau), \Omega_\tau(\nabla_k h, \circ dx_\tau)\, \varepsilon_\alpha\rangle.$$

Using (iv),

$$\langle \Gamma_{\nabla_k h}(\tau_i)\,\varepsilon_\alpha, h(\tau_i)\rangle$$
$$=\sum_\gamma \int_0^{\tau_i} \langle \Gamma_k(s)\,\varepsilon_\gamma, \dot{h}(s)\rangle \int_s^{\tau_i} \langle \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\,\varepsilon_\alpha, h(\tau_i)\rangle\, ds.$$

It follows that in $L^1(X)$

$$\sum_h \langle \Gamma_{\nabla_k h}(\tau_i)\,\varepsilon_\alpha, h(\tau_i)\rangle$$
(viii) $$=\sum_{\gamma,\gamma'} \int_0^{\tau_i} \langle \Gamma_k(s)\,\varepsilon_\gamma, \varepsilon_{\gamma'}\rangle \int_s^{\tau_i} \langle \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\,\varepsilon_\alpha, \varepsilon_{\gamma'}\rangle\, ds$$
$$=\sum_\gamma \int_0^{\tau_i} \langle \Gamma_k(s)\,\varepsilon_\gamma, \int_s^{\tau_i} \langle \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\,\varepsilon_\alpha\rangle\, ds.$$

On the other hand,

$$\Omega_\tau(\nabla_k h, \circ dx_\tau) = -\Omega_\tau\left(\int_0^\tau \Gamma_k(s)\,\dot{h}(s)\, ds, \circ dx_\tau\right)$$
$$= -\sum_\gamma \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\left\langle \int_0^\tau \Gamma_k(s)\,\dot{h}(s)\, ds, \varepsilon_\gamma\right\rangle$$
$$=\sum_\gamma \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\left\langle \int_0^\tau \Gamma_k(s)\,\varepsilon_\gamma, \dot{h}(s)\, ds\right\rangle.$$

and

$$\langle h_{n,\beta}(\tau), \Omega_\tau(\nabla_k h_{n,\beta}, \circ dx_\tau)\,\varepsilon_\alpha\rangle$$
$$=\sum_\gamma \langle \varepsilon_\beta, \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\,\varepsilon_\alpha\rangle\, c_n(\tau) \int_0^\tau \dot{c}_n(s) \langle \Gamma_k(s)\,\varepsilon_\gamma, \varepsilon_\beta\rangle\, ds.$$

It follows that

$$\sum_n \sum_\beta \int_0^{\tau_i} \langle h_{n,\beta}(\tau), \Omega_\tau(\nabla_k h_{n,\beta}, \circ dx_\tau)\,\varepsilon_\alpha\rangle$$
$$=\sum_{\gamma,\beta} \int_0^{\tau_i} \langle \varepsilon_\beta, \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\,\varepsilon_\alpha\rangle \int_0^\tau \langle \Gamma_k(s)\,\varepsilon_\gamma, \varepsilon_\beta\rangle\, ds$$
$$=\sum_\gamma \int_0^{\tau_i} \left\langle \int_0^\tau \Gamma_k(s)\,\varepsilon_\gamma\, ds, \Omega_\tau(\varepsilon_\gamma, \circ dx_\tau)\,\varepsilon_\alpha\right\rangle.$$

Now using (vii) and (viii), we see that

$$\sum_n \sum_\beta \langle \nabla_{\nabla_k h_{n,\beta}} z_{i,\alpha}, h_{n,\beta} \rangle_H = 0 \qquad in\ L^1(X).$$

In the same way, we find

$$\sum_n \sum_\beta \langle \nabla_{h_{n,\beta}} z_{i,\alpha}, \nabla_k h_{n,\beta} \rangle_H = 0 \qquad in\ \ L^1(X).$$

So we obtain (iii). ∎

LEMMA 7.4. *It holds in* $L^1(X)$

$$D_k\, \Delta F = \sum_n \sum_\alpha D_k D_{h_{n,\alpha}} D_{h_{n,\alpha}} F. \tag{7.4}$$

Proof. Denote $V_\alpha(\tau_i) = \int_0^{\tau_i} \tau \langle \mathrm{Ric}_\tau\, \varepsilon_\alpha, \circ dx_\tau \rangle$. Then by (4.3),

$$\begin{aligned} D_k\, \Delta F = &\sum_{i,j} \sum_\alpha D_k(\partial^j_{A_\alpha} \partial^i_{A_\alpha} \tilde{f})(\tau_i \wedge \tau_j) \\ &+ \sum_{i,\alpha} D_k(\partial^i_{A_\alpha} \tilde{f})\, V_\alpha(\tau_i) + \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f})\, D_k V_\alpha(\tau_i). \end{aligned} \tag{7.5}$$

On the other hand, by (4.10),

$$D_h^2 F = \sum_{i,j} \sum_\alpha (\partial^j_{A_\beta} \partial^i_{A_\alpha} \tilde{f})\, h^\beta(\tau_j)\, h^\alpha(\tau_i) - \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f}) \langle \Gamma_h(\tau_i)\, h(\tau_i), \varepsilon_\alpha \rangle.$$

Then

$$\begin{aligned} D_k D_h^2 F = &\sum_{i,j} \sum_\alpha D_k(\partial^j_{A_\beta} \partial^i_{A_\alpha} \tilde{f})\, h^\beta(\tau_j)\, h^\alpha(\tau_i) \\ &- \sum_{i,\alpha} D_k(\partial^i_{A_\alpha} \tilde{f}) \langle \Gamma_h(\tau_i)\, h(\tau_i), \varepsilon_\alpha \rangle \\ &- \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f})\, D_k \langle \Gamma_h(\tau_i)\, h(\tau_i), \varepsilon_\alpha \rangle. \end{aligned} \tag{7.6}$$

We have

$$\sum_n \sum_\beta \langle \Gamma_{h_{n,\beta}}(\tau_i)\, h_{n,\beta}(\tau_i), \varepsilon_\alpha \rangle = -V_\alpha.$$

Therefore according to (7.5) and (7.6), to prove (7.4), it is sufficient to have

$$\sum_n \sum_\beta D_k \langle \Gamma_h(\tau_i)\, h(\tau_i), \varepsilon_\alpha \rangle = -D_k V_\alpha(\tau_i). \tag{7.7}$$

By Lemma 3.4,

$$\begin{aligned} D_k \Gamma_h(\tau_i) = {} & \int_0^{\tau_i} [\Gamma_k(s), \Omega_s(h, \circ dx)] - \int_0^{\tau_i} \Omega_s(\Gamma_k h, \circ dx) \\ & + \int_0^{\tau_i} \Omega_s(h, \dot{k})\, ds + \sum_\gamma \int_0^{\tau_i} (\mathscr{L}_{A_\gamma} \Omega)_{r_x(s)}\, (h, \circ dx_s)\, k^\gamma(s). \end{aligned} \tag{7.8}$$

Therefore

$$\begin{aligned} & \sum_n \sum_\beta \langle D_k \Gamma_{h_{n,\beta}}(\tau_i)\, \varepsilon_\alpha, h_{n,\alpha}(\tau_i) \rangle \\ & \quad = \sum_\beta \Big\{ \int_0^{\tau_i} \langle [\Gamma_k(s), \Omega_s(z_{i,\beta}, \circ dx_s)\, \varepsilon_\alpha, \varepsilon_\beta \rangle \\ & \qquad - \int_0^{\tau_i} \langle \Omega_s(\Gamma_k z_{i,\beta}, \circ dx_s)\, \varepsilon_\alpha, \varepsilon_\beta \rangle + \int_0^{\tau_i} \langle \Omega_s(z_{i,\beta}, \dot{k})\, \varepsilon_\alpha, \varepsilon_\beta \rangle\, ds \\ & \qquad + \sum_\gamma \int_0^{\tau_i} \langle (\mathscr{L}_{A_\gamma} \Omega)_{r_x(s)}\, (z_{i,\beta}, \circ dx_s)\, k^\gamma(s)\, \varepsilon_\alpha, \varepsilon_\beta \rangle \Big\}. \end{aligned} \tag{7.9}$$

On the other hand, by first Bianchi identity, we have

$$V_\alpha(\tau_i) = \sum_\beta \langle \Gamma_{z_{i,\beta}} \varepsilon_\alpha, \varepsilon_\beta \rangle.$$

Therefore $D_k V_\alpha(\tau_i) = \sum_\beta \langle D_k \Gamma_{z_{i,\beta}} \varepsilon_\alpha, \varepsilon_\beta \rangle$. Replacing h by $z_{i,\beta}$ in (7.8), we find that $D_k V_\alpha(\tau_i)$ is equal to the right side of (7.9). So we obtain (7.7). ▮

PROPOSITION 7.5. *Let $k \in \mathbb{H}$. Then in $L^1(X)$,*

$$D_k(\Delta F) = \sum_n \sum_\alpha D_k \langle \nabla^2 F, h_{n,\alpha} \otimes h_{n,\alpha} \rangle. \tag{7.10}$$

Proof. By (6.4), Lemma 4.2, and Lemma 7.4, it is sufficient to prove

$$\sum_{n,\alpha} D_k \langle \nabla_{h_{n,\alpha}} z_{i,\beta}, h_{n,\alpha} \rangle_H = 0 \qquad \text{in } L^1(X). \tag{7.11}$$

We have

$$\langle \nabla_{h_{n,\alpha}} z_{i,\beta}, h_{n,\alpha} \rangle_H = -\int_0^{\tau_i} \langle \Gamma_{h_{n,\alpha}}(\tau)\, \varepsilon_\beta, \dot{h}_{n,\alpha}(\tau) \rangle \, d\tau$$
$$= -\langle \Gamma_{h_{n,\alpha}}(\tau_i)\, \varepsilon_\beta, h_{n,\alpha}(\tau_i) \rangle$$
$$+ \int_0^{\tau_i} \langle h_{n,\alpha}(\tau), \Omega_\tau(h_{n,\alpha}, \circ dx)\, \varepsilon_\beta \rangle.$$

Now as in the computation of (7.8) and (7.9), we find

$$\sum_{n,\alpha} D_k \int_0^{\tau_i} \langle h_{n,\alpha}(\tau), \Omega_\tau(h_{n,\alpha}, \circ dx)\, \varepsilon_\beta \rangle$$
$$= \sum_\alpha \left\{ \int_0^{\tau_i} \langle [\Gamma_k(s), \Omega_s(z_{i,\alpha}, \circ dx_s)\, \varepsilon_\beta, \varepsilon_\alpha \rangle \right.$$
$$- \int_0^{\tau_i} \langle \Omega_s(\Gamma_k z_{i,\alpha}, \circ dx_s)\, \varepsilon_\beta, \varepsilon_\alpha \rangle + \int_0^{\tau_i} \langle \Omega_s(z_{i,\alpha}, \dot{k})\, \varepsilon_\beta, \varepsilon_\alpha \, ds$$
$$\left. + \sum_\gamma \int_0^{\tau_i} \langle (\mathscr{L}_{A_\gamma} \Omega)_{r_x(s)} (z_{i,\alpha}, \circ dx)\, k^\gamma(s)\, \varepsilon_\beta, \varepsilon_\alpha \rangle \right\}.$$

Therefore according to (7.9), we obtain (7.11). ∎

LEMMA 7.6. *Let $k \in \mathbb{H}$. Then there exist a tangent process $\widehat{\mathrm{Ric}}^P k$ such that for all $z \in H$,*

$$\langle \widehat{\mathrm{Ric}}^P k, z \rangle = \sum_n \sum_\alpha \langle R^P(k, h_{n,\alpha})\, h_{n,\alpha} - (\nabla_{h_{n,\alpha}} T)(h_{n,\alpha}, k), z \rangle$$

converges in $L^1(X)$.

Proof. It follows easily from the explicit expression (5.7) and Lemma 3.5. ∎

LEMMA 7.7. *There exists a first order differential operator D^1 on vector fields such that in $L^1(X)$*

$$\langle D^1(\nabla F), k \rangle_H = \sum_n \sum_\alpha \{ \langle \nabla^2 F, h_{n,\alpha} \otimes T(h_{n,\alpha}, k) \rangle$$
$$+ \langle \nabla^2 F, T(h_{n,\alpha}, k) \otimes h_{n,\alpha} \rangle \}. \tag{7.12}$$

Proof. A straightforward calculus gives the result. ∎

THEOREM 7.8. *Denote by* $\Delta^1 = \Delta + D^1$. *Then for* $k \in \mathbb{H}$, *we have*

$$\langle \nabla \Delta F, k \rangle = \langle \Delta^1(\nabla F), k \rangle - \langle \widehat{\mathrm{Ric}^P}\, k, \nabla F \rangle. \tag{7.13}$$

Proof. Denote $h = h_{n,\alpha}$. Then by (6.9), (7.3), (7.4), and (7.10),

$$\begin{aligned}\langle [\Delta, \nabla]\, F, k \rangle &= \sum_h \Big(\langle \nabla^3 F, h \otimes h \otimes k \rangle - D_k \langle \nabla^2 F, h \otimes h \rangle \Big) \\ &= \sum_h (\langle \nabla^3 F, h \otimes h \otimes k \rangle - \langle \nabla^3 F, k \otimes h \otimes h \rangle) \\ &\quad - \sum_h \langle \nabla^2 F, \nabla_k (h \otimes h) \rangle \\ &= \sum_h (\langle \nabla^3 F, h \otimes h \otimes k \rangle - \langle \nabla^3 F, k \otimes h \otimes h \rangle).\end{aligned}$$

Now using (7.1), (7.2), Lemma 7.6 and Lemma 7.7, we obtain

$$\begin{aligned}\langle [\Delta, \nabla]\, F, k \rangle &= \sum_h \langle R^P(k, h)\, h - (\nabla_h T)(h, k)\, \nabla F \rangle \\ &\quad - \sum_h \langle \nabla^2 F, h \otimes T(h, k) + T(h, k) \otimes h \rangle \\ &= \langle \widehat{\mathrm{Ric}^P}\, k, \nabla F \rangle - \langle D^1(\nabla F), k \rangle. \quad \blacksquare\end{aligned}$$

8. CASE WHERE $\mathrm{RIC}^M = 0$

In this section, we shall simplify the formula (7.13) in the case where $\mathrm{Ric}^M = 0$.

PROPOSITION 8.1. *Let* $k \in \mathbb{H}$. *Then*

$$\mathbb{E}\left(\sum_{n,\alpha} D_{T(h_{n,\alpha}, k)} D_{h_{n,\alpha}} F \right) = 0. \tag{8.1}$$

Proof. We have

$$\begin{aligned}D_{T(h,k)} D_h F &= \sum_{i,j} \sum_{\beta,\gamma} (\partial^j_{A_\beta} \partial^i_{A_\gamma} \tilde{f})\, T^\beta(h, k)(\tau_j)\, h^\gamma(\tau_i) \\ &\quad - \sum_i \sum_{\beta,\gamma} (\partial^i_{A_\beta} \tilde{f}) \langle \Gamma_{T(h,k)}(\tau_i)\, \varepsilon_\gamma, \varepsilon_\beta \rangle\, h^\gamma(\tau_i).\end{aligned}$$

Then

$$\sum_{n,\alpha} D_{T(h_{n,\alpha},k)} D_{h_{n,\alpha}} F = (\partial^j_{A_\beta} \partial^i_{A_\gamma} \tilde{f})\, T^\beta(z_{i,\gamma}, k)(\tau_j)$$

$$- \sum_i \sum_{\beta,\gamma} (\partial^i_{A_\beta} \tilde{f}) \langle \Gamma_{T(z_{i,\alpha},k)}(\tau_i)\, \varepsilon_\gamma, \varepsilon_\beta \rangle$$

$$= \sum_{i,\gamma} D_{T(z_{i,\alpha},k)}(\partial^i_{A_\gamma} \tilde{f}).$$

Now by (2.6), we obtain the result. ∎

LEMMA 8.2

$$\sum_{n,\alpha} \langle \nabla F, \nabla_{T(h_{n,\alpha},k)} h_{n,\alpha} \rangle_H = 0 \qquad in\ L^1(X). \tag{8.2}$$

Proof. We have

(i)

$$\langle z_{i,\gamma}, \nabla_{T(h,k)} h \rangle_H = -\int_0^{\tau_i} \langle \Gamma_{T(h,k)}(s)\, \dot{h}(s), \varepsilon_\gamma \rangle\, ds$$

$$= \int_0^{\tau_i} \langle \Gamma_{T(h,k)}(s)\, \varepsilon_\gamma, \dot{h}(s) \rangle\, ds$$

$$= \langle \Gamma_{T(h,k)}(\tau_i)\, \varepsilon_\gamma, h(\tau_i) \rangle$$

$$- \int_0^{\tau_i} \langle h(s), \Omega_s(T(h,k), \circ dx_s)\, \varepsilon_\gamma \rangle,$$

(ii)

$$\sum_{n,\alpha} \langle \Gamma_{T(h_{n,\alpha},k)}(\tau_i)\, \varepsilon_\gamma, h_{n,\alpha}(\tau_i) \rangle = \sum_\alpha \langle \Gamma_{T(z_{i,\alpha},k)}(\tau_i)\, \varepsilon_\gamma, \varepsilon_\alpha \rangle,$$

and in $L^2(X)$,

(iii)

$$\sum_{n,\alpha} \int_0^{\tau_i} \langle h_{n,\alpha}(\tau), \Omega_\tau(T(h_{n,\alpha},k), \circ dx_\tau)\, \varepsilon_\gamma \rangle$$

$$= \sum_\alpha \left\langle \varepsilon_\alpha, \int_0^{\tau_i} \Omega_\tau(T(z_{i,\alpha},k), \circ dx_\tau)\, \varepsilon_\gamma \right\rangle$$

$$= \langle \varepsilon_\alpha, \Gamma_{T(z_{i,\alpha},k)}(\tau_i)\, \varepsilon_\gamma \rangle.$$

Combining (i)–(iii), we obtain (8.2). ∎

Let $C_{h,k} = -(\nabla_h T)(h, k) + T(h, T(h, k))$. Then $C_{h,k}(\tau)$ is a tangent process, which can be written in the form

$$dC_{h,k}(\tau) = q_{h,k}(\tau)\, dx(\tau) + \dot{r}_{h,k}(\tau)\, d\tau. \tag{8.3}$$

LEMMA 8.3. *There exists a martingale tangent process* $\xi_k(\tau)$,

$$d\xi_k(\tau) = q_k(\tau)\, dx(\tau)$$

such that for all cylindrical function F,

$$\langle \nabla F, \xi_k \rangle = \sum_{n,\alpha} \langle \nabla F, C_{h_{n,\alpha}, k} \rangle \qquad \text{in } L^2(X). \tag{8.4}$$

Proof. The convergence in (8.4) is easy to see by explicit expression (5.7) of $(\nabla_h T)(h, k)$ and of $T(h, T(h, k))$. When $\mathrm{Ric}^M = 0$, $r_{h,k}$ is given by

$$\begin{aligned} dr_{h,k}(\tau) = \tfrac{1}{2} \Big\{ &- \sum_{\alpha,\beta} (\mathscr{L}_{A_\beta} \mathscr{L}_{A_\alpha} \Omega)_{r_x(\tau)}\, (h, k)\, h^\alpha(\tau)\, \varepsilon_\beta + \sum_\beta \Omega_\tau(\Omega_\tau(h, \varepsilon_\beta)\, h, k)\, \varepsilon_\beta \\ &+ \sum_\beta \Omega_\tau(h, \Omega_\tau(h, \varepsilon_\beta))\, \varepsilon_\beta + \sum_\beta \Omega_\tau(h, \Omega_\tau(h, k)\, \varepsilon_\beta)\, \varepsilon_\beta \Big\}\, d\tau \\ &- \Omega_\tau(h, k)\, \dot{h}\, d\tau. \end{aligned}$$

Therefore

$$\begin{aligned} \dot{r}_k(\tau) &= \sum_{n,\alpha} \dot{r}_{h_{n,\alpha}, k}(\tau) \\ &= \frac{\tau}{2} \sum_{\gamma,\beta} [-(\mathscr{L}_{A_\beta} \mathscr{L}_{A_\gamma} \Omega)_{r_x(\tau)}\, (\varepsilon_\gamma, k)\, \varepsilon_\beta + \Omega_\tau(\Omega_\tau(\varepsilon_\gamma, \varepsilon_\beta)\, \varepsilon_\gamma, k)\, \varepsilon_\beta \\ &\quad + \Omega_\tau(\varepsilon_\gamma, \Omega_\tau(\varepsilon_\gamma, \varepsilon_\beta)\, k)\, \varepsilon_\beta + \Omega_\tau(\varepsilon_\gamma, \Omega_\tau(\varepsilon_\gamma, k)\, \varepsilon_\beta)\, \varepsilon_\beta]. \end{aligned} \tag{8.5}$$

Since $\sum_\beta (\mathscr{L}_{A_\beta} \Omega)_{r_x(\tau)}\, (\varepsilon_\gamma, k)\, \varepsilon_\beta = 0$, then

$$- \sum_{\gamma,\beta} (\mathscr{L}_{A_\beta} \mathscr{L}_{A_\gamma} \Omega)_{r_x(\tau)}\, (\varepsilon_\gamma, k)\, \varepsilon_\beta = \sum_{\gamma,\beta} (\mathscr{L}_{[A_\gamma, A_\beta]} \Omega)_{r_x(\tau)}\, (\varepsilon_\gamma, k)\, \varepsilon_\beta.$$

Remark that $[A_\gamma, A_\beta] = -\Omega(\varepsilon_\gamma, \varepsilon_\beta)^*$. Then

$$\begin{aligned} &\mathscr{L}_{[A_\gamma, A_\beta]} \Omega)_{r_x(\tau)}(\varepsilon_\gamma, k) \\ &\quad = [\Omega_\tau(\varepsilon_\gamma, \varepsilon_\beta), \Omega_\tau(\varepsilon_\gamma, k)] - \Omega_\tau(\Omega_\tau(\varepsilon_\gamma, \varepsilon_\beta)\, \varepsilon_\gamma, k) - \Omega_\tau(\varepsilon_\gamma, \Omega_\tau(\varepsilon_\gamma, \varepsilon_\beta)\, k). \end{aligned}$$

It follows from (8.5) that

$$\dot{r}_k(\tau) = \frac{\tau}{2} \sum_{\gamma, \beta} (\Omega_\tau(\varepsilon_\gamma, \varepsilon_\beta)\, \Omega_\tau(\varepsilon_\gamma, k)\, \varepsilon_\beta + \Omega_\tau\, (\varepsilon_\gamma, \Omega_\tau(\varepsilon_\gamma, k)\, \varepsilon_\beta)\, \varepsilon_\beta). \quad \blacksquare$$

Now we shall check that $\dot{r}_k = 0$. Let $\{\varepsilon_1, ..., \varepsilon_d\}$ be the canonical basis of $\mathbb{R}^d$. Denote

$$\Omega^\lambda_{\alpha\beta\gamma} = \langle \Omega(\varepsilon_\alpha, \varepsilon_\beta)\, \varepsilon_\gamma, \lambda \rangle.$$

Then

$$\begin{aligned}\dot{r}_k(\tau) &= \frac{\tau}{2} \left\{ \sum_{\alpha\beta\gamma} \Omega_{\gamma\beta\alpha} \Omega^\alpha_{\gamma k \beta} + \sum_{\alpha\beta\gamma} \Omega_{\gamma\alpha\beta} \Omega^\alpha_{\gamma k\beta} \right\} \\ &= \frac{\tau}{2} \left\{ \sum_{\alpha\beta\gamma} \Omega_{\gamma\beta\alpha} \Omega^\alpha_{\gamma k \beta} + \sum_{\alpha\beta\gamma} \Omega_{\gamma\beta\alpha} \Omega^\beta_{\gamma k\alpha} \right\} \\ &= \frac{\tau}{2} \cdot \sum_{\alpha\beta\gamma} \Omega_{\gamma\beta\alpha} (\Omega^\alpha_{\gamma k\beta} + \Omega^\beta_{\gamma k \alpha}) = 0. \quad \blacksquare\end{aligned}$$

THEOREM 8.4 *Assume* $\mathrm{Ric}^M = 0$. *Then for* $k \in \mathbb{H}$ *and a cylindrical function* F *on* $\mathbf{P}_{m_o}(M)$, *we have the following commutation formula*

$$\mathbb{E}(\langle [\Delta, \nabla]\, F, k \rangle_H) = 0. \tag{8.6}$$

Proof. We have

$$\begin{aligned}&\langle \nabla^2 F, h \otimes T(h, k) \rangle + \langle \nabla^2 F, T(h, k) \otimes h \rangle \\ &\quad = \langle \nabla^2 F, h \wedge T(h, k) \rangle + 2 \langle \nabla^2 F, T(h, k) \otimes h \rangle \\ &\quad = -\langle \nabla F, T(h, T(h, k)) \rangle + 2 D_{T(h, k)}\, D_h F - 2 \langle \nabla F, \nabla_{T(h, k)} h \rangle_H.\end{aligned}$$

Using Lemma 8.1 and Lemma 8.2 and the fact that $\mathbb{E}(D_{\xi_k} F) = 0$, we have

$$\mathbb{E}(\langle D^1(\nabla F), k \rangle) = -\mathbb{E}\left(\sum_{n, \alpha} \langle \nabla F, T(h_{n,\alpha}, T(h_{n,\alpha}, k)) \rangle \right).$$

Now by Theorem 7.8 and Lemma 8.3, we obtain

$$\mathbb{E}(\langle \nabla \Delta F, k \rangle_H) = \mathbb{E}(\langle \Delta(\nabla F), k \rangle_H). \quad \blacksquare$$

Remark 8.5. An analogous formula has been proved in [CM2]. The vector field k in (8.6) can be extended to the category of adapted vector fields.

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